$$\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times (0,1)$$

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$$

$$\eta_1 = \eta_1(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$\vdots$

$$\frac{\partial \eta}{\partial \theta}$$
(5×5)

$\cancel{\eta}$

CONSISTENCIA

Def Una sucesión $\delta_1, \delta_2, \ldots$ de estimadores es (debilmente) consistente para $q(\theta)$ si

$$\delta_n \xrightarrow{P} q(\theta)$$

es decir $\forall \varepsilon > 0$

$$P\{|\delta_n - q(\theta)| > \varepsilon\} \to 0.$$

¿Cómo ver consistencia?

1) Por definición

Ej $X_1 \ldots X_n \overset{iid}{\sim} U(0, \theta)$ $\quad q(\theta) = \theta \quad \delta_n = X_{(n)}$

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = P(X_1 \leq x \ldots X_n \leq x) = P(X_1 \leq x)^n$$

$$= \begin{cases} \left(\frac{x}{\theta}\right)^n & 0 < x < \theta \\ 1 & x \geq \theta \\ 0 & x \leq 0 \end{cases}$$

Tomo $\varepsilon > 0$ $\quad P[|X_{(n)} - \theta| > \varepsilon] = P[-(X_{(n)} - \theta) > \varepsilon]$

$= P[X_{(n)} < -\varepsilon + \theta] = P[X_{(n)} \leq \theta - \varepsilon]$

$= F_{X_{(n)}}(\theta - \varepsilon)$

$\varepsilon > \theta \quad F_{X_{(n)}}(\theta - \varepsilon) = F_{X_{(n)}}(-) = 0$

$0 < \varepsilon < \theta$

$$F_{X_{(n)}}(\theta - \varepsilon) = \left(\frac{\theta - \varepsilon}{\theta}\right)^n \to 0$$

$\forall \varepsilon > 0 \quad P[|X_{(n)} - \theta| > \varepsilon] \to 0$

$\therefore$ $X_{(n)}$ consistente para θ.

Ej $\quad X_1 \dots$ iid $N(\mu, 1)$

$\delta_1 = X_1$

$\delta_2 = me\{X_1, X_2\}$

$\delta_3 = \dfrac{X_1 + X_2 + X_3}{3}$

medias para impar
medianas para par.
Consistente.

2.- Por teoremas anteriores

Ej $\quad X_1, X_2, \dots$ v.a. iid $EX_1 = \mu < \infty$.

$$\delta_n = \frac{1}{n}(X_1 + \dots + X_n)$$

$\delta_n \xrightarrow{P} \mu \qquad$ LGN.

Teorema (Mapeo continuo)

Supongamos una sucesión de v.a. $Y_1, Y_2, \dots$ y una constante a tal que

$$Y_n \xrightarrow{P} a \quad \Big| \; \forall \varepsilon \; P[|Y_n - a| > \varepsilon\} \to 0$$

y sea g una función continua en a.
Entonces

$$g(Y_n) \xrightarrow{P} g(a)$$

Ej $X_1, X_2, \dots$ iid $B(\theta)$

$$\hat{p}_n = \frac{1}{n} \sum X_i = \frac{\#\{i : X_i = 1\}}{n}$$

$$\hat{p}_n \xrightarrow{P} p \quad \text{LGN}$$

$$(\hat{p}_n)^2 \to p^2 \quad \text{por teorema mapeo continuo}$$

Ej $X_1, X_2, \dots \overset{iid}{\sim} \quad EX = \mu \neq 0 \quad Var X = \sigma^2$

$$\delta_n = \frac{S_n}{\bar{X}_n} \xrightarrow{P} \left(\frac{\sigma}{\mu}\right)$$

$$\bar{X}_n \xrightarrow{P} \mu \qquad S_n^2 \to \sigma^2$$

$$g(\bar{X}_n, S_n^2) = \frac{\sqrt{S_n^2}}{\bar{X}_n} \xrightarrow{P} g(\mu, \sigma^2) = \frac{\sigma}{\mu}$$

3.- Teorema $\delta_1, \delta_2, \ldots$ sucesión estimadores tales que

i) $Var_\theta(\delta_n) \to 0$ ii) $E_\theta(\delta_n) \to g(\theta)$

entonces δ_n es consistente para $g(\theta)$.

