Metodo de Maxima Verosimilitud

Motivamos

Ej Tenemos un dado con θ lados rojos y el resto blancos.

Observamos r, b, r

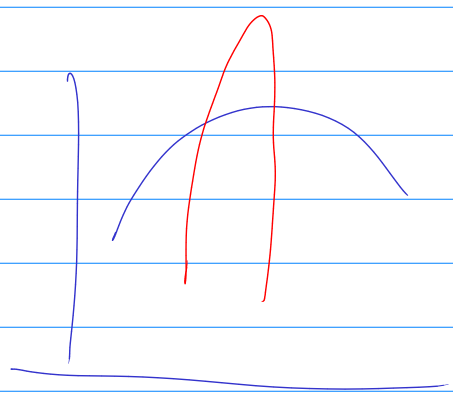
Objetivo estimar θ = # caras rojas

2 o r 1 o b

θ	$P(rbr \mid \theta)$		$P(rrr \mid \theta)$
0	0		0
1	$(1/6)^2 (5/6) = 5/216$	1	$(1/6)^3 = 1/6^3$
2	$(2/6)^2 (4/6) = 16/216$	2	$(2/6)^3 = 8/6^3$
3	$(3/6)^2 (3/6) = 27/216$	3	$(3/6)^3 = 27/$
4	$(4/6)^2 (2/6) = 32/216$	1000000 $\leftarrow$ Max	$\vdots$
5	$(5/6)^2 (1/6) = 25/216$	3 2	$\vdots$
6	0		$(6/6)^3 = 1$

Θ

$\hat{\theta}^{MV} = 4$



Idea: dado un modelo de probabilidad (que estoy tirando un dado con 6 lados) ¿cuál debe ser el valor del parametro para que la muestra sea lo más típica (probable) posible.

Def $X = (X_1, \dots, X_n) \sim F_\Theta \in \mathcal{F} = \{ F_\Theta : \Theta \in \Theta \subset \mathbb{R}^q \}$

$\sim f_\Theta(x_1, \dots, x_n)$

$$\hat{\Theta}^{MV} = \underset{\Theta \in \Theta}{\arg\max}\; f_\Theta(x_1, \dots, x_n).$$

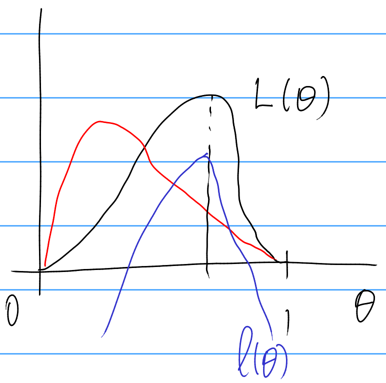
Ej 1 $\quad X_1, X_2, \dots, X_n \overset{iid}{\sim} B(\theta) \qquad X_i = \begin{cases} 0 & cp \quad 1-\theta \\ 1 & cp \quad \theta \end{cases}$

$$f_{X_i}(x_i;\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$$

$$f_{X}(x_1, x_2, \dots, x_n;\theta) = \prod_{i=1}^{n} f_{X_i}(x_i;\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$= \theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i} \quad =: L_n(\theta)$$

Función de Verosimilitud $L_n(\theta)$.



$$l_n(\theta) = \log_{(e)} L(\theta) =$$

$$= (\Sigma x_i) \log \theta + (n-\Sigma x_i) \log (1-\theta)$$

$$\frac{\partial}{\partial\theta} l_n(\theta) = \frac{\Sigma x_i}{\theta} - \frac{n-\Sigma x_i}{1-\theta} = 0$$

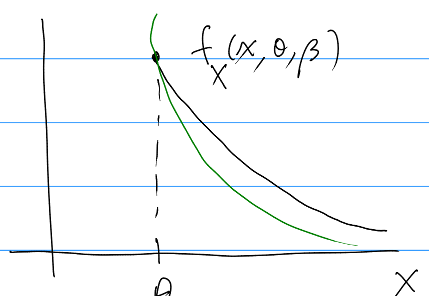
$$\iff \frac{\Sigma x_i}{\theta} = \frac{n-\Sigma x_i}{1-\theta}$$

