Intervalos (regiones) de confianza

Ej Población normal $X_1 \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$$

Afirmamos

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad \Big| \qquad \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n$$

Para verlo

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left[(X_i-\bar{X}) + (\bar{X}-\mu)\right]^2$$

$$= \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu)^2 + 2(\bar{X}-\mu)\underbrace{\sum_{i=1}^{n}(X_i-\bar{X})}_{0}$$

$\sum X_i - n\bar{X}$

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$\underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2}_{U \sim \chi^2_n} = \underbrace{\frac{(n-1)S^2}{\sigma^2}}_{V} + \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{W \sim \chi^2_1}$$

$V \perp W$

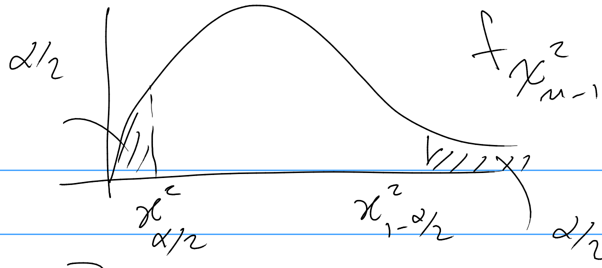
$$M_U(t) = M_V(t)\, M_W(t)$$

$$M_V(t) = \frac{M_U(t)}{M_W(t)} = \frac{\left(\frac{1}{1-2t}\right)^{n/2}}{\left(\frac{1}{1-2t}\right)^{1/2}} = \underbrace{\left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}}}$$

$$M_X(t) := E\left(e^{Xt}\right)$$

$$V \sim \text{Gamma}\left(\frac{n-1}{2}, 2\right) = \chi^2_{n-1} \quad \diamond$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$



$$P\left[\chi^2_{\alpha/2} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{1-\alpha/2}\right] = 1-\alpha$$

IC de cobertura $(1-\alpha)$ para σ^2 es:

$$\boxed{\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}} ; \frac{(n-1)S^2}{\chi^2_{\alpha/2}}\right)}$$

Recordar $Z \sim N(0,1) \perp W \sim \chi^2_k$

$$T_k = \frac{Z}{\sqrt{\frac{W}{k}}} \sim t_k \quad \text{t-student}$$

En nuestro caso

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \perp W = \frac{(n-1)S^2}{\sigma^2}$$

$$T_{n-1} = \frac{\frac{\bar{X}-\mu}{\cancel{\sigma}/\sqrt{n}}}{\sqrt{\frac{\frac{\cancel{(n-1)}S^2}{\cancel{\sigma^2}}}{\cancel{n-1}}}} = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

IC para μ

$$\bar{X} \pm t_{n-1;1-\alpha/2} \frac{S}{\sqrt{n}}$$

Observar si $X_1 \dots X_n$ son iid con $\sigma^2 < \infty$.
y n es suficientemente grande

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \overset{A}{\sim} N(0,1)$$

Tentación $\quad \bar{X} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \quad$ Valido σ conocido

$$\bar{X} \pm t_{n-1;1-\alpha/2} \frac{S}{\sqrt{n}}$$

si esta bien

$$\frac{1}{\sigma} \sqrt{n} (\bar{X} - \mu) \Rightarrow N(0,1)$$

S^2 consistente $\quad S^2 \xrightarrow{P} \sigma^2$

$$S \xrightarrow{P} \sigma$$

$$\underbrace{\frac{S}{\sigma}}_{\xrightarrow{P} 1} \boxed{\frac{1}{S} \sqrt{n} (\bar{X} - \mu) \Rightarrow N(0,1)} \quad \text{Slutzky}$$

IC $\quad \bar{X} \pm z_{1-\alpha/2} \frac{S}{\sqrt{n}} \quad$ Intervalo asintotico o aproximado

Simulación: es mejor

$$\bar{X} \pm t_{n-1;1-\alpha/2} \frac{S}{\sqrt{n}}$$

$(a;b)$ σ^2

$(\sqrt{a},\sqrt{b})$ σ

Pensar IC σ

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}\,;\,\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}\right)$$? MAL Valido Sim

