Estimadores Minimax

Recordar $\underset{\sim}{X} = (X_1, \dots X_n) \sim F_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta) \qquad \theta \in \Theta$

$\lambda = q(\theta)$ a estimar

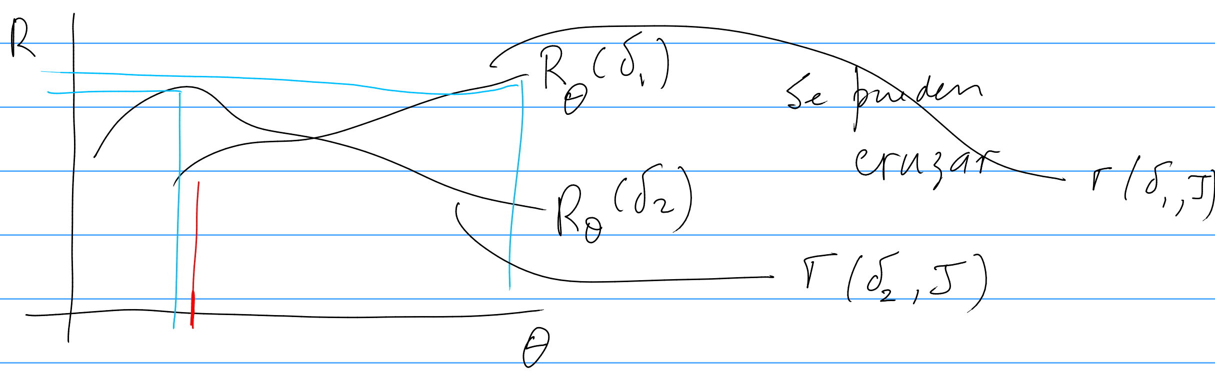
$\ell(\theta, d)$ perdida

$\delta = \delta(\underset{\sim}{X})$ estimador de $q(\theta)$

$R_\theta(\delta) = E_\theta\, \ell(\theta, \delta)$ riesgo clasico

$= E\left[\ell(\Theta, \delta) \mid \Theta = \theta\right]$ enfoque Bayesiano

Selección de Estimadores. Minimizar R



Soluciones 1) IMVU. Admitir solo δ insesgados y en ese subconjunto minimizar uniformemente

2) Bayesiana. Minimizar riesgo promedio según una priori J

$$\min \int R_\theta(\theta, \delta)\, J(\theta)\, d\theta$$

3) Minmax. Hoy

Def Maximo riesgo de un estimador δ

$$MR(\delta) = \sup_{\theta \in \Theta} R_\theta(\delta)$$

Def δ^* es minimax ssi $\forall$ estimador δ

$$MR(\delta^*) \leq MR(\delta)$$

Recordar Riesgo Bayes para el estimador δ con priori J es

$$r(\delta, J) = \int R(\theta, \delta)\, J(\theta)\, d\theta$$

El estimador Bayesiano δ_J lo minimiza, o sea

$\forall \delta$ $$r(\delta_J; J) \leq r(\delta, J)$$

Def Una priori J_0 es 'menos favorable' si $\forall J$

$$r(\delta_J, J) \leq r(\delta_{J_0}, J_0)$$

Teorema Supongamos que para un estimador Bayesiano δ_{J_0}, su $R(\delta_{J_0}, \theta)$ es constante en θ.

Entonces
a) δ_{J_0} es minimax
b) si δ_{J_0} es único como Bayesiano, también lo es como minimax.
c) J_0 es la menos favorable.

Dem

$$\underbrace{r(\delta_{J_0}, J_0)} = \int \underbrace{R(\delta_{J_0}, \theta)}\, J_0(\theta)\, d\theta$$

$$= R(\delta_{J_0}, \theta) \underbrace{\int J_0(\theta)\, d\theta}_{=1} \quad \text{por cte}$$

$$= \underbrace{\sup_{\theta \in \Theta} R(\delta_{J_0}, \theta)} \quad (*)$$

a) Tomemos $\delta \neq \delta_{J_0}$

$$MR(\delta) = \sup_{\theta \in \Theta} R(\delta, \theta) \geq R(\delta, \theta)$$

tomar E_{J_0}

$$MR(\delta) = \sup_{\theta \in \Theta} R(\delta; \theta) \geq \underbrace{\int R(\delta, \theta) J_0(\theta)\, d\theta}_{r(\delta, J_0)}$$

$$MR(\delta) \geq r(\delta, J_0)$$

Aparte, como δ_{J_0} es Bayesiano $r(\delta_{J_0}, J_0) \leq r(\delta, J_0)$
entonces