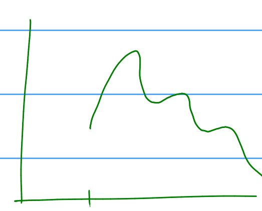
$$\iff \frac{1-\theta}{\theta} = \frac{n-\Sigma x_i}{\Sigma x_i} \iff \frac{1}{\theta} - \cancel{1} = \frac{n}{\Sigma x_i} - \cancel{1} \iff \boxed{\hat{\theta}_{MV} = \frac{\Sigma x_i}{n} = \hat{p}}$$

$$\frac{\partial^2 l_n(\theta)}{\partial\theta^2} = -\frac{\Sigma x_i}{\theta^2} - \frac{n-\Sigma x_i}{(1-\theta)^2} < 0$$

Ej 2 $\quad X_1 \dots X_n \overset{iid}{\sim} f_{X_i}(x_i;\theta,\beta) = \begin{cases} \beta\theta^\beta \frac{1}{x_i^{\beta+1}} & x \geq \theta \\ 0 & c.c. \end{cases}$

PARETO $\qquad \theta, \beta > 0$





$$f_{X_i}(x;\theta,\beta) = \beta\theta^\beta x_i^{-\beta-1} \mathbb{1}(x_i \geq \theta)$$

$$f_{X_1 \dots X_n}(x_1, \dots, x_n;\theta,\beta) = \prod_{i=1}^{n} \beta\theta^\beta x_i^{-\beta-1} \mathbb{1}(x_i \geq \theta)$$

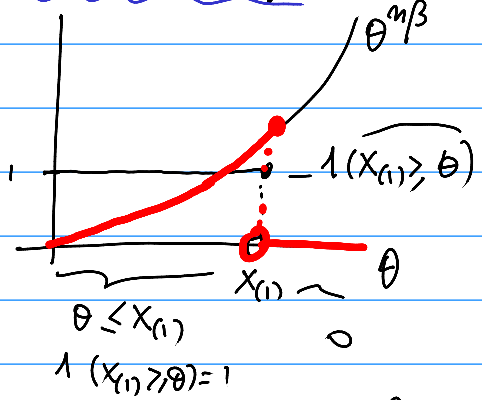
$$f_{X_1,\dots,X_n}(x_1,\dots,x_n;\theta,\beta) = \beta^n \theta^{n\beta} \left[\prod_{i=1}^n x_i\right]^{-\beta-1} \underbrace{\prod_{i=1}^n 1(x_i \geq \theta)}_{=}$$

$$= \beta^n \left[\prod_{i=1}^n x_i\right]^{-\beta-1} \theta^{n\beta} \underbrace{1(x_1 \geq \theta, \dots, x_n \geq \theta)}_{=}$$

$$= \underbrace{\beta^n \left(\prod x_i\right)^{-\beta-1}}_{I} \; \underbrace{\theta^{n\beta} \, 1(x_{(1)} \geq \theta)}_{II}$$

$x_{(1)} = \min\{x_1, \dots, x_n\}$

II, para cualquier β tiene la forma



$$\hat{\theta}^{MV} = X_{(1)}$$

$$L_n(\beta, \hat\theta) = \beta^n \left(\prod x_i\right)^{-\beta-1} \hat\theta^{n\beta} \, \overbrace{1(X_{(1)} \geq \hat\theta)}^{1}$$

