Familias Exponenciales

Recordar $X \sim f$ con flia exponencial ssi

$$f(x;\theta) = A(\theta)\exp\left\{\sum_{i=1}^{k} c_i(\theta)\, r_i(x)\right\} h(x)$$

Lema 1

Si $m(X)$ tal que $\overbrace{E_\theta |m(X)|}^{\int\cdots\int \overset{1}{|m(x)|}\, A(\theta)\, e^{c(\theta) r(x)} h(x)\,dx} < \infty$ entonces

$$\frac{\partial}{\partial\theta} E_\theta\, m(X) = E_\theta \frac{\partial}{\partial\theta} m(X)$$

Teorema En una familia exponencial

i) $A(\theta)$ es infinitamente derivable

ii) $$E_\theta(r(X)) = -\frac{A'(\theta)}{A(\theta)\, c'(\theta)}$$

iii) $$Var_\theta(r(X)) = \frac{\frac{\partial}{\partial\theta} E_\theta(r(X))}{c'(\theta)}$$

Dem $$\int\cdots\int A(\theta)\, e^{c(\theta) r(x)} h(x)\, dx_1 \cdots dx_q = 1 \qquad k=1$$

$$\frac{1}{A(\theta)} = \underbrace{\int\cdots\int e^{c(\theta) r(x)} h(x)\, dx_1 \cdots dx_q}_{\text{satisface lema 1 con } m(X)=1} \qquad (*)$$

derivando

$$\frac{\partial}{\partial\theta}\, A(\theta) \int\cdots\int e^{c(\theta) r(x)} h(x)\, dx_1 \cdots dx_q = \frac{\partial}{\partial\theta} 1$$

$$\underbrace{A'(\theta) \int\cdots\int e^{c(\theta) r(x)} h(x)\, dx_1 \cdots dx_q} + A(\theta) \int\cdots\int e^{c(\theta) r(x)}\, r(x)\, c'(\theta)\, h(x)\, dx_1 \cdots dx_q = 0$$

$$\frac{A'(\theta)}{A(\theta)} + c'(\theta)\, E_\theta(r(X)) = 0$$

$$E_\theta(r(X)) = -\frac{A'(\theta)}{c'(\theta)\,A(\theta)}$$

iii) Ej.

Estadísticos Completos

Def Un estadístico $T = r(X)$ se dice completo si la ecuación

$$E_\theta\, g(T) = 0 \Rightarrow P_\theta\left[g(T) = 0\right] = 1 \quad \forall \theta \in \Theta$$

Ej $X \sim B(n,\theta)$. ¿Es $T = X$ completo?

Supongamos $\underbrace{E_\theta\, g(X)}$ $= 0$

$$\sum_{x=0}^{n} g(x)\binom{n}{x}\theta^x(1-\theta)^{n-x} = 0 \quad \forall \theta \in [0,1]$$

$$(1-\theta)^n \sum_{x=0}^{n} g(x)\left(\frac{\theta}{1-\theta}\right)^x \binom{n}{x} = 0 \quad \forall \theta \in (0,1)$$

$$\sum_{x=0}^{n} \binom{n}{x} g(x)\left(\frac{\theta}{1-\theta}\right)^x = 0 \quad \forall \theta \in (0,1)$$

$$\boxed{\forall \lambda \in (0,\infty) \quad \sum_{x=0}^{n} \binom{n}{x} g(x)\,\lambda^x = 0} \qquad \frac{\theta}{1-\theta} = \lambda \in (0,\infty)$$

$$E_\theta(g(X)) = 0 \quad \forall \theta \in [0,1]$$

$$P_n(\lambda) \equiv 0 \quad \forall \lambda \in (0,\infty)$$

$$\underbrace{\binom{n}{x} g(x)}= a_x \qquad a_0 + a_1\lambda + \dots + a_n\lambda^n = 0$$

$$g(x) = 0 \qquad x = 0, 1, \dots, n \qquad \forall x \; \cancel{\binom{n}{x}} g(x) = 0$$

$$P_\theta\left[g(X) = 0\right] = 1 \quad \therefore \quad T = X \text{ es completo.}$$

Ej 3 $X \sim U(0,\theta)$ $\quad T = X \quad$ ¿es completo?

