# IMVU (cont)

Ej $X_1 \ldots X_n \overset{iid}{\sim} P(\lambda)$ $\boxed{g(\lambda) = e^{-\lambda}} = P_\lambda(X_i = 0)$

$T = \sum X_i \sim P(n\lambda)$

Proponemos $\eta(T)$ y intentamos

$$E_\lambda \eta(T) = e^{-\lambda}$$

$$\sum_{t=0}^{\infty} \eta(t) \frac{e^{-n\lambda}(n\lambda)^t}{t!} = e^{-\lambda}$$

$$\sum_{t=0}^{\infty} \eta(t) \frac{(n\lambda)^t}{t!} = e^{-\lambda+n\lambda} = \underbrace{e^{(n-1)\lambda}}$$

$$\underbrace{\sum_{t=0}^{\infty} \eta(t) n^t \frac{\lambda^t}{t!}} = \sum_{t=0}^{\infty} \frac{[(n-1)\lambda]^t}{t!} = \underbrace{\sum_{t=0}^{\infty} (n-1)^t \frac{\lambda^t}{t!}}$$

$\forall t$ $$\eta(t) n^t = (n-1)^t$$

$$\boxed{\eta(t) = \left(\frac{n-1}{n}\right)^t}$$

Ver

$$E_\lambda \eta(T) = \sum_{t=0}^{\infty} \left(\frac{n-1}{n}\right)^t \frac{e^{-n\lambda}(n\lambda)^t}{t!}$$

$$= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{1}{t!} [(n-1)\lambda]^t \, e^{-(n-1)\lambda} \, \underbrace{e^{(n-1)\lambda}}$$

$$= e^{-n\lambda} e^{(n-1)\lambda} \underbrace{\sum_{t=0}^{\infty} e^{-(n-1)\lambda} \frac{[(n-1)\lambda]^t}{t!}}_{1}$$

$$= e^{-\lambda} \quad \diamond$$

Ejercicio $X_1 \ldots X_n \overset{iid}{\sim} U(0,\theta)$ objetivo IMVU

a) $\delta = 2\bar{X}$ $\quad E\delta = \theta$ $\quad T = X_{(n)}$

$$\eta(X_{(n)}) = E\left[2\bar{X} \mid X_{(n)}\right]$$

intentar?

b) $\tilde{\delta} = 2X_1$ $\quad E\tilde{\delta} = \theta$

$$\eta(X_{(n)}) = E\left[2X_1 \mid X_{(n)}\right]$$

Intuyamos $X_1 \mid X_{(n)} = x_{(n)} \sim U(0, x_{(n)})$ $\cancel{X}$

$$E\left[X_1 \mid X_{(n)} = x_{(n)}\right] = \frac{x_{(n)}}{2}$$

$$\eta(X_{(n)}) = 2\frac{X_{(n)}}{2} = X_{(n)}$$

Mal

c) $X_{(n)}$

$$F_{X_{(n)}}(v) = P(X_{(n)} \leq v) = \left(\frac{v}{\theta}\right)^n$$

$$f_{X_{(n)}}(v) = \frac{n}{\theta}\left(\frac{v}{\theta}\right)^{n-1}$$

$$E_\theta X_{(n)} = \int_0^\theta v \, \frac{n}{\theta}\left(\frac{v}{\theta}\right)^{n-1} dv = \ldots = \frac{n}{n+1}\theta$$

$$\boxed{\delta^* = \frac{n+1}{n} X_{(n)}}$$

$E\delta^* = \theta$ fn est suf.

IMVU

Observar IMVU no siempre existe

Ej $X_1 \dots X_n \overset{iid}{\sim} B(k,\theta)$ $T = \sum X_i \sim B(nk, \theta)$

$$g(\theta) = E_\theta\, \delta(T) = \sum_{t=0}^{nk} \delta(t) \binom{nk}{t} \theta^t (1-\theta)^{nk-t}$$

$$= (1-\theta)^{nk} \underbrace{\sum_{t=0}^{nk} \delta(t) \binom{nk}{t} \left(\frac{\theta}{1-\theta}\right)^t}_{\text{Polinomio grado } nk \text{ en } \left(\frac{\theta}{1-\theta}\right)}$$

La ecuación
no tiene solución si
$g(\theta) = e^{-\theta}$ ni cuando
$g(\theta)$ no se corresponde con un polinomio de grado nk!

