Teorema En una familia exponencial el estadístico
$$T(X) = (T_1(X); \dots ; T_k(X))$$ es suficiente.

Dem Teorema de factorización con

$$g(T(x);\theta) = \exp\left\{\sum_{i=1}^{k} c_i(\theta)\, T_i(x)\right\} A(\theta) \qquad \diamond$$

Teorema Sean $X_1, X_2, \dots, X_n$ iid con $f(x_i;\theta)$ perteneciente a una familia exponencial con k parámetros (canónicos)

Entonces la distribución conjunta de $(X_1, \dots, X_n)$ también pertenece a una familia exponencial con k parámetros (canónicos) y estadístico suficiente

$$T^* = (T_1^*, \dots, T_k^*) \quad \text{con} \quad T_i(X_1 \dots X_n) = \sum_{j=1}^{n} T_i(X_j)$$

Dem $f(x_j;\theta) = A(\theta)\exp\left\{\sum_{i=1}^{k} c_i(\theta) T_i(x_j)\right\} h(x_j)$

$$f(x_1, \dots, x_n;\theta) = \prod_{j=1}^{n} A(\theta)\exp\left\{\sum_{i=1}^{k} c_i(\theta) T_i(x_j)\right\} h(x_j)$$

$$= [A(\theta)]^n \exp\left\{\sum_{j=1}^{n}\left[\sum_{i=1}^{k} c_i(\theta) T_i(x_j)\right]\right\} \cdot \prod_{j=1}^{n} h(x_j)$$

$$= \underbrace{[A(\theta)]^n}_{A^*(\theta)} \exp\left\{\sum_{i=1}^{k} \underbrace{c_i(\theta)}_{c_i(\theta)} \underbrace{\sum_{j=1}^{n} T_i(x_j)}_{T_i^*(x_1 \dots x_n)}\right\} \underbrace{\prod_{j=1}^{n} h(x_j)}_{h^{**}(x_1, \dots, x_n)}$$

$\triangle$

$$\exp\left\{ \cancel{f}\left(c\cdot(\theta)\, r\cdot(x) \right) \right\} = \log(1+\theta x)$$

Recordar $X \sim f(x;\theta) = \begin{cases} \frac{1}{2}(1+\theta x) & -1<x<1 \\ 0 & \text{c.c.} \end{cases} \qquad \theta \in [-1,1]$

· Muestra aleatoria simple

$X_1 \dots X_n \qquad f(x_1 \dots x_n;\theta) = \frac{1}{2^n}\prod(1+\theta x_{(i)})$

$$T(X_1 \dots X_n) = (X_{(1)}; \dots ; X_{(n)})$$

Ej $X_1 \dots X_n \overset{iid}{\sim} \mathcal{P}(\lambda)$

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = \underbrace{e^{-\lambda}}_{A(\lambda)} \exp\{\underbrace{(\log\lambda)}_{c(\lambda)}\underbrace{x}_{r(x)}\} \underbrace{\frac{1}{x!}}_{h(x)}$$

$$T(X) = X$$

$$f(x_1 \dots x_n;\lambda) = \prod_{j=1}^{n} \frac{e^{-\lambda}\lambda^{x_j}}{x_j!} = (e^{-\lambda})^n \lambda^{\sum x_j} \prod_{j=1}^{n}\frac{1}{x_j!}$$

$$= \underbrace{(e^{-\lambda})^n}_{[A(\lambda)]^n} \exp\left\{ \underbrace{(\log\lambda)}_{c(\lambda)} \underbrace{\sum_{j=1}^{n} x_j}_{T(X_1 \dots X_n) = \sum X_i} \right\} \underbrace{\prod_{j=1}^{n}\frac{1}{x_j!}}_{h^*(x_1 \dots x_n)}$$

$X_1 \dots X_n \overset{iid}{\sim} U(0,\theta) \quad T = X_{(n)}$

$X_1 \dots X_n \overset{iid}{\sim} \mathcal{P}(\lambda) \quad T = \sum X_i \sim \mathcal{P}(n\lambda)$

Teorema Sea X un vector aleatorio perteneciente a una familia exponencial con k parametros (canónicos) Luego la densidad (o prob. puntual) de los estadísticos suficientes

$$T(X) = (T_1(X), \ldots, T_k(X)) \text{ es de la forma}$$

$$f_{T_1 \ldots T_k}(t_1 \ldots t_k; \theta) = A(\theta) \exp\left\{\sum_{i=1}^k c_i(\theta) t_i\right\} h^0(t_1, \ldots, t_k)$$

$$X_1 \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2) \qquad T = (\Sigma X_i, \Sigma X_i^2)$$

$$\bar{X} \sim N(\mu, \tfrac{\sigma^2}{n}) \qquad \text{1-1} \updownarrow \quad \tilde{T} = (\bar{X}, S^2)$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad (\bar{X}, S^2) \sim F_{\bar{X}, S^2} \in \text{familia exponencial}$$

Dem para el caso discreto

$$f(x, \theta) = A(\theta) \exp\left\{\sum_{i=1}^k c_i(\theta) T_i(x)\right\} h(x)$$

$$T(x) = (T_1(x), \ldots, T_k(x)) \quad \text{si} \quad t = (t_1, \ldots, t_k)$$

$$f_{T_1 \ldots T_k}(t_1, \ldots, t_k; \theta) = \sum_{\{x : T(x) = t\}} f_X(x; \theta)$$

$$= \sum_{\underbrace{\{x : T(x) = t\}}} \left[A(\theta) \exp\left\{\sum_{i=1}^k c_i(\theta) \underbrace{T_i(x)}_{t_i}\right\} h(x) \right]$$

$$\underset{\sim}{\text{FE}} \quad \underset{\sim}{\text{FE}}$$

$$X = (X_1, X_2) \qquad X_1 \sim N(\mu, 1) \qquad X_2 = 2X_1 \sim N(2\mu, 4)$$

$$(X_1, X_2)$$

Dem cont

$$f_{T_1 \ldots T_k}(t_1, \ldots, t_k, \theta) = A(\theta) \exp\left\{ \sum_{i=1}^{k} c_i(\theta) t_i \right\} \underbrace{\sum_{\{x : t(x) = t\}} h(x)}_{h^{o}(x)}$$

$\square$

$$F_{X_1 X_2}(x_1, x_2) = P(X_1 \le x_1 ; X_2 \le x_2) \qquad X_2 = 2X_1$$

$$= \begin{cases} P(X_1 \le x_1) & x_2 \le 2x_1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(u, v) \quad dx_1 \, dx_2$$