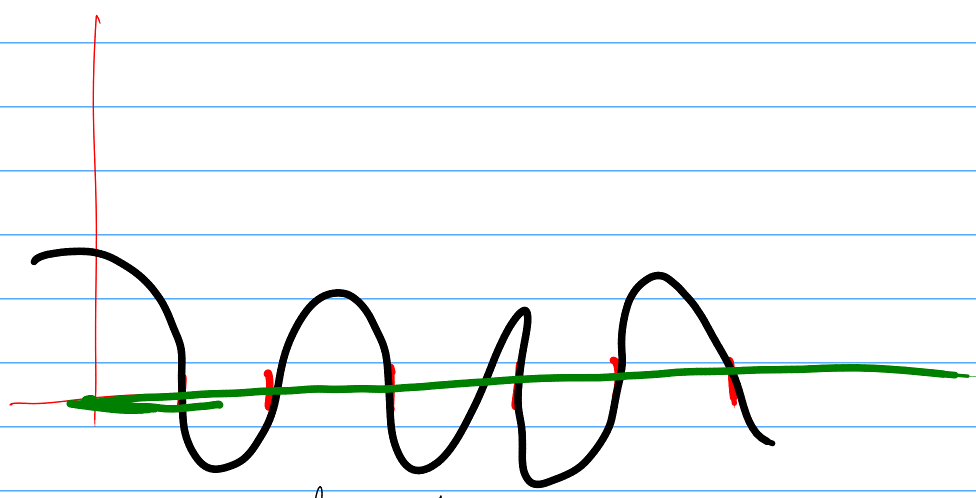
Suponga $E_\theta\, g(X) = 0 \quad \forall \theta \in (0,\infty)$

$$\int_0^\theta g(x)\,dx = 0 \qquad \forall \theta \in (0,\infty)$$

$$\underbrace{\frac{\partial}{\partial\theta}\int_0^\theta g(x)\,dx}_{} = 0 \qquad \forall \theta \in (0,\infty)$$

$$g(\theta) = 0 \qquad \forall \theta \in (0,\infty)$$

$$\frac{\partial}{\partial x}\int_0^x f(u)\,du = f(x) \qquad \text{TFC}$$



Teorema En una familia exponencial de rango completo $T(X) = (T_1(X), \dots, T_k(X))$ es suficiente minimal y completo

Que la familia exponencial sea de rango completo significa

1) que los estadísticos no satisfacen ninguna restricción lineal

2) que

$$C(\theta) = (C_1(\theta), \dots, C_k(\theta)) \in \Xi \subset \mathbb{R}^k$$

Θ y Ξ contiene algún rectángulo abierto k dimensional

ξ Ξ

z_i

Ej $X \sim N(\theta, \theta)$ $f(x,\theta) = \left(\frac{1}{2\pi\theta}\right)^{1/2} \exp\left\{-\frac{1}{2}\frac{(x-\theta)^2}{\theta}\right\}$ $\theta > 0$

$$f(x;\theta) = \left(\frac{1}{2\pi}\right)^{1/2} \theta^{-1/2} \exp\left\{-\frac{1}{2\theta}(x^2 - 2\theta x + \theta^2)\right\}$$

$$= (2\pi)^{-1/2} \theta^{-1/2} \exp\left\{-\frac{1}{2\theta}x^2 + x - \frac{\theta}{2}\right\}$$

$$= (2\pi)^{-1/2} \theta^{-1/2} e^{-\theta/2} \exp\left\{\underbrace{-\frac{1}{2\theta}}\underbrace{x^2}\right\} e^{x}$$

$C(\theta) = -\frac{1}{2\theta}$ $T(x) = x^2$

$C(\theta) \in (-\infty, 0)$ no satisface restricción lineal

La familia es de rango completo

Ej 2 $X \sim N(\sqrt{\theta}, \theta)$ $\theta > 0$

$$f(x;\theta) = (2\pi\theta)^{-1/2} \exp\left\{-\frac{1}{2\theta}(x - \sqrt{\theta})^2\right\}$$

