Cuestiones Pendientes

$X_1 \ldots X_n \underset{\sim}{\text{iid}} \mathcal{E}(\theta)$ $\quad f_X(x;\theta) = \theta e^{-\theta x}$ $\quad x \geq 0$

$\theta > 0$

$EX = \frac{1}{\theta}$

1) $\underset{\smile}{V} = \theta X \quad EV = \theta EX = \theta \frac{1}{\theta} = 1$

$$F_V(v) = P(V \leq v) = P(\theta X \leq v) = P\left(X \leq \frac{v}{\theta}\right)$$

$$= F_X\left(\frac{v}{\theta}\right) = 1 - e^{-\theta\, v/\theta} = 1 - e^{-v}$$

$$f_V(v) = e^{-v} \qquad V \sim \mathcal{E}(1)$$

2) $\bar{X} = \frac{1}{n}(X_1 + \ldots + X_n)$ $\qquad R = \frac{X_{(n)}}{X_{(1)}}$

a) $T = \sum X_i$

$\bar{X}$ minimal suficiente completo $\quad \perp \quad R = \frac{V_{(n)}/\cancel{\theta}}{V_{(1)}/\cancel{\theta}}$ ancilar

Ancilar $\quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$\boxed{X_1 \ldots X_n \underset{\sim}{\text{iid}}} \; \mathcal{E}(\theta) \quad \overset{D}{=} \quad \boxed{\frac{V_1}{\theta}, \ldots \frac{V_n}{\theta}} \sim \mathcal{E}(1)$

$f_{X_{(1)}, X_{(n)}}(x, y) \qquad f_{X_{(n)}/X_{(1)}}(w)$

Sin Basu

$$f_{\bar{X},R}(u,v) = f_{\bar{X}}(u)\, f_R(v)$$

$$R = \frac{X_{(n)}}{X_{(1)}}$$

Usando Basu

1) Est suficiente completo $\bar{X}$ ✓

2) Est Ancilar dist $R = \dfrac{X_{(n)}}{X_{(1)}}$

no depende de θ.

$V_1 \sim \mathcal{E}(1)$ $\qquad V \overset{\mathcal{D}}{=} \theta X$

$$(V_1 \dots V_n) \overset{\mathcal{D}}{=} (\theta X_1, \dots, \theta X_n)$$

$$\left(\frac{V_1}{\theta}, \dots, \frac{V_n}{\theta}\right) \overset{\mathcal{D}}{=} (X_1, \dots, X_n)$$

$$\left[\frac{V_{(1)}}{\theta}, \dots, \frac{V_{(n)}}{\theta}\right] \overset{\mathcal{D}}{=} \left[X_{(1)}, \dots, X_{(n)}\right]$$

$$\frac{V_{(n)}/\cancel{\theta}}{V_{(1)}/\cancel{\theta}} \overset{\mathcal{D}}{=} R = \frac{X_{(n)}}{X_{(1)}} \overset{\mathcal{D}}{=} \frac{V_{(n)}}{V_{(1)}}$$

no dep θ

Ancilar

$$F_{V_{(n)}}(v) = P(V_{(n)} \le v)$$

$$= P(V_1 \le v, \dots, V_n \le v) = \left[F_V(v)\right]^n = \left[1 - e^{-v}\right]^n$$

R ancilar $\quad R = \dfrac{X_{(n)}}{X_{(1)}}$

$f_{X_{(1)}, X_{(n)}}(u,v) =$

$W = X_{(1)}$

$Q = \dfrac{X_{(n)}}{X_{(1)}}$

$X, Y \sim f_{X,Y}(x,y)$

$U = U(X,Y) \qquad X_{(1)}$

$V = V(X,Y)$

Insesgamiento es una propiedad de un estimador

$$E\,\delta(X) = \theta$$

Consistencia es una propiedad de una sucesión

$$\delta_1(X),\ \delta_2(X), \dots$$

$$\forall \varepsilon > 0 \qquad P(|\delta_n(X) - \theta| > \varepsilon) \xrightarrow{n\to\infty} 0$$

