Small Perturbations on Artificial Satellites as an Inverse Problem

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The geocentric motion of a satellite is mathematically simulated by a system of second order ordinary differential equations involving two perturbing functions. The first one represents the second term of the gravitational potential of the Earth and the second is due to the atmospheric drag. Assuming that the solutions of the differential equations and their first derivatives are known from measurements, a stepwise computation of the perturbations is made through a deterministic method. Two examples illustrate our method. In a real case our method should help to design an appropriate maneuver to correct the motion of a satellite.

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I. INTRODUCTION

Let us consider a dynamic system of three dimensions represented by a system of differential equations of the form

$$\ddot{y} = f(y,\dot{y},t) + P(t) \tag{1}$$

where y is a vector of components y_1, y_2, and y_3; f is a vectorial function depending on the mathematical laws governing the system and $P(t)$ is a small function to be determined on the basis of a set of measurements $\tilde{y}(t_n)$ and $\tilde{\dot{y}}(t_n)$ on a discrete set of points t_n $(n = 1,2,\ldots)$ with a constant step $h = |t_{n+1} - t_n|$.

Now let us assume that the solution of (1) may be represented by a convergent Taylor expansion with the remainder expressed in integral form, such that

$$y(t_k) = y(t_j) + h\dot{y}(t_j) + \cdots + \frac{h^\nu}{\nu!}y^{(\nu)}(t_j) + \frac{1}{\nu!}\int_{t_j}^{t_k} y^{(\nu+1)}(u)(t_k-u)^\nu du \tag{2}$$

where $|t_k - t_j| = h$.

For $\nu = 1$ and by virtue of (1)

$$y(t_k) = y(t_j) + h\dot{y}(t_j) + \int_{t_j}^{t_k}[f(y,\dot{y},u) + P(u)](t_k-u)du. \tag{3}$$

Now let us consider a reference problem

$$\ddot{y}^j = f(y^j,\dot{y}^j,t) \tag{4}$$

obtained by dropping from (1) the unknown perturbation $P(t)$ and assuming the osculating initial conditions

$$\begin{aligned} y^j(t_j) &= y(t_j) \\ \dot{y}^j(t_j) &= \dot{y}(t_j). \end{aligned} \tag{5}$$

Then we have

$$y^j(t_k) = y^j(t_j) + h\dot{y}^j(t_j) + \int_{t_j}^{t_k} f(y^j,\dot{y}^j,u)(t_k-u)du. \tag{6}$$

Comparing with (4) and by virtue of (5) we obtain

$$y(t_k) - y^j(t_k) = \int_{t_j}^{t_k}[f(y,\dot{y},u) - f(y^j,\dot{y}^j,u) + P(u)](t_k-u)du. \tag{7}$$

The quantities $y(t_k)$ are assumed as known from measurements and affected in consequence by random measurement errors while the quantities $y^j(t_k)$ are solutions of the differential equation (4) that can be obtained through convenient analytical or numerical methods.

II. NUMERICAL SOLUTIONS

From now on we put for simplicity

$$
\begin{aligned}
y(t_k) &= y_k \\
y^j(t_k) &= y_{jk} \\
R_{jk} &= y_k - y_{jk} \\
P(t_j) &= P_j.
\end{aligned} \tag{8}
$$

Equation (7) can be considered as a Fredholm integral equation of the first kind for our unknown perturbing function $P(t)$. To solve such integral equation we applied a numerical method based on the hypothesis that the function $P(t)$ admits a sufficiently accurate polynomial piecewise representation. Such method has been inspired and justified from the classical theory of FREDHOLM (see [1]). In this way our problem is reduced to a system of algebraic linear equations with unknowns P_j, $j = 1,2,\ldots,n$ ($n = 3$ or 5, say) and the procedure is repeated on successive sets of 3 or 5 data points. A complete analysis of the influence of truncation and measurement errors has been made thus allowing the development of procedures and formulas that reduce such influence to a minimum ([4–8]).

R_{jk} is the difference, or residual, between the value of the actual solution y_k of (1) at point t_k and the corresponding value y_{jk} of the reference solution fulfilling the osculating conditions (5) at point t_j.