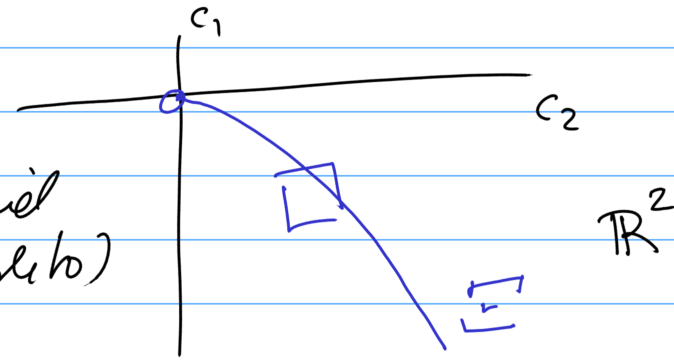
$$f(x;\theta) = (2\pi\theta)^{-1/2} \exp\left\{-\frac{1}{2\theta}\left(x^2 - 2x\sqrt{\theta} + \theta\right)\right\}$$

$$= (2\pi\theta)^{-1/2} \exp\left\{-\frac{1}{2\theta}x^2 + \frac{x}{\sqrt{\theta}} - \frac{1}{2}\right\}$$

$c_1(\theta) = -\frac{1}{2\theta}$ $\quad T_1(x) = x^2$

$c_2(\theta) = \frac{1}{\sqrt{\theta}}$ $\quad T_2(x) = x$

$$c_1 = -\frac{1}{2}c_2^2$$



Llamada familia exponencial curva (no de rango completo)

Teorema Si para determinada familia un estadístico $T = T(X)$ es i) suficiente
ii) completo
entonces es minimal suficiente.

Dem dimensión 1

P1 Sea $U = U(X)$ minimal suficiente para θ.

$= m(T)$ para alguna m (def minimal)

P2 Sea $\psi(t)$ la función $\operatorname{arctg}(t)$

$$\psi : (-\infty, \infty) \overset{1-1}{\longleftrightarrow} (0, 2\pi)$$

ψ: continua y acotada, estrictamente creciente.

P3 $E_\theta(\psi(T)) < \infty$ (queremos mostrar que $T = \psi^{-1}(U)$)

P4) Def $\quad \eta(U) = \cancel{E_\theta}\left[\psi(T) \mid U\right]$

$= E\left[\psi(T) \mid U\right]$ (suficiencia)

P5) Tómese $g(T) = \psi(T) - \eta[m(T)] = \psi(T) - \eta(U)$

$$E_\theta\left[g(T)\right] = E_\theta\left[\psi(T) - \eta(U)\right]$$

$$= E_\theta\left\{ E_\theta\left[\psi(T) - \eta(U) \mid U\right]\right\}$$

$$= E_\theta\left\{ \underbrace{E_\theta\left[\psi(T) \mid U\right]}_{(P4)\ \eta(U)} - \underbrace{E_\theta\left[\eta(U) \mid U\right]}_{\eta(U)} \right\} = 0$$

P6) Por completitud $\quad g(T) = 0 \quad \forall \theta$

$$\psi(T) = \eta(U)$$

$$T = \psi^{-1}(\eta(U)) \qquad \diamond$$

Teorema de Basu

Sea T un estadístico i) suficiente } para una familia
ii) Completo } determinada marcada por θ

U un estadístico cuya distribución no depende de θ

Entonces U es independiente de T

Dem P1 Sea A un evento

$$P_\theta(U \in A) = P(U \in A) = p_A$$

P2 Sea $\eta_A(t) = P_{\cancel{\theta}}(U \in A \mid T=t) = P(U \in A \mid T=t)$

P3 Considere

$$E_\theta[\eta_A(T) - p_A] = E_\theta\{E_\theta[\eta_A(T) - p_A] \mid T\}$$
$$= 0$$

P4 Por completitud

$$\eta_A(T) = p_A$$
$$P(U \in A \mid T=t) = P(U \in A)$$
$$U \perp T. \qquad \diamond$$