Suficiencia es una propiedad de una familia de distribuciones

$$X_1 \dots X_n \overset{iid}{\sim} \mathcal{P}(\lambda) \qquad f_{X_1 \dots X_n}(x_1 \dots x_n; \lambda) = \frac{e^{-\lambda n}\lambda^{\sum x_i}}{\prod x_i}$$

$$\mathcal{F} = \left\{ f_{X_1 \dots X_n}(x_1 \dots x_n / \lambda) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod x_i}\ ;\ \lambda > 0 \right\} \qquad T = \sum X_i$$

$$\mathcal{F}_1 = \mathcal{F} \cup \{ f_{X_1 \ldots X_n}(x_1 \ldots x_n, \lambda) = \frac{1}{\lambda^n} \mathbb{1}(X_{(n)} \leq \lambda) \}$$

Para $\mathcal{F}_1$ $T = \sum X_i$ no es más nada

Recordar Completitud

$$E_\theta \, g(T) = 0 \quad \Rightarrow \quad g(T) = 0 \quad \text{a.e.}$$

$\forall \theta \in \Theta$

$$E_f \, g(T) = 0 \quad \Rightarrow \quad g(T) = 0$$

$$\forall f \in \mathcal{F}$$

g no puede depender de θ salvo que

$$\# \Theta = 1$$

<u>Ej</u>

$$X_1 \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$\mathcal{F} = \{ N_n(\mu, \sigma^2) : \mu \in \mathbb{R}, \ \sigma^2 \in \mathbb{R}^+ \}$$

$$\tilde{\mathcal{F}} = \{ N_n(\mu, \sigma^2) : \mu \in \mathbb{R}, \ \sigma^2 = \sigma_0^2 \}$$

g puede depender de σ_0^2.

CORRECTO

$\forall f \in \mathcal{F} = \{ P(\lambda) \, ; \, \mathcal{E}(\lambda) \, ; \, U(0,\lambda) \}$

$f(\cancel{X}, \theta) = g(T(\cancel{X}), \theta) \, h(X)$

$T = \Sigma X_i$

$T(X)$ es suficiente para $\mathcal{F} = \{ N_n(\mu, 1) \sim \}$

Estimación Bayesiana

Notación

$\Theta \sim f(\theta)$ $\quad X|\Theta=\theta \sim f(x;\theta)$ $\quad \Theta|X=x \sim f(\theta;x)$

$\quad\;\; f_{\Theta} = \xi \qquad\qquad\qquad\quad f_{X|\Theta} \qquad\qquad\qquad\qquad f_{\Theta|X}$

Priori $\qquad\qquad\qquad$ modelo $\qquad\qquad\qquad$ posteriori

¿Como resumir la posteriori para Estimación puntual?

En teoría de la decisión se toma una función de perdida

$$L(\theta, d)$$

perdida de estimar $g(\theta)$ mediante d

$$L(a,a) = 0$$

$$\geq 0$$

El riesgo frecuentista es

$$R(\theta,\delta) = E_\theta\left[L(\theta,\delta)\right]$$

$$= E\left[L(\Theta,\delta) \mid \Theta = \theta\right]$$

En un tratamiento Bayesiano $L(\Theta,\delta)$ es aleatoria en los dos variables

riesgo Bayesiano $\quad r(\delta,\xi) = E\left[L(\Theta,\delta)\right]$

Def: Estimador puntual Bayesiano de $g(\theta)$

$$\hat{\delta}^B = \arg\min_{\delta} r(\delta, \mathcal{J})$$

Teorema: Sea $\mathcal{J}$ la priori de Θ y $F(x;\theta)$ la distribución condicional de X dado $\Theta = \theta$.
Supongamos que se quiere estimar $g(\theta)$ y la perdida es $l(\theta, d)$. Supongamos que se cumple

a) Existe un algún estimador $\delta_0(X)$ de $g(\theta)$ con riesgo finito.

b) Para todo $X = x$ existe un valor $\delta_{\mathcal{J}}(x)$ que minimiza

notar esperanza bajo posteriori

$$E[l(\Theta, \delta(X)) \mid X = x]$$

Entonces $\delta_{\mathcal{J}}(X)$ es el estimador Bayesiano de $g(\theta)$.