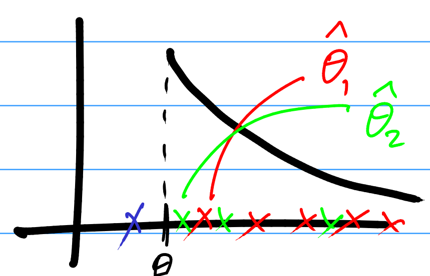
$$\ell_n(\beta, \hat\theta) = n \log \beta - (\beta+1) \sum \log x_i + n\beta \log \hat\theta$$

$$\frac{\partial}{\partial \beta} \ell_n(\beta, \hat\theta) = \frac{n}{\beta} - \sum \log x_i + n \log \hat\theta = 0$$

$$\frac{n}{\beta} = \sum \log x_i - \sum \log x_{(1)}$$

$$\frac{1}{\beta} = \frac{1}{n} \sum \log \left(\frac{x_i}{x_{(1)}}\right)$$

$$\boxed{\hat\theta = X_{(1)}} \qquad \boxed{\hat\beta^{MV} = \left[\frac{1}{n} \sum \log \left(\frac{x_i}{x_{(1)}}\right)\right]^{-1}}$$



Ej 3 Regresión de Poisson

$$Y_1, Y_2, \dots, Y_n \overset{indep}{\sim} P(\lambda_i) \qquad \log \lambda_i = \underset{(1\times k)}{\beta'} \underset{(k\times 1)}{x_i}$$

$$f_{Y_i}(y_i, \lambda_i) = \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i!}$$

i	y_i	x_{i1}	x_{i2}	$\dots$	$x_{i\,k-1}$
		2	1	7	8
		1	1	1	1

$$f_{Y_i}(y_i, \beta) = \frac{e^{-e^{\beta' x_i}} \left(e^{\beta' x_i}\right)^{y_i}}{y_i!}$$

$$2\beta_1 + \beta_2 + 7\beta_3 + 8\beta_4$$

$$Y_i \sim P(\lambda_i) \qquad \lambda_i \overset{1-1}{\longleftrightarrow} (1, x_{i1}, \dots, x_{ik})$$

$$\log \lambda_i = \beta' x_i$$

$$\mathbb{R}^+$$

$$X = \begin{cases} 0 & M \\ 1 & H \end{cases} \qquad \log \hat{\lambda}_i = 0.5 + 0.2\, X_i + 0.1\, V_i$$

si i es M $\quad \log \hat{\lambda}_i = 0.5$

$$\log \hat{\lambda}_i = 0.5 + 0.2 = 0.7$$

$$0.2 = \log \frac{\hat{\lambda}_H}{\hat{\lambda}_M} \qquad e^{0.2} = 1.22$$

$$e^{0.1} = 1.10$$

$$\lambda \in (0, \infty)$$

$$\log \lambda \in (-\infty, \infty) \qquad \beta' x \in (-\infty, \infty)$$

$$f_{Y_i}(y_i; \beta) = \frac{\exp\{-e^{\beta' x_i}\} \left(e^{\beta' x_i}\right)^{y_i}}{y_i!} \qquad \left(e^{y_i \beta' x_i}\right)$$

$$f_{Y_1 \dots Y_n}(y_1 \dots y_n; \beta) = \frac{1}{\prod y_i!} \cdot \prod_{i=1}^{n} \exp\{-e^{\beta' x_i}\} \prod_{i=1}^{n} e^{y_i \beta' x_i}$$

$$\prod(y_i!) \neq \left(\prod y_i\right)! \qquad = \frac{1}{\prod y_i!} \exp\left\{-\sum_{i=1}^{n} e^{\beta' x_i}\right\} \exp\left\{\sum_{i=1}^{n} y_i \beta' x_i\right\}$$

$$l_n(\beta) = -\sum_{i=1}^{n} e^{\beta' x_i} + \sum_{i=1}^{n} y_i \beta' x_i - \log \prod_{i=1}^{n} y_i!$$

$$\frac{\partial}{\partial \beta} l_n(\beta) = -\sum_{i=1}^{n} e^{\beta' x_i} x_i + \sum_{i=1}^{n} y_i x_i = 0 \qquad (*)$$

$\hat{\beta}^{MV}$ es la solución numérica de $(*)$

$$\lambda_i \quad \overset{\log}{\underset{1-1}{\longleftrightarrow}} \quad \beta' x_i$$

$$(0, \infty) \longleftrightarrow (-\infty, \infty)$$

Invarianza

Sea θ un parametro con EMV $\hat{\theta}$

$g(\theta)$ una fn del parametro.