Ej $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\mathcal{F} = \{f(x_1, \dots x_n; \mu, \sigma^2) : \mu \in \mathbb{R}, \ \sigma^2 \text{ fijo}\}$$

σ^2 fijo conocido

$$f_{X_1 \dots X_n}(x_1, \dots x_n; \mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$$
$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\mu \sum x_i + n\mu^2\right)\right\}$$
$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{\frac{\mu}{\sigma^2} \sum x_i\right\} \exp\left\{-\frac{n\mu^2}{2\sigma^2}\right\} \exp\left\{-\frac{1}{2\sigma^2} \sum x_i^2\right\}$$
$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{n\mu^2}{2\sigma^2}\right\}}_{A(\mu)} \exp\left\{\frac{\mu}{\sigma^2} \sum x_i\right\} \underbrace{\exp\left\{-\frac{1}{2\sigma^2} \sum x_i^2\right\}}_{h(x_1, \dots x_n)}$$

$c = \frac{\mu}{\sigma^2}$ $\quad T = \sum x_i$

$T = \sum x_i$ es suficiente minimal y completo

$T = \bar{X}$ es suficiente minimal y completo

elijo
$$U = \underbrace{\frac{(n-1)S^2}{\sigma^2}} = \frac{\sum (X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

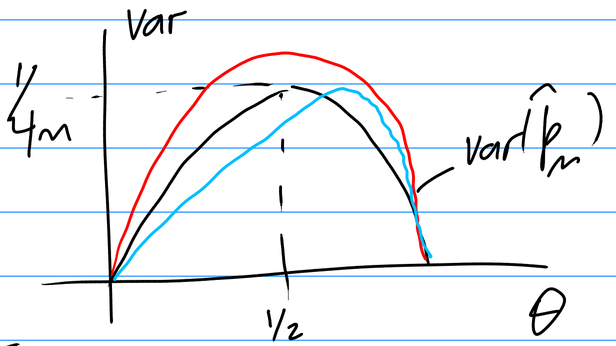
tiene una distribución que no depende de μ

y S^2 también tiene una distribución que no depende de μ

Por Basu

$$\bar{X} \perp S^2 \quad \forall \sigma^2$$

Estimadores Insesgados de Mínima Varianza Uniformemente (IMVU)



$\hat{p}_n$ $\quad E\,\hat{p}_n = \theta$

$$\text{Var}\,\hat{p}_n = \frac{\theta(1-\theta)}{n}$$

Teorema de Lehmann - Scheffé

sea X un vector aleatorio con distribución $F(x;\theta)$ $\theta \in \Theta$
T un estadístico suficiente y completo
$q(\theta) : \Theta \to \mathbb{R}$

i) $\exists$ a lo sumo un estimador insesgado de $q(\theta)$ basado en T

ii) Si $\delta(T)$ es insesgado para $g(\theta)$, entonces $\delta(T)$ es IMVU para $g(\theta)$

iii) Si $\delta(X)$ es insesgado para $g(\theta)$ entonces

$$\delta^*(T) = E\left[\delta(X) \mid T\right] \text{ es IMVU para } g(\theta)$$

Ej $X_1 \dots X_n \overset{iid}{\sim} \mathcal{P}(\lambda)$

$$f_{X_1 \dots X_n}(x_1 \dots x_n; \lambda) = \frac{e^{-n\lambda}\, \lambda^{\sum x_i}}{\prod x_i!}$$

$T = \sum X_i$ es suficiente minimal completo

$$\underset{\lambda}{E}\, T = \underset{\lambda}{E} \sum X_i = \sum \underset{\lambda}{E} X_i = n\lambda$$

$$\delta(T) = \frac{1}{n} \sum X_i \qquad E\,\delta(T) = \lambda \quad \text{es IMVU}$$