Furthermore, let us write for the expression in brackets of (7)

$$\varphi^j(u) = f(y,\dot{y},u) - f(y^j,\dot{y}^j,u) + P(u) \tag{9}$$

so (8) takes the form

$$R_{jk} = \int_{t_j}^{t_k} \varphi^j(u)(t_k - u)du. \tag{10}$$

Let us consider three successive points t_1,t_2,t_3 and define a quadratic interpolating function

$$z(u) = a + b(t_k - u) + c(t_k - u)^2 \tag{11}$$

such that $z(u) = \varphi(u)$ at the three points. The coefficients a, b, and c depend on the reference point t_k. For instance, if we take $k = 1$, we have

$$
\begin{aligned}
a &= \varphi_{j1} \\
b &= \frac{1}{2h}[3\varphi_{j1} - 4\varphi_{j2} + \varphi_{j3}] \\
c &= \frac{1}{2h^2}[\varphi_{j1} - 2\varphi_{j2} + \varphi_{j3}]
\end{aligned}
$$

where $\varphi^j(t_k) = \varphi_{jk}$ ($k = 1,2,3$).

Replacing $z(u)$ by $\varphi(u)$ in (10) and integrating, we obtain, for $j = 2$,

$$R_{21} = h^2\left[\tfrac{1}{8}\varphi_{21} + \tfrac{5}{12}\varphi_{22} - \tfrac{1}{24}\varphi_{23}\right] + \delta I \tag{12}$$

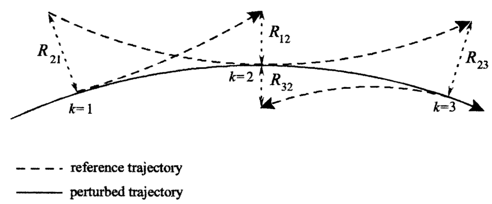

Fig. 1. Illustration of algorithm to calculate P_2.

where δI is the error introduced by replacing $z(u)$ for $\varphi^j(u)$. Now, let us put

$$\Delta f_{jk} = f[y_k,\dot{y}_k,t_k] - f[y_{jk},\dot{y}_{jk},t_k] \tag{13}$$

and

$$\tilde{R}_{jk} = \frac{R_{jk}}{h^2}. \tag{14}$$

Obviously $\Delta f_{jk} = 0$ for $j = k$, and, by virtue of (9), (12) reduces to the form

$$\tilde{R}_{21} + \tfrac{1}{24}\Delta f_{23} - \tfrac{1}{8}\Delta f_{21} - \frac{\delta I}{h^2} = \tfrac{1}{8}P_1 + \tfrac{5}{12}P_2 - \tfrac{1}{24}P_3 \tag{15}$$

where we have $P(t_i) = P_i$ ($i = 1,2,3$).

The same reasoning can be applied by combining the three points in several different pairs; by taking, for instance (see Fig. 1),

$$
\begin{aligned}
k &= 1 \qquad j = 2 \\
k &= 2 \qquad j = 1 \\
k &= 2 \qquad j = 3 \\
k &= 3 \qquad j = 2
\end{aligned} \tag{16}
$$

we obtain a system of four linear equations for P_1, P_2, and P_3 as follows:

$$
\begin{bmatrix}
\frac{1}{8} & \frac{5}{12} & -\frac{1}{24} \\
\frac{7}{24} & \frac{1}{4} & -\frac{1}{24} \\
-\frac{1}{24} & \frac{1}{4} & \frac{7}{24} \\
-\frac{1}{24} & \frac{5}{12} & \frac{1}{8}
\end{bmatrix}
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}
=
\begin{bmatrix}
\tilde{R}_{21} - \frac{1}{8}\Delta f_{21} + \frac{1}{24}\Delta f_{23} - \frac{\delta I}{h^2} \\
\tilde{R}_{12} - \frac{7}{24}\Delta f_{12} + \frac{1}{24}\Delta f_{13} - \frac{\delta I}{h^2} \\
\tilde{R}_{32} + \frac{1}{24}\Delta f_{31} - \frac{7}{24}\Delta f_{32} - \frac{\delta I}{h^2} \\
\tilde{R}_{23} + \frac{1}{24}\Delta f_{21} - \frac{1}{8}\Delta f_{23} - \frac{\delta I}{h^2}
\end{bmatrix} \tag{17}
$$

where the first equation is precisely (15). If we write this system in the form

$$MP = \tilde{R} \tag{18}$$

then $M \in \mathbb{R}^{4\times 3}$ is a rectangular matrix, P is a vector $[P_1,P_2,P_3]^T$ and $\tilde{R}$ is the vector at the right-hand side member of (17). This linear system is overdetermined, and we may obtain the generalized inverse of M,

$$M^+ = (M^TM)^{-1}M^T \tag{19}$$

which, in this case, is exactly

$$
M^+ = \begin{bmatrix}
-0.9 & 3.7 & 1.3 & -2.1 \\
1.5 & -0.5 & -0.5 & 1.5 \\
-2.1 & 1.3 & 3.7 & -0.9
\end{bmatrix}. \tag{20}
$$

