Estimadores basados en Estadísticos Suficientes

Teorema de Rao-Blackwell

Sea $X \sim F(x;\theta)$, $\theta \sim \Theta$. Y sea $T(X)$ un estadístico suficiente para θ y sea $\delta(X)$ un estimador del $q(\theta)$.

Construyamos $\delta^*(T) = E_{\cancel{\theta}}[\delta(X) \mid T]$

Luego i) $EMC_\theta(\delta^*) \leq EMC_\theta(\delta) \quad \forall \theta \in \Theta$

ii) hay $=$ en (i) ssi

$$P_\theta(\delta^*(T) = \delta(X)) = 1 \quad \forall \theta \in \Theta$$

iii) Si $\delta(X)$ es insesgado $\delta^*(T)$ también lo es.

Dem $EMC_\theta(\delta) = E_\theta\{[\delta(X) - q(\theta)]^2\} \qquad q: \Theta \to \mathbb{R}$

$$= E_\theta\{[\delta^*(T) - q(\theta)] + [\delta(X) - \delta^*(T)]\}^2$$

$$= \underbrace{E_\theta\{[\delta^*(T) - q(\theta)]^2\}} + E\{[\delta(X) - \delta^*(T)]^2\}$$

$$+ 2 \cdot E\{[\delta^*(T) - q(\theta)][\delta(X) - \delta^*(T)]\}$$

$$= EMC_\theta(\delta^*(T)) + \underbrace{E_\theta[\delta(X) - \delta^*(T)]^2}_{\geq 0}$$

$$+ 2E_\theta\{[\delta^*(T) - q(\theta)][\delta(X) - \delta^*(T)]\}$$

calculamos

$$E_\theta\{[\delta^*(T) - q(\theta)][\delta(X) - \delta^*(T)]\} =$$

$$= E_\theta\left\langle E_\theta\{\underbrace{[\delta^*(T) - q(\theta)]}[\delta(X) - \delta^*(T)] \mid T\}\right\rangle$$

$$= E_\theta\left\langle [\delta^*(T) - q(\theta)]\ \underbrace{E_\theta\{[\delta(X) - \delta^*(T)] \mid T\}}_{0}\right\rangle.$$

$$E\{[\delta(X)-\delta^*(T)]\,|\,T\} = \underbrace{E[\delta(X)\,|\,T]} - \underbrace{E[\delta^*(T)\,|\,T]}$$
$$= \delta^*(T) - \delta^*(T) = 0$$

$$EMC_\theta(\delta^*(T)) \leq EMC_\theta(\delta(X))$$

Con igualdad ssi $E_\theta[\delta(X)-\delta^*(T)]^2 = 0$

ssi $P_\theta[\delta(X) = \delta^*(T)] = 1.$

Si $E_\theta(\delta(X)) = q(\theta)$

$$\delta^*(T) = E_\theta[\delta(X)\,|\,T]$$

$$E_\theta\,\delta^*(T) = E_\theta\{E_\theta[\delta(X)\,|\,T]\} = E_\theta\,\delta(X) = q(\theta) \quad \Diamond$$

<u>Ej</u> $X_1 \ldots X_n \overset{iid}{\sim} B(\theta)$ $\qquad \delta(X_1 \ldots X_n) = X_1$

$E\,\delta(X_1 \ldots X_n) = EX_1 = \theta \qquad EMC(\delta) = Var(\delta)$

$= Var\ X_1 = \theta(1-\theta)$

$T = \Sigma X_i$ suficiente

$$\delta^*(X_1 \ldots X_n) = E[\delta(X_1 \ldots X_n)\,|\,T]$$
$$= E[X_1\,|\,\Sigma X_i]$$

$$X_1 + X_2 + \ldots + X_n = \Sigma X_i$$

$$E\{X_1 + \ldots + X_n\,|\,\Sigma X_i\} = E[\Sigma X_i\,|\,\Sigma X_i]$$

$$\underbrace{E\{X_1\,|\,\Sigma X_i\} + \ldots + E\{X_n\,|\,\Sigma X_i\}} = \Sigma X_i$$

$$n\,E[X_1\,|\,\Sigma X_i] = \Sigma X_i$$

$$\delta^*(T) = E[X_1 \mid \sum X_i] = \frac{1}{n} \sum X_i = \hat{p}_n$$

$$E\,\delta^*(T) = \frac{1}{n} n\theta = \theta$$

$$Var\,\delta^*(T) = \frac{1}{n^2} \sum \theta(1-\theta) = \frac{\theta(1-\theta)}{n} < \theta(1-\theta)$$

Familias Exponenciales

Def. Una familia de distribuciones $F(x,\theta)$ con $x = (x_1, \dots, x_q)$ con $\theta \in \Theta \subset \mathbb{R}^p$ es una familia exponencial con k parametros (canónicos) si su densidad tiene la forma

$$f(x;\theta) = A(\theta) \exp\left\{ \sum_{i=1}^{k} c_i(\theta) r_i(x) \right\} h(x)$$

con $c_1(\theta), \dots, c_k(\theta)$ funciones $\Theta \to \mathbb{R}$

$$A(\theta) : \Theta \to \mathbb{R}^+$$

$$r_1(x) \dots r_k(x) : \mathbb{R}^q \to \mathbb{R}^k$$

$$h(x) : \mathbb{R}^q \to \mathbb{R}^+$$

$\exp\{\log \quad \}$

Ej Sea $X \sim B(n,\theta)$ $\quad \theta \in (0,1) = \Theta \subset \mathbb{R}^1$

$x = 0, \dots, n$

$$f(x,\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} = \binom{n}{x} \left[\frac{\theta}{1-\theta}\right]^x (1-\theta)^n$$

$q=1$ $\quad p=1$
$k=1$

$$= \underbrace{(1-\theta)^n}_{A(\theta)} \exp\{ \underbrace{(\log \frac{\theta}{1-\theta})}_{c(\theta)} \underbrace{x}_{r(x)} \} \underbrace{\binom{n}{x}}_{h(x)}$$

$A(\theta) = (1-\theta)^n \quad c(\theta) = \log\left(\frac{\theta}{1-\theta}\right) \quad r(x) = x \quad h(x) = \binom{n}{x}$

$$(X_1 \cdots X_q) \sim f(x_1 \cdots x_q ; \theta)$$

Ep 2 $\quad X \sim N(\mu, \sigma^2)$

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}}(\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}}(\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2}\right\}$$

$q=1$

$k=2$

$p=2$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}(\sigma^2)^{-1/2}\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}}_{A(\mu,\sigma^2)} \exp\left\{\left(-\frac{1}{2\sigma^2}\right)x^2 + \left(\frac{\mu}{\sigma^2}\right)x\right\} \underbrace{1}_{h(x)}$$

$c_1(\mu,\sigma^2) = \frac{-1}{2\sigma^2}$ $\quad T_1(x) = x^2$ $\quad c_2(\mu,\sigma^2) = \frac{\mu}{\sigma^2}$ $\quad T_2(x) = x$