Medida de las n-Bolas

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1. Introducción

Sea $B_n = B_n(0,1) \subseteq \mathbb{R}^n$. Nuestro objetivo es calcular $|B_n| = \int_{\mathbb{R}^n} \chi_{B_n}$

2. Llevarlo a una dimension menos

$$\int_{\mathbb{R}^n} \chi_{B_n} = \int_{\mathbb{R}} \int_{\mathbb{R}^{(n-1)}} \chi_{B_n}(x,y) dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \chi_{(B_n)_x}(y) dy dx = \int_{\mathbb{R}} |(B_n)_x| dx$$

Pero

$$B_n = \{(x,y) \in \mathbb{R}^n : x^2 + \sum_{i=1}^{n-1} y_i^2 \leq 1\}$$

y si $|x| \leq 1$

$$(B_n)_x = \{y \in \mathbb{R}^{n-1} : (x,y) \in B_n\} = \{y \in \mathbb{R}^{n-1} : x^2 + \sum_{i=1}^{n-1} y_i^2 \leq 1\} =$$

$$= \{y \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} y_i^2 \leq 1 - x^2\} = B_{n-1}(0, \sqrt{1-x^2})$$

Si $|x| > 1 \Rightarrow (B_n)_x = \emptyset$. Ademas ya sabemos que $|B_n(0,r)| = r^n |B_n(0,1)|$.
Con lo cual:

$$\int_{\mathbb{R}^n} \chi_{B_n} = \int_{\mathbb{R}} |(B_n)_x| dx = \int_{-1}^{1} |B_{n-1}| (\sqrt{1-x^2})^{n-1} =$$

$$= |B_{n-1}| \int_{-1}^{1} (1-x^2)^{\frac{n-1}{2}}$$

Y obtuvimos una linda recursión

3. Muchas cuentas

La idea es ver si podemos calcular una formula cerrada.

$$\int_{-1}^{1}(1-x^2)^{\frac{n}{2}} = 2\int_{0}^{1}(1-x^2)^{\frac{n}{2}} = 2\int_{0}^{\frac{\pi}{2}}(1-\sin^2 t)^{\frac{n}{2}}\cos t\ dt = 2\int_{0}^{\frac{\pi}{2}}\cos^{n+1} t\ dt$$

Si hacemos la sustituición $x = \sin t$, $dx = \cos t\ dt$, llamemos ahora:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n t\ dt = \int_0^{\frac{\pi}{2}} \cos^{n-1} t \cos t\ dt =$$

(Integrando por partes)

$$= [\sin t \cos^{n-1} t]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t \sin^2 t\ dt =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} t(1-\cos^2 t)\ dt = (n-1)(I_{n-2} - I_n)$$

Luego:

$$I_n = (n-1)(I_{n-2} - I_n) \Rightarrow I_n = \frac{n-1}{n} I_{n-2}\ , I_0 = \frac{\pi}{2}\ , I_1 = 1$$

$$I_{2n} = I_0 \prod_{i=0}^{n-1} \frac{2n-1-2i}{2n-2i} = \frac{\pi}{2} 2n! \prod_{i=0}^{n-1} \frac{1}{(2n-2i)^2} = \frac{\pi 2n!}{2^{2n+1}(n!)^2}$$

$$I_{2n+1} = I_1 \prod_{i=0}^{n-1} \frac{2n-2i}{2n-2i+1} = \frac{\prod_{i=0}^{n-1}(2n-2i)^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

4. Muchas mas cuentas

Hasta ahora sabemos:

$$|B_n| = 2|B_{n-1}|I_n$$

$$|B_{2n}| = \pi|B_{2n-1}|\frac{2n!}{2^{2n}(n!)^2}$$

$$|B_{2n-1}| = |B_{2n-2}|\frac{2^{2n-1}((n-1)!)^2}{(2n-1)!}$$

$$|B_{2n}| = \pi|B_{2n-2}|\frac{2^{2n-1}((n-1)!)^2}{(2n-1)!}\frac{2n!}{2^{2n}(n!)^2} = \frac{\pi}{n}|B_{2n-2}|$$

$$|B_{2n}| = \frac{\pi^n}{n!}$$

$$|B_{2n+1}| = |B_{2n}|\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{\pi^n}{n!}\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \pi^n\frac{2^{2n+1}n!}{(2n+1)!}$$