Dem: Sea $\delta(X)$ un estimador con riesgo finito

$$E[L(\Theta, \delta(X))] < \infty$$

$$E\Big[\underbrace{E[L(\Theta, \delta(X)) \mid X]}\Big] < \infty$$

$$E[L(\Theta, \delta_{\mathcal{J}}(X)) \mid X = x] \leq E[L(\Theta, \delta(X)) \mid X = x] < \infty \quad \text{a.e.}$$

por b)

$$E[L(\Theta, \delta_{\mathcal{J}}(X)) \mid X] \leq E[L(\Theta, \delta(X)) \mid X]$$

$$\underbrace{E\left[L(\theta,\delta_J(X))\right]}_{r(\delta_J,\,J)} \leq E\left[L(\theta,\delta(X))\right]$$

$$r(\delta_J, J) \leq r(\delta, J)$$

δ_J es el estimador Bayesiano. $\diamond$

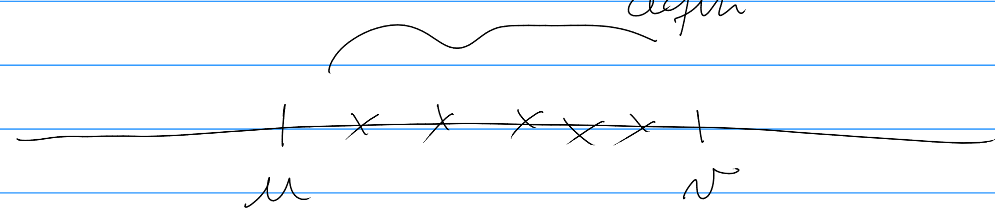
Distribución conjunta de $X_{(1)}$ y $X_{(n)}$

$$F_{X_{(1)},X_{(n)}}(u,v) = P(X_{(1)} \leq u\,;\, X_{(n)} \leq v)$$

$$\underbrace{P(X_{(n)} \leq v)}_{\substack{\text{conozco} \\ F_{X_{(n)}}(v)}} = \overbrace{P(X_{(1)} \leq u\,;\, X_{(n)} \leq v)}^{F_{X_{(1)},X_{(n)}}(u,v)} + P(X_{(1)} > u\,;\, X_{(n)} \leq v)$$

$$F_{X_{(1)},X_{(n)}}(u,v) = F_{X_{(n)}}(v) - \underbrace{P(X_{(1)} > u, X_{(n)} \leq v)}$$

todos los X_i aquí



$$P(X_{(1)} > u\,;\, X_{(n)} \leq v) = P(u < X_1 \leq v\,;\, \ldots\,;\, u < X_n \leq v)$$

$$= \left[F_X(v) - F_X(u)\right]^n$$

$$F_{X_{(1)};X_{(n)}}(u,v) = F_X^n(v) - \left[F_X(v) - F_X(u)\right]^n$$

$$\frac{\partial^2}{\partial u\, \partial v} F_{X_{(1)},X_{(n)}}(u,v) = f_{X_{(1)},X_{(n)}}(u,v) = n(n-1)$$ anterior

$$R = \frac{X_{(n)}}{X_{(1)}}$$

$$Y = Y(X_{(1)}, X_{(n)}) = X_{(1)}$$

$$W = W(X_{(1)}, X_{(n)}) = \frac{X_{(n)}}{X_{(1)}}$$

$$\frac{\partial\, Y, W}{\partial\, X_{(1)}, X_{(n)}} = \begin{bmatrix} 1 & 0 \\ -\frac{X_{(n)}}{X_{(1)}^2} & \frac{1}{X_{(1)}} \end{bmatrix}$$

$$\det |J| = \frac{1}{X_{(1)}}$$

$$f_{Y,W}(y, w) = f_{X_{(1)}, X_{(n)}}(y, wy)\, y$$

$$f_W(w) = \int f_{Y,W}(y, w)\, dy$$

Corolario Supongamos que Θ tiene priori $\mathcal{J}$ y se cumplen las condiciones del teorema.
Entonces

a) si $l(\theta, d) = w(\theta)(g(\theta) - d)^2$

donde $w(\theta) > 0$ y $E(w(\theta)) < \infty$

$$\delta_{\mathcal{J}}(X) = \frac{E[g(\theta)w(\theta) \mid X]}{E[w(\theta) \mid X]}$$

en particular si $w(\theta) = 1$ $\delta_{\mathcal{J}}(X) = E[g(\theta) \mid X]$

Dem a)

$$\delta_{\mathcal{J}}(x) \text{ minimiza } E[L(\theta, \delta) \mid X = x] =$$

$$= E\left[w(\theta)(g(\theta) - \delta(x))^2 \mid X = x\right]$$

$$g(\delta(x)) = E\left[w(\theta) g(\theta)^2 \mid X = x\right] - 2E\left[w(\theta) g(\theta) \delta(x) \mid X = x\right]$$
$$+ E\left[w(\theta)\delta(x)^2 \mid X = x\right]$$

$$\frac{\partial}{\partial \delta} g = -2E\left[w(\theta)g(\theta) \mid X = x\right] + 2\delta(x) E\left[w(\theta) \mid X = x\right]$$