Para $g(\theta) = e^{-\theta}$ no existe IMVU!

## Desigualdad de Cramer-Rao

$(X_1 \dots X_n) \sim f(x_1 \dots x_n ; \theta)$ $\theta \in \overset{\circ}{\Theta} \subset \mathbb{R}$

Supuestos

i) $S = \{ \mathbf{x} : f(\mathbf{x}; \theta) > 0 \}$ independiente de $\theta$

ii) $\forall \mathbf{x}$ $\frac{\partial}{\partial\theta} f(\mathbf{x};\theta)$ existe

iii) Si $h(\mathbf{X})$ es tal que $E_\theta |h(\mathbf{X})| < \infty$

entonces $\frac{\partial}{\partial\theta} \int \dots \int h(\mathbf{x}) f(\mathbf{x};\theta)\, d\mathbf{x} = \int \dots \int h(\mathbf{x}) \frac{\partial}{\partial\theta} f(\mathbf{x},\theta)\, d\mathbf{x}$

iv) $0 < I(\theta) := E_\theta \left( \frac{\partial \log f(\mathbf{x},\theta)}{\partial\theta} \right)^2 < \infty$

Información de Fischer

**Lema 1** con i) a iv) sea $\psi(\mathbf{x};\theta) = \dfrac{\frac{\partial}{\partial\theta} f(\mathbf{x};\theta)}{f(\mathbf{x};\theta)}$

entonces

$$= \frac{\partial}{\partial\theta} \log f(\mathbf{x};\theta)$$

A) $E_\theta\, \psi(\mathbf{x};\theta) = 0$ $\quad\quad$ $Var_\theta\, \psi(\mathbf{x};\theta) = I(\theta)$

B) Si aparte i)–iv) tambien vale que

$$\frac{\partial^2}{\partial\theta^2} \int\cdots\int h(\mathbf{x}) f(\mathbf{x};\theta)\, d\mathbf{x} = \int\cdots\int h(\mathbf{x}) \frac{\partial^2}{\partial\theta^2} f(\mathbf{x},\theta)\, dx$$

entonces tambien:

$$I(\theta) = -E_\theta \left\{ \frac{\partial^2 \log f(X;\theta)}{\partial\theta^2} \right\}$$

**Dem**

$$\int_S\cdots\int f(x_1 \ldots x_n;\theta)\, dx_1 \ldots dx_n = 1$$

$$\frac{\partial}{\partial\theta} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} 1_S(x_1 \ldots x_n; \cancel{\theta})\, f(x_1 \ldots x_n;\theta)\, dx_1 \ldots dx_n = \frac{\partial}{\partial\theta} 1$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} 1_S(x_1 \ldots x_n) \frac{\partial}{\partial\theta} f(x_1 \ldots x_n;\theta)\, dx_1 \ldots dx_n = 0$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} 1_S(\mathbf{x}) \boxed{\frac{\frac{\partial}{\partial\theta} f(\mathbf{x};\theta)}{f(\mathbf{x};\theta)}}\, f(\mathbf{x};\theta)\, d\mathbf{x} = 0$$

$$E_\theta\left[\frac{\frac{\partial}{\partial\theta} f(X;\theta)}{f(X;\theta)}\right] = E_\theta\left[\frac{\partial}{\partial\theta} \log f(X;\theta)\right] = E_\theta\, \psi(X;\theta) = 0$$

$$\text{Var } W = E W^2 - \underbrace{(EW)^2}_{0}$$

$$\text{Var}_\theta\, \psi(X;\theta) = E_\theta\left[\psi(X;\theta)\right]^2 = E_\theta\left[\frac{\partial}{\partial\theta}\log f(X;\theta)\right]^2 = I(\theta)$$

Aparte

$$\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\log f(X;\theta) = \frac{\partial}{\partial\theta}\left\{\frac{f'(X,\theta)}{f(X,\theta)}\right\}$$

$$= \frac{f''f - (f')^2}{f(X;\theta)^2} = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2$$

$$E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right\} = \underbrace{E_\theta\left\{\frac{\frac{\partial^2}{\partial\theta^2}f(X;\theta)}{f(X;\theta)}\right\}}_{0} - \underbrace{E_\theta\left[\frac{\partial}{\partial\theta}\log f(X,\theta)\right]^2}_{I(\theta)}$$

$$E_\theta\left\{\frac{\partial}{\partial\theta^2}\log f(X;\theta)\right\} = 0 - I(\theta)$$