El EMV de $g(\theta)$ es $g(\hat{\theta})$

Ej $X_1 \dots X_n \overset{iid}{\sim} B(\theta)$ $\quad \hat{\theta} = \frac{\sum x_i}{n} = \hat{p}_n$

Supongamos $a = P(X_{n+1} = 1 ; X_{n+2} = 1) = \theta^2$

$a = g(\theta) = \theta^2$

$\hat{a}^{MV} = \hat{\theta}^2 = \hat{p}^2$ por invarianza

Método de Mínimos Cuadrados

Supongamos V. A. de la forma

$$Y_i = \underbrace{\underbrace{S_i}_{\text{función conocida}} \underbrace{(\theta_1, \dots, \theta_p)}_{\text{parametros}}}_{\text{parte 'deterministica'}} + \underbrace{\varepsilon_i}_{\text{pertubación aleatoria}}$$

i) $E\varepsilon_i = 0$ $\quad$ ii) $\operatorname{Var} \varepsilon_i = \sigma^2$ $\quad$ iii) $\varepsilon_1 \dots \varepsilon_n$ indep.

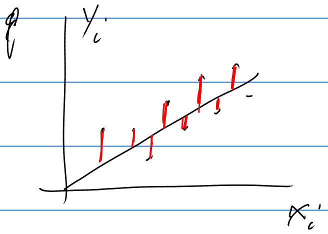
Ej 1 $\quad Y_i = \mu + \varepsilon_i \qquad S_i(\mu) = \mu$

Ej 2 $\quad Y_i = \alpha + \beta \underbrace{x_i}_{\text{ctes conocidos}} + \varepsilon_i \qquad (S_i(\alpha,\beta) = \alpha + \beta x_i$

Ej 3
$$Y_i = \underbrace{\alpha e^{\beta x_i}} + \varepsilon_i \qquad \text{alt} \quad Y_i = \alpha e^{\beta x_i} e^{\varepsilon_i}$$
$$S_i(\alpha,\beta) = \alpha e^{\beta x_i}$$

Def Los EMC de θ son
$$\underset{\theta \in \Theta}{\operatorname{argmin}} \sum_{i=1}^{n} \left[Y_i - S_i(\theta) \right]^2$$

Ej $\quad Y_i = \beta x_i + \varepsilon_i \qquad$ Regresión por el origen



$$E Y_i = \beta x_i$$
$$|Y_i - \beta x_i|$$
$$h(\beta) = \sum_{i=1}^{n} (Y_i - \beta x_i)^2$$

$$\frac{\partial h(\beta)}{\partial \beta} = (-2) \sum_{i=1}^{n} (Y_i - \beta x_i) x_i = 0$$
$$\sum_{i=1}^{n} x_i Y_i - \beta \sum_{i=1}^{n} x_i^2 = 0$$

$$\boxed{\hat{\beta}^{MC} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}}$$

$\varepsilon_i \sim N \qquad \hat{\beta}^{MC} = \hat{\beta}^{MV}$

Criterios para evaluar estimadores

$L(\theta, \delta)$ pérdida de estimar θ mediante δ

$L(\theta,\theta) = 0$

$L(\ ,\ ) \geq 0$

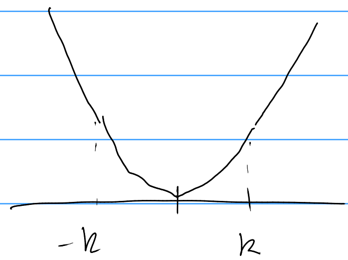
$L(a,b) = L(b,a)$

Def $R_\theta(\delta) = E_\theta L(\theta, \delta)$

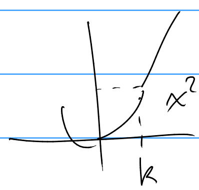
Pérdidas comunes $L(\theta,\delta) = (\theta - \delta)^2$

$L(\theta,\delta) = |\theta - \delta|$

etc



$$L(\theta,\delta) = \begin{cases} (\theta-\delta)^2 & |\theta-\delta| < k \\ |\theta-\delta| k & \text{c.c.} \end{cases}$$