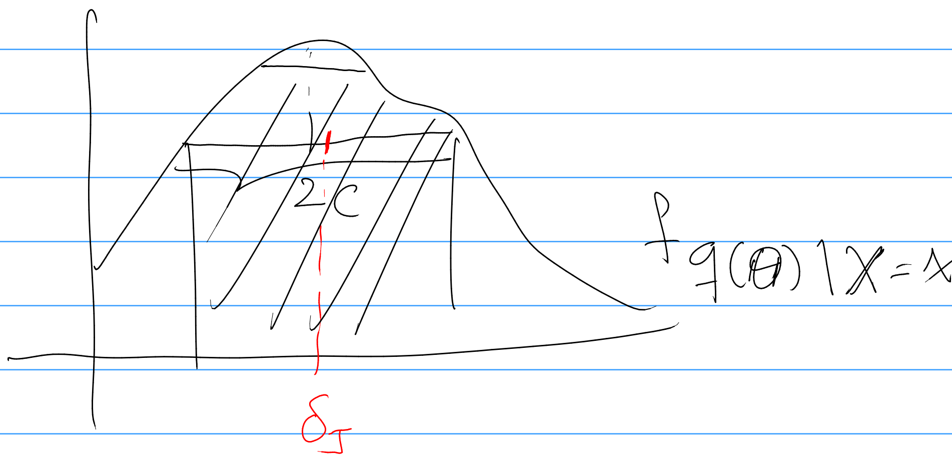
$$= 0 \iff \delta(x) = \frac{E[w(\theta)g(\theta) \mid X = x]}{E[w(\theta) \mid X = x]}$$

b) si $L(\theta, d) = |g(\theta) - d|$ entonces
$\delta_{\mathcal{J}}(X) = \text{mediana}(g(\theta) \mid X)$

c) Si $L(\theta, d) = I(|g(\theta) - d| > c)$

Suponiendo que la posteriori es unimodal,

δ_I es el punto medio del Intervalo I de longitud $2c$ que maximiza $P(g(\theta) \in I \mid X = x)$



Ej $X_1 \ldots X_n \overset{iid}{\sim} P(\lambda)$ $\quad X \mid \Lambda = \lambda \sim P(\lambda)$

$\Lambda \sim \mathcal{E}(\theta)$

$$f_{X_1 \ldots X_n \mid \Lambda}(x_1 \ldots x_n ; \lambda) = \frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod x_i!} \qquad f_{\Lambda}(\lambda) = \theta e^{-\theta\lambda}$$

$$f_{X_1 \ldots X_n ; \Lambda}(x_1 \ldots x_n, \lambda) = \theta \frac{\lambda^{\Sigma x_i}}{\prod x_i!} e^{-(n+\theta)\lambda}$$

$$f_{\Lambda \mid X_1 \ldots X_n}(\lambda; x_1 \ldots x_n) = \frac{\cancel{\theta} \frac{\lambda^{\Sigma x_i}}{\cancel{\prod x_i!}} e^{-(n+\theta)\lambda}}{\frac{1}{\cancel{\prod x_i!}} \int_0^\infty \lambda^{\Sigma x_i} \cancel{\theta} e^{-(n+\theta)\lambda} \, d\lambda}$$

$$\underset{\Lambda|X_1\dots X_n}{f}(\lambda; x_1 \dots x_n) = \frac{\lambda^{\Sigma x_i} e^{-(n+\theta)\lambda}}{\int_0^\infty \lambda^{\Sigma x_i} e^{-(n+\theta)\lambda}\, d\lambda} \cdot \frac{\Gamma(\Sigma x_i+1)\left(\frac{1}{n+\theta}\right)^{\Sigma x_i+1}}{\Gamma(\Sigma x_i+1)\left(\frac{1}{n+\theta}\right)^{\Sigma x_i+1}}$$

$$\alpha = \Sigma x_i + 1 \qquad \beta = \frac{1}{n+\theta}$$

$$U \sim G(\alpha, \beta) \qquad f_U(u) = \frac{u^{\alpha-1} e^{-u/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