¿porque 1er termino es 0?

$$E_\theta\left\{\frac{\frac{\partial^2}{\partial\theta^2}f(X;\theta)}{f(X;\theta)}\right\} = \int_S\cdots\int \frac{\frac{\partial^2}{\partial\theta^2}f(X;\theta)}{\cancel{f(X;\theta)}}\,\cancel{f(X,\theta)}\,dX$$

$$= \frac{\partial^2}{\partial\theta^2}\underbrace{\int_S\cdots\int f(X;\theta)\,dX}_{1} = 0$$

$$I(\theta) = -E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right\}$$

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Teorema Cramer-Rao

Sean i) a iv) y $\delta(X)$ insesgado de $q(\theta)$

con $E_\theta\,\delta(X) < \infty$

Entonces A) $\boxed{\text{Var}_\theta[\delta(X)] \geq \frac{|q'(\theta)|^2}{I(\theta)}}$

B) = ssi $\delta(X)$ es est suf en flia exponencial.

Dem P1) Por lema $E_\theta \frac{\partial}{\partial\theta} \log f(X;\theta) = 0$

$$Var_\theta \left(\frac{\partial}{\partial\theta} \log f(X;\theta)\right) = I(\theta)$$

P2) Como $\delta(X)$ es insesgado

$$E_\theta\, \delta(X) = \int_S \cdots \int \delta(x) f(x;\theta)\, dx = g(\theta)$$

por hipótesis $E_\theta |\delta(X)| < \infty$ entonces puedo derivar dentro

$$\frac{\partial}{\partial\theta} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} 1_S(x)\, \delta(x)\, f(x;\theta)\, dx = \frac{\partial}{\partial\theta} g(\theta)$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} 1_S(x)\, \delta(x) \frac{\partial}{\partial\theta} f(x;\theta)\, dx = g'(\theta)$$

$$\int_S \cdots \int \left[\frac{\delta(x) \frac{\partial}{\partial\theta} f(x;\theta)}{f(x;\theta)}\right] f(x;\theta)\, dx = g'(\theta)$$

$$E_\theta \left\{ \delta(X) \frac{\partial}{\partial\theta} \log f(X;\theta) \right\} = g'(\theta)$$

$$\operatorname{cov}(U,V) = EUV - EU \underbrace{EV}_{=0} \qquad V = \frac{\partial}{\partial\theta} \log f(X;\theta)$$

$$\operatorname{Cov}_\theta \left\{ \delta(X); \frac{\partial}{\partial\theta} \log f(X;\theta) \right\} = g'(\theta)$$

$$\operatorname{Cov}(U,V)^2 \leq \operatorname{var} U \; \operatorname{Var} V$$

$$[g'(\theta)]^2 = \operatorname{Cov}_\theta \left\{ \delta(X); \frac{\partial}{\partial\theta} \log f(X;\theta) \right\}^2 \leq \operatorname{Var}_\theta [\delta(X)]\, \overbrace{\operatorname{Var}_\theta \left\{ \frac{\partial}{\partial\theta} \log f(X;\theta) \right\}}^{I(\theta)}$$

$$\mathrm{Var}_\theta\, \delta(X) \geq \frac{|q'(\theta)|^2}{I(\theta)}$$

Parte B = en Cauchy Schwartz cdo $V = a + bU$

en nuestro caso

$$\frac{\partial}{\partial\theta} \log f(X,\theta) = a(\theta) + \overbrace{b(\theta)}^{c(\theta)}\ \overbrace{\delta(X)}^{r(X)}$$

$$f(X;\theta) = \underbrace{e^{a(\theta)}}_{A(\theta)}\ e^{b(\theta)\delta(X)}$$

en forma exponencial

Recíprocamente supongamos

$$f(x;\theta) = A(\theta)\, e^{c(\theta) r(X)}\, h(x)$$

$$\log f(x;\theta) = \log A(\theta) + c(\theta) r(X) + \log h(x)$$

$$\frac{\partial}{\partial\theta} \log f(X,\theta) = \underbrace{\frac{A'(\theta)}{A(\theta)}}_{a(\theta)} + \underbrace{c'(\theta)}_{b(\theta)} r(X)$$

volvemos a = en Cauchy-Schwartz y por lo tanto hay igualdad en A.