$$MR(\delta) \geq r(\delta, J_0) \geq r(\delta_{J_0}, J_0) = \sup_{\theta \in \Theta} R(\delta_{J_0}, \theta) \quad (*)$$

$$MR(\delta) \geq MR(\delta_{J_0}) \qquad (\text{def } MR)$$

$\therefore \quad \delta_{J_0}$ es minimax (def minimax)
(parte a) demostrada)

b) Supongamos unicidad en el sentido Bayesiano osea

$$\forall \delta \neq \delta_{J_0} \quad r(\delta, J_0) > r(\delta_{J_0}, J_0)$$

$$MR(\delta) > MR(\delta_0) \qquad \delta_0 \text{ único como minimax.}$$

c) Sea J otra priori y δ_J su estimador de Bayes

$$r(\delta_J, J) \leq r(\delta_{J_0}, J) = \int R(\delta_{J_0}, \theta) J(\theta)\, d\theta$$

$$\leq \sup_{\theta \in \Theta} R(\delta_{J_0}, \theta) \underbrace{\int J(\theta)\, d\theta}_{1}$$

$$r(\delta_J, J) \leq \sup_{\theta \in \Theta} R(\delta_{J_0}, \theta) \overset{(*)}{=} r(\delta_{J_0}, J_0)$$

$\therefore$ J_0 es menos favorable (por def) $\diamond$

Ej $X_1, \dots X_n$ tales que dado $\Theta = \theta$ son indep $B(\theta)$

$\Theta \sim \text{Beta}(a,b)$ $\qquad T = \sum X_i$

$$\delta_{a,b} = \frac{T + a}{n + a + b} \quad \text{bajo pérdida cuadrática}$$

$$R_\theta(\delta_{a,b}) = \text{Var}_\theta(\delta_{a,b}) + [\theta - E_\theta \delta_{a,b}]^2$$

$$E_\theta \delta_{a,b} = \frac{E_\theta T + a}{n + a + b} = \frac{n\theta + a}{n + a + b} \qquad T \sim B(n, \theta)$$

$$\text{Var}_\theta \delta_{a,b} = \frac{1}{(n+a+b)^2} \text{Var}_\theta T = \frac{n\theta(1-\theta)}{(n+a+b)^2}$$

$$R_\theta(\delta_{a,b}) = \frac{n\theta(1-\theta)}{(n+a+b)^2} + \left[\theta - \frac{n\theta + a}{n+a+b}\right]^2$$

$$= \frac{n\theta(1-\theta) + [(n+a+b)\theta - (n\theta + a)]^2}{(n+a+b)^2}$$

$$= \frac{n\theta - n\theta^2 + [\cancel{n\theta} + (a+b)\theta - \cancel{n\theta} - a]^2}{(n+a+b)^2}$$

$$= \frac{n\theta - n\theta^2 + (a+b)^2\theta^2 + a^2 - 2(a+b)\theta a}{(n+a+b)^2}$$

$$\text{Num} = \underbrace{[-n + (a+b)^2]}_{0}\theta^2 + \underbrace{[n - 2(a+b)a]}_{0}\theta + a^2$$

$$\begin{cases} -n + (a+b)^2 = 0 \\ n - 2(a+b)a = 0 \end{cases}$$

$(a+b)^2 = n$

$2(a+b)a = n$

$4(a+b)^2 a^2 = n^2$ sust (1)

$4 \cancel{n} a^2 = n^{\cancel{2}}$ $\qquad a^2 = \frac{n}{4}$ $\qquad a = \frac{\sqrt{n}}{2}$

vuelvo (1) $\qquad a+b = \sqrt{n}$

$$\frac{\sqrt{n}}{2} + b = \sqrt{n} \Rightarrow b = \frac{\sqrt{n}}{2}$$

$$\delta_{MINMAX} = \frac{T + \frac{\sqrt{n}}{2}}{n + \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2}} = \frac{\Sigma X_i + \sqrt{n}/2}{n + \sqrt{n}}$$

$$R(\delta_{MINMAX}, \theta) = \frac{(\sqrt{n}/2)^2}{\left(n + \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{2}\right)^2} = \frac{n/4}{(n+\sqrt{n})^2}$$

$=$

$X_1 \dots X_n \overset{iid}{\sim} B(\theta)$

Pregunta $\qquad \delta = \sum_{i=1}^{n} a_i X_i$ $\qquad$ lineal

$l(\theta, \Sigma a_i X_i) \qquad \theta \in \Theta$

$$R(\theta, \Sigma a_i X_i) = \int l(\theta, \Sigma a_i X_i) f_X(x, \theta) dx$$

Teorema 2 Supongamos que $\delta(X)$ es un estimador tal que:

i) $R(\delta,\theta) = c$ (cte) $\forall \theta \in \Theta$

ii) $\exists\ \tau_1, \tau_2, \ldots$ tales que

$$\lim_{k\to\infty} r(\delta_k, \tau_k) = c$$

Entonces δ es minimax.