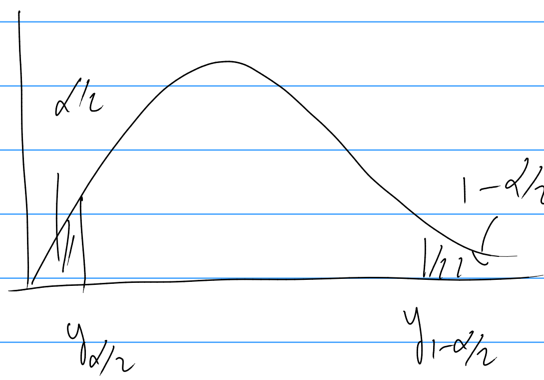
No interesantes.

$$V=\frac{(n-1)S^2}{\sigma^2}\sim\chi^2_{n-1}\qquad f_V(v)=\frac{v^{\frac{n-1}{2}-1}e^{-v/2}}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}$$

$$Y=\sqrt{V}=\frac{\sqrt{n-1}\,S}{\sigma}$$

$$F_Y(y)=P(Y\le y)=P(\sqrt{V}\le y)=P(V\le y^2)=F_V(y^2)$$

$$f_Y(y)=f_V(y^2)\,2y\qquad f_Y(y)=\frac{(y^2)^{\frac{n-1}{2}-1}e^{-y^2/2}\,2y}{\Gamma\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}}$$



$$P\left[y_{\alpha/2}\le\frac{\sqrt{n-1}\,S}{\sigma}\le y_{1-\alpha/2}\right]=1-\alpha$$

$$\left(\frac{\sqrt{n-1}\,S}{y_{1-\alpha/2}}\,;\,\frac{\sqrt{n-1}\,S}{y_{\alpha/2}}\right)$$

Selección de un tamaño muestral

Ej $X_1,\dots,X_n \overset{iid}{\sim} B(\theta)$ n gde

$$\sqrt{n}\,(\hat{p}_n-\theta)\Rightarrow N(0,\theta(1-\theta))\qquad \hat{p}_n=\frac{\sum X_i}{n}$$

IC $\quad \hat{p}_n \pm z_{1-\alpha/2} \underbrace{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}}$

$\hat{p}_n \xrightarrow{P} \theta$

$\hat{p}_n(1-\hat{p}_n) \to \theta(1-\theta)$

$$\text{longitud} = 2 z_{1-\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}} \leq c$$

$$\frac{2 z_{1-\alpha/2}}{c} \sqrt{\hat{p}_n(1-\hat{p}_n)} \leq \sqrt{n}$$

$$n \geq \left(\frac{\cancel{2} z_{1-\alpha/2}}{c}\right)^2 \underbrace{\hat{p}_n}_{1/2} \underbrace{(1-\hat{p}_n)}_{\cancel{1/2}}$$

Ej

$$n \geq \left(\frac{2 \cdot 1.96}{0.1}\right)^2 \underbrace{\hat{p}_n}_{1/2} \underbrace{(1-\hat{p}_n)}_{1/2}$$

1) conservador $\hat{p}_n = 1/2$

$$n \geq \left(\frac{z_{1-\alpha/2}}{c}\right)^2 \qquad \left(\frac{1.96}{0.1}\right)^2 \qquad \left(\frac{2}{0.1}\right)^2 = 400$$

2) utilizar otro $\hat{p}_n$

3) utiliza $\hat{p}_n$ preliminar.

Esto mismo para μ

$$\bar{X} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Intervalos (regiones) aproximados
basados en distribuciones asintóticas.