Lema $X_1 \ldots X_n \overset{iid}{\sim} f_{X_i}(x_i;\theta)$

$$\underbrace{I_n(\theta)} = n\, I_1(\theta)$$

Dem

$$f_{X_1 \ldots X_n}(x_1 \ldots x_n;\theta) = \prod_{i=1}^{n} f_X(x_i;\theta)$$

$$\log f_{X_1 \ldots X_n}(x_1 \ldots x_n;\theta) = \sum_{i=1}^{n} \log f_X(x_i;\theta)$$

$$E_\theta \; -\frac{\partial^2}{\partial\theta^2}\log f_{X_1\cdots X_n}(x_1\cdots x_n;\theta) = -\sum_{i=1}^{n} E_\theta \frac{\partial^2}{\partial\theta^2}\log f_X(x_i;\theta)$$

$$\underbrace{-E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f_{X_1\cdots X_n}(X_1\cdots X_n;\theta)\right\}}_{I_n(\theta)} = \sum_{i=1}^{n}\underbrace{-E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f_X(x_i;\theta)\right]}$$

$$I_n(\theta) = n\,I_1(\theta)$$

Ejemplo $X_1 \cdots X_n \overset{iid}{\sim} B(\theta)$ $\quad f_X(x_i;\theta) = \theta^{x_i}(1-\theta)^{1-x_i}$

$$\log f_X(x_i;\theta) = x_i\log\theta + (1-x_i)\log(1-\theta)$$

$$\frac{\partial}{\partial\theta}\log f_X(x_i;\theta) = \frac{x_i}{\theta} + \frac{1-x_i}{1-\theta}(-1)$$

$$\frac{\partial^2}{\partial\theta^2}\log f_X(x_i;\theta) = -\frac{x_i}{\theta^2} - \frac{1-x_i}{(1-\theta)^2}$$

$$\frac{\partial}{\partial\theta}\left\{-\frac{1-X}{1-\theta}\right\} = -(1-X)\frac{\partial}{\partial\theta}\left(\frac{1}{1-\theta}\right)$$

$$= -(1-X)\frac{(\cancel{-}1)}{(1-\theta)^2}(\cancel{-}1)$$

$$E_\theta\left\{\frac{\partial^2}{\partial\theta^2}\log f_X(X_i;\theta)\right\} = -\frac{E_\theta X_i}{\theta^2} - \frac{1-E_\theta X_i}{(1-\theta)^2}$$

$$= -\frac{\cancel{\theta}}{\theta^{\cancel{2}}} - \frac{\cancel{1-\theta}}{(1-\theta)^{\cancel{2}}} = -\frac{1}{\theta} - \frac{1}{1-\theta}$$

$$I(\theta) = \frac{1-\theta+\theta}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}$$

Supongamos que queremos estimar $g(\theta) = \theta$ entonces

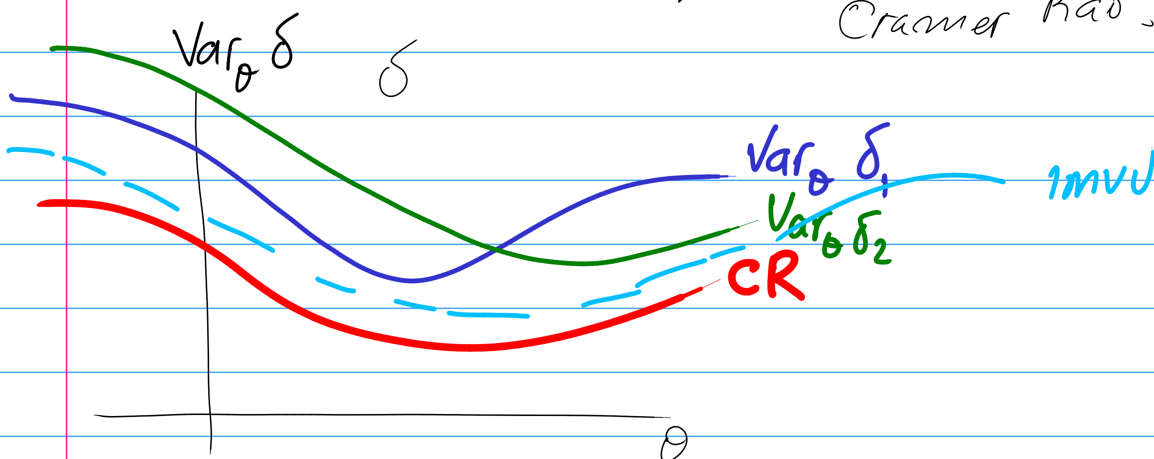
$$\frac{\partial}{\partial \theta} q(\theta) = \frac{\partial}{\partial \theta} \theta = 1$$