Dem (notas)

Aplicación $X = (X_1 \ldots X_n) \overset{iid}{\sim} N(\theta, \sigma^2)$ σ^2 conocida

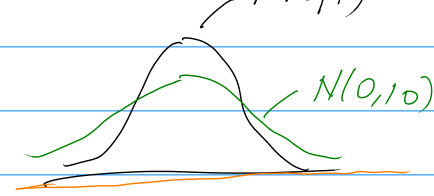
Tomemos $\delta(X) = \bar{X}$ y pérdida cuadrática

$$R(\delta,\theta) = Var_\theta(\delta) + [\theta - E_\theta \delta]^2$$

$$= Var_\theta(\bar{X}) + \underbrace{[\theta - E_\theta \bar{X}]}_{0}$$

$$= \frac{\sigma^2}{n}$$

$\tau_k \sim N(0, \rho_k^2)$ $\quad \rho_k^2 \to \infty$



$$\delta_{\tau_k} = w_k \bar{X} = \underbrace{\frac{n/\sigma^2}{(n/\sigma^2) + (1/\rho_k^2)}}_{w_k} \bar{X}$$

$$w_k \xrightarrow[k\to\infty]{} 1$$

$$R(\delta_{\tau_k}, \theta) = Var_\theta(w_k \bar{X}) + [\theta - E_\theta w_k \bar{X}]^2$$

$$= w_k^2 \, Var_\theta(\bar{X}) + [\theta - w_k E_\theta \bar{X}]^2$$

$$R(\delta_{J_k}, \theta) = w_k^2 \frac{\sigma^2}{n} + (1-w_k)^2 \theta^2$$

$$r(\delta_{J_k}, J_k) = E_{J_k}\left[w_k^2 \frac{\sigma^2}{n} + (1-w_k)^2\theta^2\right]$$

$$= w_k^2 \frac{\sigma^2}{n} + (1-w_k)^2 \rho_k^2$$

$$E_{J_k}\theta^2 = \mathrm{Var}_{J_k}\theta = \rho_k^2$$

$$\lim_{k\to\infty} r(\delta_{J_k}, J_k) = \frac{\sigma^2}{n}$$

$$\bar{X} = \hat{\theta}_{MINMAX}$$

$$(1-w_k)^2\rho_k^2 = \left[1 - \frac{n/\sigma^2}{(n/\sigma^2)+(1/\rho_k^2)}\right]^2 \rho_k^2$$

$$(1-w_k)\rho_k = \frac{\left[1-\frac{n/\sigma^2}{n/\sigma^2 + 1/\rho_k^2}\right]}{1/\rho_k} = \frac{1-\frac{a}{a+1/y^2}}{1/y}$$

$y = \rho_k$

$$\approx \frac{-\frac{n/\sigma^2}{(n/\sigma^2+1/\rho_k^2)^2}\left(-\frac{1}{(\rho_k^2)^2}\right)}{-1/\rho_k^2}$$

$$\approx \frac{-\frac{a}{(a+1/y^2)^2}\left[-\frac{2}{y^3}\right]}{-1/y^2}$$

$$\approx -\frac{a}{[a+1/y^2]^2}\frac{1}{y} \to 0 \quad y\to\infty$$

$\diamond$

Regiones de Confianza

Sea $X \sim F_X(x;\theta) \quad \theta \in \Theta$

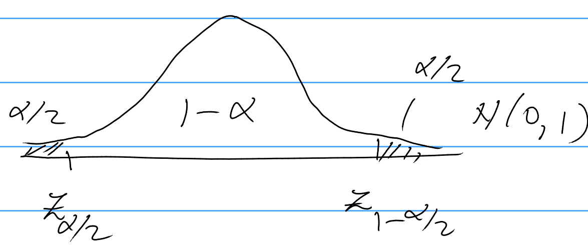
Def $S(X)$ es una región de confianza con nivel de confianza $1-\alpha$ ssi

$$\forall \theta \quad P_\theta\left(S(X) \ni \theta\right) = 1-\alpha$$

convención gralmente se toma $\alpha = 0{,}1\ ;\ 0.05\ ;\ 0.01$

Ej $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma_0^2) \qquad \sigma_0^2$ conocido