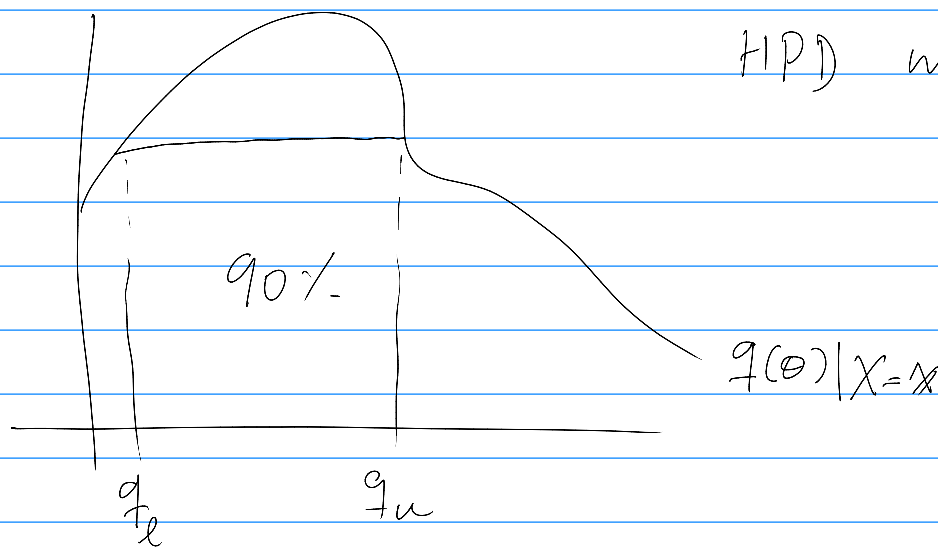
$$\underset{\Lambda|X_1\dots X_n}{f}(\lambda, x_1 \dots x_n) = \left[\frac{1}{\Gamma(\Sigma x_i+1)\left(\frac{1}{n+\theta}\right)^{\Sigma x_i+1}}\right] \cdot \lambda^{(\Sigma x_i+1)-1} e^{-\lambda/\left(\frac{1}{n+\theta}\right)}$$

$$\lambda \mid X_1 = x_1 \ldots X_n = x_n \sim \text{Gamma}\left(\sum X_i + 1 ; \frac{1}{n+\theta}\right)$$

Bajo $L(\lambda, d) = (\lambda - d)^2$

$$\delta^B(X_1 \ldots X_n) = \frac{\sum X_i + 1}{n + \theta}$$

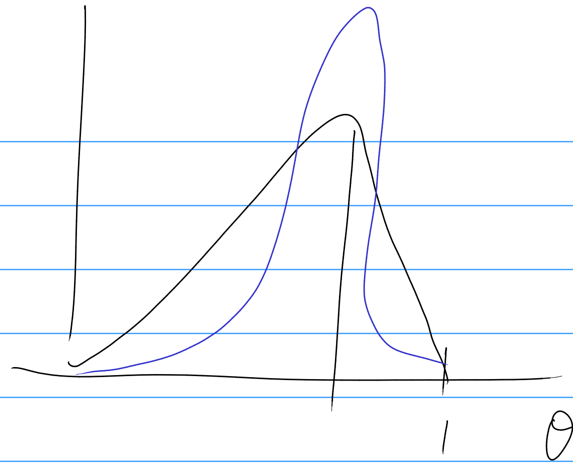
$$= \underbrace{\frac{n}{n+\theta}}_{\to 1} \underbrace{\frac{\sum X_i}{n}}_{\bar{X}} + \underbrace{\frac{1}{n+\theta}}_{\to 0}$$



HPD interval

$$P\left(q_\ell \leq g(\theta) \leq q_u \mid X = x\right) = 0.90$$

$$P(0.7 \leq \mu \leq 2 \mid X = x) = 0.90$$



$$\delta^B = \frac{\sum X_i + 1}{n + \theta}$$

$$\delta^c = \frac{\sum X_i}{n}$$