Dem $\varepsilon > 0$ $P_\theta\{|\delta_n - g(\theta)| > \varepsilon\} \leq \dfrac{E_\theta[|\delta_n - g(\theta)|]^2}{\varepsilon^2}$

$$= \frac{1}{\varepsilon^2}\{\underbrace{Var_\theta\,\delta_n}_{\to 0} + \underbrace{[E_\theta\delta_n - g(\theta)]^2}_{\to 0}\}$$

$$\therefore \delta_n \xrightarrow{P} g(\theta) \quad \triangledown.$$

$\bar{X}_n$ $\bar{X}_n^2$ $\dfrac{\sqrt{S_n^2}}{\bar{X}_n}$ Me

Convergencia en distribución

Recordar $X_1 \; X_2, \ldots \quad X$

$\wr \qquad \wr \qquad\qquad \wr$

$F_1(\cdot) \; F_2(\cdot) \qquad F(\cdot)$

$$X_n \xrightarrow{\mathcal{D}} X \; ; \quad X_n \xrightarrow{\mathcal{D}} F(\cdot)$$

$F_n(x) \longrightarrow F(x)$ $\forall x$ donde $F(\cdot)$ es continua

Ej $X_1\, X_2 \ldots$ iid $B(\theta)$

$$\sqrt{n}\,(\hat{p}_n - \theta) \Rightarrow N(0, \theta(1-\theta))$$

TCL $\hat{p}_n = \frac{1}{n}\sum X_i$

Distribución Asintótica De los Estimadores Máximo Verosímiles

Situación más regular.

$X_1, X_2, \ldots \overset{iid}{\sim} F_X(x;\theta) \qquad \theta \in \Theta \subset \mathbb{R}$

$$f_{X_1 \cdots X_n}(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = L_n(\theta)$$

$$\ell_n(\theta) = \log L_n(\theta) = \sum_{i=1}^{n} \log f_X(x_i;\theta)$$

$$\ell_n'(\theta) = \frac{\partial}{\partial\theta} \ell_n(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f_X(x_i;\theta)$$

En el caso regular $\ell_n'(\hat{\theta}^{MV}) = 0$.

Expandamos $\quad \ell_n'(\hat{\theta}^{MV}) \cong \ell_n'(\theta) + \ell_n''(\theta)(\hat{\theta}^{MV} - \theta)$

$$0 \cong \ell_n'(\theta) + \ell_n''(\theta)(\hat{\theta}^{MV} - \theta)$$

$$\sqrt{n}\ \frac{1}{n} - \ell_n''(\theta)(\hat{\theta}^{MV} - \theta) \cong \sqrt{n}\frac{\ell_n'(\theta)}{n} \Rightarrow N(0, I(\theta))$$

Pero $\ell_n'(\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f_X(x_i;\theta)$

antes de sacar la muestra

$$\sqrt{n}\ \frac{\ell_n'(\theta)}{n} = \sqrt{n}\ \frac{1}{n}\sum_{i=1}^{n} \underbrace{\frac{\partial}{\partial\theta} \log f_X(X_i;\theta)}_{T_i = \frac{\partial}{\partial\theta} \log f_X(X_i;\theta)}$$

$$E_\theta T_i = E_\theta\left\{\frac{\partial}{\partial\theta} \log f_X(X_i,\theta)\right\} = 0 \text{ bajo regularidad}$$

(2) $$\sqrt{n}\ \frac{\ell_n'(\theta)}{n} = \sqrt{n}\left(\bar{T}_n - 0\right) \overset{\mathcal{D}}{\Rightarrow} N(0, I(\theta))$$

$$\operatorname{Var} T_i = \operatorname{Var}_\theta \frac{\partial}{\partial\theta} \log f(X_i;\theta) = I(\theta)$$

O sea

$$\underbrace{-\frac{\ell_n''(\theta)}{n}}_{I(\theta)^{-1}} \sqrt{n}\left(\hat{\theta}^{MV}-\theta\right) \Longrightarrow \underbrace{N\left(0, \overbrace{I(\theta)}^{I(\theta)^{-1}}\right)}_{\quad a^{-1} \qquad a^{-2}a}$$

Recordar Slutzky $X_n \xrightarrow{P} a$

$Y_n \overset{\mathcal{D}}{\Rightarrow} Y$

$X_n Y_n \overset{\mathcal{D}}{\Rightarrow} aY$

$$I(\theta)^{-1}\left[-\frac{\ell_n''(\theta)}{n}\right] \sqrt{n}\left(\hat{\theta}^{MV}-\theta\right) \Rightarrow N\left(0, I(\theta)^{-1}\right)$$

$X \sim N(a, b)$ $\qquad c = I^{-1} \quad b = I$

$cX \sim N(ca, c^2 b)$

Notar $\displaystyle -\frac{\ell_n''(\theta)}{n} = \frac{1}{n}\sum_{i=1}^{n} -\frac{\partial^2}{\partial\theta^2}\log f_X(X_i,\theta)$

$$\xrightarrow{P} E_\theta\left\{-\frac{\partial^2}{\partial\theta^2}\log f_X(X_i,\theta)\right\} = I(\theta)$$