$$\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} \sim N(0,1)$$



$Z \sim N(0,1)$

$$z_{\alpha/2} = -z_{1-\alpha/2}$$

$$P\left(-z_{1-\alpha/2} \le Z \le z_{1-\alpha/2}\right) = 1-\alpha$$

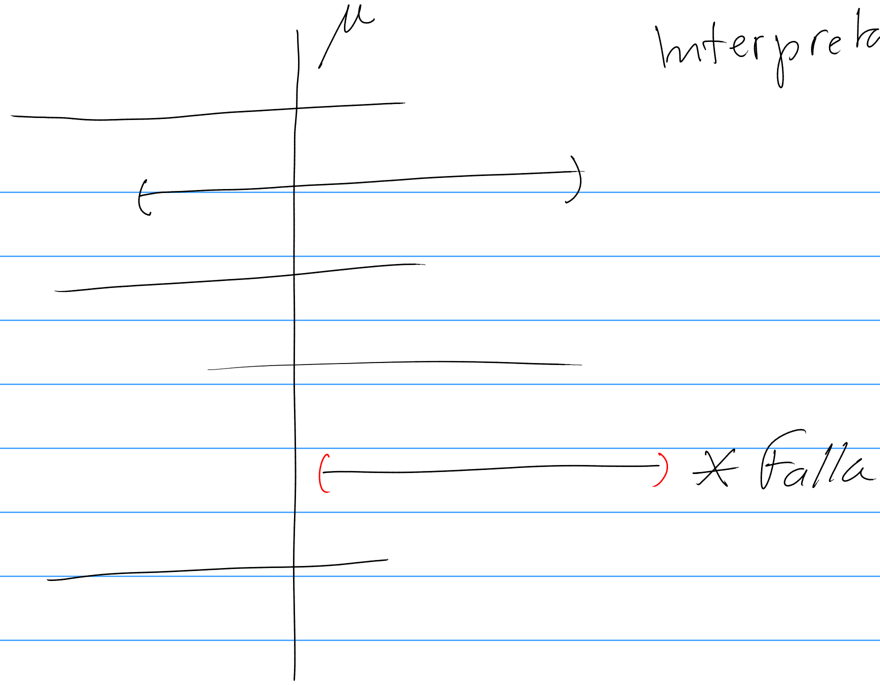
$$P\left(-z_{1-\alpha/2} \le \frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} \le z_{1-\alpha/2}\right) = 1-\alpha$$

$$P\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma_0}{\sqrt{n}} \le \mu \le \bar{X} + z_{1-\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right) = 1-\alpha$$

$$P\Bigg(\underbrace{\left(\bar{X} - z_{1-\alpha/2}\frac{\sigma_0}{\sqrt{n}}\ ;\ \bar{X} + z_{1-\alpha/2}\frac{\sigma_0}{\sqrt{n}}\right)}_{S(X)} \ni \mu\Bigg) = 1-\alpha$$

IC de $(1-\alpha)100\%$.

Interpretación ...



Metodos para obtener regiones de Confianza

a) Metodo del Pivote ✓

b) ✓ aproximado o asintótico ✓

c) Invertir un test de hipótesis

d) Metodo Bayesiano

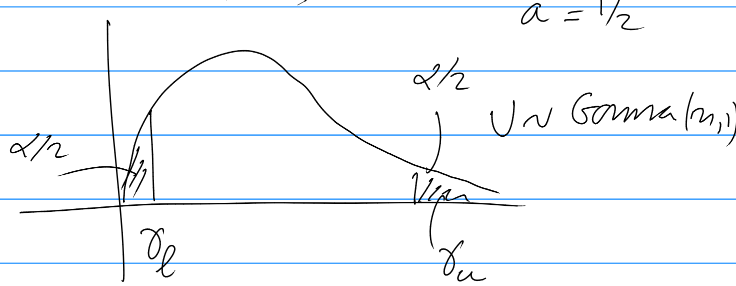
a) Método del Pivote

Ej $X_1 \ldots X_n \overset{iid}{\sim} \mathcal{E}(\theta)$ $\quad f_X(x;\theta) = \theta e^{-\theta x}$ $\quad x \geq 0$, $\theta > 0$

$\theta X_i \sim \mathcal{E}(1)$

$\underbrace{\sum_{i=1}^{n} (\theta X_i)}_{\text{ancilar}} \sim \text{Gamma}(n, 1)$