Def ECM $E_\theta (\theta - \delta)^2$

Def Un estimador $\delta = \delta(X)$ se dice insesgado para un parametro θ si

$$\forall \theta \in \Theta \qquad E_\theta\, \delta(X) = \theta$$

Def $\text{Sesgo}_\theta(\delta) = E_\theta\, \delta(X) - \theta$

Prop $EMC_\theta(\delta) = Var_\theta(\delta) + \text{sesgo}_\theta(\delta)^2$

$$EMC(\delta) = E[\delta - \theta]^2 = E\{[\delta - E\delta] + [E\delta - \theta]\}^2$$

$$= \underbrace{E\{(\delta - E\delta)^2\}}_{Var\ \delta} + E\{\underbrace{E\delta - \theta}_{\text{Sesgo}(\delta)}\}^2 + 2\underbrace{E\{[\delta - E\delta][E\delta - \theta]\}}_{=0}$$

$$E\{[\delta - E\delta][E\delta-\theta]\} = [E\delta-\theta]\,\underbrace{E[\delta-E\delta]}_{=0}$$

$$EMC_\theta(\delta) = Var_\theta(\delta) + [Sesgo_\theta(\delta)]^2 \quad \diamond$$

Ej $X_1 \ldots X_n$ iid $U(0,\theta)$

a) MM θ

$$EX_i = \frac{1}{n}\sum X_i$$

$$\frac{\theta}{2} = \frac{1}{n}\sum X_i \Rightarrow \hat{\theta}^{MM} = 2\bar{X}$$

$$E\hat{\theta}^{MM} = E\,2\bar{X} = 2\,E\bar{X} = 2\,E\left[\frac{1}{n}\sum X_i\right]$$

$$= 2\,\frac{1}{n}\sum EX_i = 2\,\frac{1}{n}\,n\,\frac{\theta}{2} = \theta$$

$$Var\,\hat{\theta}^{MM} = Var(2\bar{X}) = 4\,Var\,\bar{X} = 4\,\frac{Var\,X}{n}$$

$$= \frac{4}{n}\,\frac{\theta^2}{12} = \frac{\theta^2}{3n}$$

$U \sim U(a,b)$

$$Var\,U = \frac{(b-a)^2}{12}$$

$$EMC(\hat{\theta}^{MM}) = \frac{\theta^2}{3n}$$

b) Maximaverosimilitud

$$\hat{\theta}^{MV} = X_{(n)}$$

$$F_{\hat{\theta}}(a) = P(\hat{\theta} \le a) = P(X_{(n)} \le a) = P(X_1 \le a \ldots X_n \le a) = P(X_1 \le a)\ldots P(X_n \le a)$$

$$= \left(\frac{a}{\theta}\right)^n$$

$$f_{\hat{\theta}}(a) = \frac{n}{\theta}\left(\frac{a}{\theta}\right)^{n-1}$$

$a/\theta = u$

$$E\hat{\theta} = \theta\,\frac{n}{\theta}\int_0^\theta \frac{a}{\theta}\left(\frac{a}{\theta}\right)^{n-1} da = \frac{\theta\, n}{n+1}\int_0^\theta (n+1)\left(\frac{a}{\theta}\right)\left(\frac{a}{\theta}\right)^{n-1}\frac{da}{\theta}$$

$$= \theta\,\frac{n}{n+1}\int_0^1 (n+1)\,u^n\,du = \theta\,\frac{n}{n+1}\left(u^{n+1}\right)\Big|_0^1$$

$$E\hat{\theta} = \theta\,\frac{n}{n+1} \quad \text{sesgada}$$

$$\frac{n+1}{n} E\hat{\theta} = \frac{\cancel{n}}{n\cancel{+1}}\theta \quad \frac{n\cancel{+1}}{\cancel{n}}$$

Mobar $\tilde{\theta} = \frac{n+1}{n} X_{(n)}$ $\hat{\theta} = 2\bar{X}$

$$E\tilde{\theta} = \frac{n+1}{n} E X_{(n)} = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta$$