Entonces

$$I^{-1}(\theta)\,\frac{1}{n}\left(-\ell_n''(\theta)\right) \xrightarrow{P} 1$$

Con lo cual por slutzky

$$\boxed{\sqrt{n}\left(\hat{\theta}^{MV}-\theta\right) \overset{\mathcal{D}}{\Rightarrow} N\left(0, I(\theta)^{-1}\right)}$$

Falla cuando $\hat{\theta}^{MV}$ no es un cero de $\ell_n'(\theta)$

$I(\theta)$ no es regular

Ej $\quad X_1 X_2 \ldots$ iid $P(\lambda)$ $\quad f_X(x_i;\lambda) = \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

$$\log f_X(x_i;\lambda) = -\lambda + x_i \log\lambda - \log x_i!$$

$$\frac{\partial}{\partial\lambda}\log f_X(x_i,\lambda) = -1 + \frac{x_i}{\lambda}$$

$$\ell_n'(\lambda) = \sum_{i=1}^{n}\frac{\partial}{\partial\lambda}\log f_X(x_i;\lambda) = -n + \frac{\sum x_i}{\lambda} = 0$$

$$\boxed{\hat{\lambda}^{MV} = \frac{1}{n}\sum X_i}$$

$$\frac{\partial^2}{\partial\lambda^2}\log f_X(x_i,\lambda) = (-1)\frac{1}{\lambda^2}X_i$$

$$E\left\{-\frac{\partial^2}{\partial\lambda^2}\log f_X(X_i,\lambda)\right\} = \frac{1}{\lambda^2}E_\lambda X_i = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda} = I(\lambda)$$

$$\sqrt{n}\left(\hat{\lambda}^{mv} - \lambda\right) \Rightarrow N(0,\lambda)$$

Ej no regular

$X_1, X_2, \ldots$ iid $U(0,\theta)$

$$L_n(\theta) = \frac{1}{\theta^n}\mathbb{1}(X_{(n)} < \theta) \qquad \hat{\theta}^{MV} = X_{(n)}$$

Vimos que $\quad F_{X_{(n)}}(x) = \begin{cases} 0 & x \leq 0 \\ \left(\frac{x}{\theta}\right)^n & 0 < x < \theta \\ 1 & x \geq \theta \end{cases}$

Def $\quad V_n = n\left[\theta - X_{(n)}\right]$

$$F_{V_n}(v) = P_\theta\left[n(\theta - X_{(n)}) \leq v\right] = P_\theta\left[X_{(n)} \geq \theta - \frac{v}{n}\right]$$

$$= P_\theta\left[X_{(n)} > \theta - \frac{v}{n}\right] = 1 - F_{X_{(n)}}\left(\theta + \frac{-v}{n}\right) = 1 - \left[\frac{\theta + \frac{-v}{n}}{\theta}\right]^n$$

$$F_{V_n}(v) = 1 - \left(1 + \frac{-v/\theta}{n}\right)^n \xrightarrow[n\to\infty]{} 1 - e^{-v/\theta}$$

$$\left(1 + \frac{a}{n}\right)^n \to e^a$$

$$F_{V_n}(v) = \begin{cases} 0 & v < 0 \\ 1 - \left(1 + \frac{-v/\theta}{n}\right)^n & v \geq 0 \end{cases} \rightarrow F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - e^{-v/\theta} & v \geq 0 \end{cases}$$

$$\boxed{n(\theta - X_{(n)}) \Rightarrow \mathcal{E}(1/\theta)}$$

Extensión: CASO MULTIPARAMETRICO

$X_1 \text{----}$ iid $F_X(x_i; \theta)$ $\quad \theta \in \Theta \subset \mathbb{R}^k$

$$\sqrt{n}\left(\hat{\theta}^{MV} - \theta\right) \Rightarrow N_k\left(0, I^{-1}(\theta)\right)$$

Observar límite alcanza Cramer-Rao

Aplicación

Regresión logística $X_1 \text{---}$ indep $B(\theta_i)$

$$\log\left(\frac{\theta_i}{1-\theta_i}\right) = \beta_0 + \beta_1 X_{1i} + \dots + \beta_k X_{ki}$$

$L_n(\beta_0, \beta_1, \dots \beta_k)$ Verosimilitud

$l_n(\hat{\beta}_0^{MV}; \dots, \hat{\beta}_k^{MV}) = 0$ $\quad \hat{\beta}$ implícito

$$\sqrt{n}\left(\hat{\beta}^{MV} - \beta\right) \Rightarrow N\left(0, I^{-1}(\beta)\right)$$