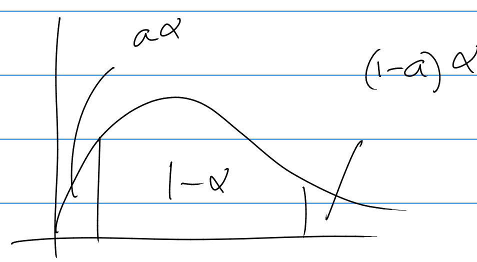
$a = 1/2$



$P[\gamma_\ell \leq U \leq \gamma_u] = 1-\alpha$

$P[\gamma_\ell \leq \theta \sum X_i \leq \gamma_u] = 1-\alpha$

$\left(\frac{\gamma_\ell}{\sum X_i} ; \frac{\gamma_u}{\sum X_i} \right)$



Ej $X_1 \dots X_n \overset{iid}{\sim} N(0, \sigma^2)$

$$\frac{X_i}{\sigma} \sim N(0,1) \qquad \sum_{i=1}^{n} \frac{X_i}{\sigma} \sim N(0, n)$$

$$\frac{1}{\sqrt{n}} \sum \left(\frac{X_i}{\sigma}\right) \sim N(0,1)$$

$$\frac{1}{\sigma} \left(\frac{1}{\sqrt{n}} \sum X_i\right) \sim N(0,1)$$

$$P\left[z_l \leq \frac{1}{\sigma} \frac{1}{\sqrt{n}} \sum X_i \leq z_u \right] = 1-\alpha$$

$$\left(\frac{\frac{1}{\sqrt{n}} \sum X_i}{z_u} ; \frac{\frac{1}{\sqrt{n}} \sum X_i}{z_l} \right) \quad 1-\alpha \quad \text{para } \sigma$$

Opcion 2

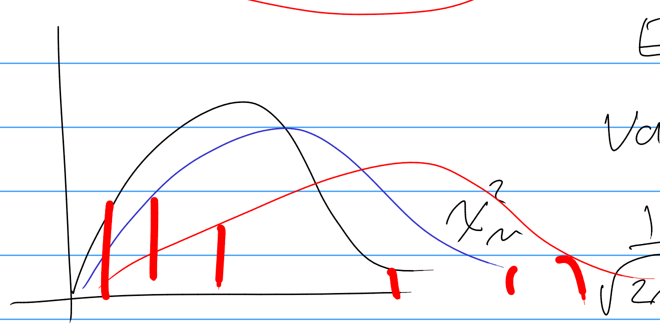
$$\left(\frac{X_i}{\sigma}\right)^2 \sim \chi^2_1$$

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i)^2 \sim \chi^2_n$$

$$P\left[\left(\frac{\sum X_i^2}{\chi^2_u} ; \frac{\sum X_i^2}{\chi^2_l} \right) \ni \sigma^2 \right] = 1-\alpha.$$

Proponer

$$P\left[\left(\sqrt{\frac{\sum X_i^2}{\chi^2_u}} ; \sqrt{\frac{\sum X_i^2}{\chi^2_l}} \right) \ni \sigma \right] = 1-\alpha$$

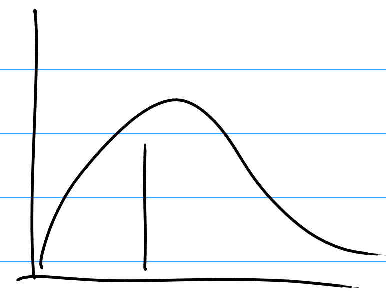


$$E\, \chi^2_n = n$$

$$Var\, \chi^2_n = 2n$$

$$\frac{1}{\sqrt{2n}} (\chi^2_n - n) \Rightarrow N(0,1)$$

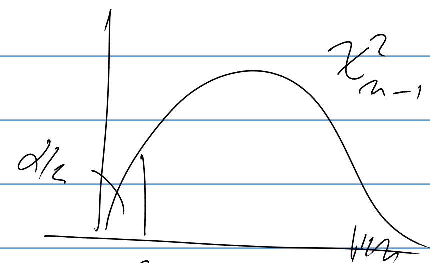
$\hat{\theta} \pm$ margen error



Ej $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Hecho $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$



$$P\left[\chi^2_l \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_u\right] = 1-\alpha$$

$$\left(\frac{(n-1)S^2}{\chi^2_u}; \frac{(n-1)S^2}{\chi^2_l}\right) \neq S^2 \pm a$$

$$\text{Centro} = (n-1)S^2 \, \frac{1}{2}\left[\frac{1}{\chi^2_u} + \frac{1}{\chi^2_l}\right] \neq S^2$$