CR dice que $\boxed{\operatorname{Var}_\theta \delta(X_1 \dots X_n) \geq \frac{1}{I_n(\theta)} = \frac{\theta(1-\theta)}{n}}$

$$I_n(\theta) = n I_1(\theta) = \frac{n}{\theta(1-\theta)}$$

Tomemos $\delta(X_1 \dots X_n) = \frac{1}{n} \sum X_i \qquad E\delta = \theta$

$\operatorname{Var}_\theta \delta = \frac{\theta(1-\theta)}{n}$ y alcanza la cota de Cramer Rao.




Mas info Hammersley - Chapman - Robbins

## CASO MULTIPARAMETRICO

Ahora $\theta = (\theta_1 \dots \theta_k) \in \Theta \subset \mathbb{R}^k$

$$q(\theta) = q(\theta_1, \dots \theta_k) : \mathbb{R}^k \to \mathbb{R}$$

$$f_X(x, \theta) = f_X(x; \theta_1 \dots \theta_k)$$

la información $I(\theta)$ es ahora una matriz con elementos

$$\underset{(k \times k)}{I(\theta)_{ij}} = \operatorname{Cov}_\theta \left[ \left( \frac{\partial}{\partial \theta_i} \log f_X(X; \theta) \right); \left( \frac{\partial}{\partial \theta_j} \log f_X(X; \theta) \right) \right]$$

Bajo regularidad similar al caso $k=1$

$$I(\theta)_{ij} = E_\theta \left\{ \left( \frac{\partial}{\partial \theta_i} \log f_X(X;\theta) \right) \left( \frac{\partial}{\partial \theta_j} \log f_X(X;\theta) \right) \right\}$$

$$= -E_\theta \left\{ \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f_X(X;\theta) \right\}$$

$$\nabla q(\theta) = \begin{bmatrix} \frac{\partial}{\partial \theta_1} q(\theta) \\ \vdots \\ \frac{\partial}{\partial \theta_k} q(\theta) \end{bmatrix}$$

CR
$$\boxed{\underset{1\times 1}{\operatorname{Var}_\theta \delta(X)} \geq \underset{1\times k}{\nabla q(\theta)^T} \underset{k \times k}{\left[ I(\theta) \right]^{-1}} \underset{k\times 1}{\nabla q(\theta)}}$$

$k=1$

$$q'(\theta) \frac{1}{I(\theta)} q'(\theta)$$

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Continuación Ejemplo

$X_1 - X_n \overset{iid}{\sim} U(0,\theta)$ $\quad T = X_{(n)}$ suf. minimal completo

b) $\tilde{\delta} = 2X_1$

$$\eta(X_{(n)}) = \cancel{E_\theta}\left[ 2X_1 \mid X_{(n)} \right]$$

Sostengo que dado $X_{(n)} = x_{(n)}$ la v.a. $X_1$

$$2X_1 = \begin{cases} 2x_{(n)} & \text{con prop } \frac{1}{n} \\ U(0, 2x_{(n)}) & \checkmark\checkmark \ \frac{n-1}{n} \end{cases}$$

$$2\,E\left[X_1 \mid X_{(n)} = x_{(n)}\right] = 2x_{(n)}\frac{1}{n} + \frac{n-1}{n}x_{(n)}$$

$$= x_{(n)}\left\{\frac{2+n-1}{n}\right\} = x_{(n)}\frac{n+1}{n}$$

$$\boxed{g(X_{(n)}) = \frac{n+1}{n}X_{(n)}}$$

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