Recordar Bajo regularidad

$$\sqrt{n}\left(\hat{\theta}^{MV} - \theta\right) \Rightarrow N\left(0;\, I_1^{-1}(\theta)\right) \qquad (1)$$

$$I_1^{1/2}(\hat\theta^{MV})\, I_1^{-1/2}(\hat\theta)\, \underbrace{I_1^{1/2}(\theta)}_{\Rightarrow \mathbb{I}}\, \sqrt{n}\left(\hat{\theta}^{MV} - \theta\right) \Rightarrow N\left(0;\, \mathbb{I}\right)$$

$$I_1^{1/2}(\hat{\theta}^{MV})\, \sqrt{n}\left(\hat{\theta}^{MV} - \theta\right) \Rightarrow N\left(0;\, \mathbb{I}\right)$$

Revisar (1) Distr asintótica EMV

$$\hat{\theta}^{MV} - \theta \Rightarrow \frac{1}{\sqrt{n}} N = 0 \text{ con prob } 1$$

$$\hat{\theta}^{MV} \xrightarrow{P} \theta$$

$$I_1^{1/2}(\hat{\theta}^{MV}) \xrightarrow{P} I_1^{1/2}(\theta)$$

$$I^{-1/2}(\hat{\theta}^{MV})\, I^{1/2}(\theta) \xrightarrow{P} \mathbb{I}$$

Entonces

$$\hat{\theta}^{MV} \overset{A}{\sim} N\left(\theta;\, I^{-1/2}(\hat{\theta})\right)$$

lo puedo usar para regiones de confianza

Caso $k=1$ IC $\boxed{\hat{\theta}^{MV} \pm z_{1-\alpha/2} \dfrac{1}{\sqrt{I(\hat{\theta})\, n}}}$

Analogía $X_1 \dots X_n \overset{iid}{\sim} B(p)$ $\quad I^{-1}(p)$

$$\sqrt{n}(\hat{p}_n - p) \Rightarrow N(0, \overbrace{p(1-p)}^{I^{-1}(p)})$$

$$\hat{p}_n \pm z\sqrt{\frac{p(1-p)}{n}}$$

$$\sim N(0, I^{-1}(\hat{p}))$$

$$\hat{p}_n \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Regiones simultaneas

Supongamos $S_1(X)$ de $(1-\tilde{\alpha})$ para θ_1
$S_2(X)$ " $(1-\tilde{\alpha})$ " θ_2

$$P(S_1(X) \ni \theta_1) = 1-\tilde{\alpha} = P(S_2(X) \ni \theta_2)$$

Si $S_1(X)$ y $S_2(X)$ fueran indep.

$$P(S_1(X) \ni \theta_1 ; S_2(X) \ni \theta_2) = (1-\tilde{\alpha})^2$$

Entonces $P\Big(\underbrace{(S_1(X) \times S_2(X))}_{\text{region simultanea de cobertura } (1-\tilde{\alpha})^2} \ni (\theta_1, \theta_2)\Big) = (1-\tilde{\alpha})^2$

Entonces resuelvo

$$1-\alpha = (1-\tilde{\alpha})^2$$

$$1-0.05 = (1-\tilde{\alpha})^2$$

$$\sqrt{0.95} = 1-\tilde{\alpha} \qquad \tilde{\alpha} = 1-\sqrt{0.95} =$$

$$1-\tilde{\alpha} = 0.9747$$

Regiones simultanes aproximadas

Consideremos

$$P(\theta_1 \in S_1) = 1 - \alpha/2 = P(\theta_2 \in S_2)$$

$$P(\theta_1 \in S_1 \overset{\cap}{;} \theta_2 \in S_2) = 1 - \underbrace{P[(\theta_1 \notin S_1) \cup (\theta_2 \notin S_2)]}_{\geq P(\theta_1 \notin S_1) + P(\theta_2 \notin S_2)}$$

$$\leq 1 - \frac{\alpha}{2} - \frac{\alpha}{2}$$

$$P(\theta_1 \in S_1 ; \theta_2 \in S_2) \leq 1 - \alpha$$

o inductivamente

$$P(\theta_1 \in S_1 \; --- \; \theta_k \in S_k) \leq 1 - \alpha \quad \text{con} \quad P(\theta_i \in S_1) = 1 - \frac{\alpha}{k}$$

Metodo Bonferroni ———.