Supongamos que quiero estimar

$$g(\lambda) = P\left[X = 0\right]$$

$$\tilde{\delta}(X_1 \dots X_n) = \frac{1}{n} \sum_{i=1}^{n} 1(x_i = 0)$$

$$E\,\tilde{\delta}(X_1 \dots X_n) = \frac{1}{n} \sum_{i=1}^{n} E\left\{1(X_i = 0)\right\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} P(X_i = 0) = \frac{1}{n}\, n\, P(X_i = 0)$$

$$= P(X_i = 0) \quad \text{insesgado}$$

$$\delta^{\circ}(x_1 \dots x_n) = e^{-\bar{x}} \qquad q(\lambda) = P(X=0) = e^{-\lambda}$$

1) $$\underbrace{\delta^{*}(\bar{x}) = E\left[\tilde{\delta}(x_1 \dots x_n) \mid \bar{x}\right]}_{\text{buena suerte!}}$$

$$\begin{array}{cc} \lambda & \bar{x} \\ e^{-\lambda} & e^{-\bar{x}} \end{array}$$

Idea 2)

$$\delta^{\circ}(x_1 \dots x_n) = e^{-\bar{x}} = \underbrace{e^{-\frac{1}{n}\sum x_i}}$$

$$Y = \sum X_i \sim P(n\lambda)$$

$$E\,\delta^{\circ}(x_1 \dots x_n) = \sum_{y=0}^{\infty} \frac{e^{-\frac{y}{n}} e^{-n\lambda} \cdot (n\lambda)^y}{y!}$$

$$= e^{-n\lambda} \sum_{y=0}^{\infty} \frac{\left(e^{-\frac{1}{n}} n\lambda\right)^y}{y!} \qquad a = e^{-\frac{1}{n}} n\lambda$$

$$= e^{-n\lambda} \exp\left\{ e^{-\frac{1}{n}} n\lambda \right\} \qquad ?$$

PENSAR

Ej $X_1 \dots X_n \overset{iid}{\sim} B(\theta)$ $\quad T = \sum X_i$ suficiente minimal completo

quiero estimar

$$g(\theta) = P(X_{n+1} = 1 ; X_{n+2} = 1)$$
$$= \theta^2$$

$\hat{p}_n$ es estimador insesgado de θ

$\delta(x_1 \dots x_n) = (\hat{p}_n)^2$ no es insesgado

$\delta^0(x_1 \dots x_n) = \mathbb{1}(X_1 = 1)\, \mathbb{1}(X_2 = 1)$

$E\, \delta^0(x_1 \dots x_n) = P(X_1 = 1)\, P(X_2 = 1) = \theta^2$

$\delta^*(x_1 \dots x_n) = E\left[\delta^0(x_1 \dots x_n) \mid \delta\right]$

o $E\left[\delta^0(X_1 \dots X_n) \mid T\right]$ $\quad T = \sum X_i$

$$p_{X_1, X_2 \mid T}(1, 1, t) = \frac{P(X_1 = 1;\ X_2 = 1;\ \sum X_i = t)}{P(\sum X_i = t)}$$

$$= \frac{\cancel{\theta} \cdot \cancel{\theta} \binom{n-2}{t-2} \cancel{\theta^{t-2}} \cancel{(1-\theta)^{n-2-(t-2)}}}{\binom{n}{t} \cancel{\theta^{t}} \cancel{(1-\theta)^{n-t}}}$$

$$= \frac{\binom{n-2}{t-2}}{\binom{n}{t}} = \frac{(n-2)!}{(t-2)!\, \cancel{(n-t)!}} \cdot \frac{t!\, \cancel{(n-t)!}}{n!}$$

$$= \frac{t\,(t-1)}{n\,(n-1)}.$$

$$E\left[\mathbb{1}(X_1 = 1)\, \mathbb{1}(X_2 = 1) \mid T\right] = \boxed{\delta^* = \frac{\sum X_i}{n} \frac{\sum X_i - 1}{n-1} \quad \text{IMVU}}$$

$$E(\delta^*) = E\left[\frac{\sum x}{n} \quad \frac{\sum x - 1}{n-1}\right]$$

$$E\,\delta^* = E\left(\frac{\sum x}{n}\right) \frac{1}{n-1} E\left(\sum x_i - 1\right)$$

MAL

$$= \frac{n\theta}{n} \frac{1}{n-1} (n\theta - 1) = \theta \frac{n\theta - 1}{n-1}$$