Then, we have

$$P = M^+\tilde{R} \tag{21}$$

as a least squares solution of (18).

III. ERROR BOUNDS

In (21), the generalized inverse M^+ has no errors and, (if $\tilde{R}$ is affected by some errors $\delta\tilde{R}$, which we are going to analyze), then we have for the errors in the unknowns,

$$\delta P = M^+\delta\tilde{R}. \tag{22}$$

1) *Inherent Errors*: This kind of error derives from the approximations introduced by the method. The integral of (10) may be written, by the generalized mean value theorem for integrals, in the form

$$I_{32} = (t_3 - \xi)\int_{t_2}^{t_3}\varphi^2(u)du \tag{23}$$

with $\xi \in (t_2,t_3)$, so that $\max(t_3-\xi) \le h$. The replacement of the integrand by the quadratic interpolant (11) is equivalent to Simpson's rule for integrals, and it is well known that the approximation error has the form$[-h^5\varphi^{2(iv)}(h)/90]$ with $h \in (t_2,t_3)$; the total error in I_{32} is then $\|\delta I\| = \|h^6\varphi^{2(iv)}(h)/90\|$ and, in view of (22), the inherent error is

$$\|\varepsilon P \text{ inherent}\| \le \|M^+\|\left(\frac{\delta\tilde{I}}{h^2}\right) \tag{24}$$

where $\delta\tilde{I}/h^2$ is the norm of a vector of four elements of the form

$$\left|\frac{h^4\varphi_{hi}^{(iv)}(h)}{90}\right| \qquad (i = 1,2,3,4). \tag{25}$$

2) *Measurements Errors*: Here y_k and $\dot{y}_k$ may be given as quantities measured in a set of points t_k, so that

$$\begin{aligned} y_k &= \tilde{y}_k - \varepsilon_k \\ \dot{y}_k &= \dot{\tilde{y}}_k - \dot{\varepsilon}_k \qquad k = 1,2,\ldots \end{aligned} \tag{26}$$

where ε_k and $\dot{\varepsilon}_k$ are measurement errors. These errors may affect the right-hand member of (17) in two ways.

In fact, if in (6) we replace y_k, y_{jj} and $\dot{y}_{jj}$ by the measured quantities $\tilde{y}_k$, $\tilde{y}_{jj}$, and $\dot{\tilde{y}}_{jj}$, respectively, we introduce in $\tilde{R}_{jk}$ (defined by (14)) an error of the form

$$\varepsilon_I = \frac{\varepsilon_k + \varepsilon_j + h\dot{\varepsilon}_j}{h^2}. \tag{27}$$

Similarly, if instead of (13), we put

$$\Delta f_{jk} \simeq f[\tilde{y}_k,\dot{\tilde{y}}_k,t_k] - f[\tilde{y}_{jk},\dot{\tilde{y}}_{jk},t_k] \tag{28}$$

we introduce an error of the form

$$\varepsilon_{II} = \frac{\delta f}{\delta y}(\varepsilon_k + \varepsilon_j + h\dot{\varepsilon}_j) - \frac{\delta f}{\delta\dot{y}}(\dot{\varepsilon}_k + \dot{\bar{\varepsilon}}_k) \tag{29}$$

where, by virtue of (1)

$$\dot{\bar{\varepsilon}}_h = \dot{\varepsilon}_j + h[f(y_{jj},\dot{y}_{jj},t_j) + P_j]. \tag{30}$$

Summarizing, we may say that the inherent error is proportional to h^4; the measurement errors ε_I are proportional in part to $1/h^2$, while ε_{II} is proportional to h. This indicates that, when possible, in order to maintain the effect of these errors within acceptable limits, one should choose for the interval h a value of compromise. Equations (24)–(30) may help to make a proper analysis in any particular problem or situation.

IV. NUMERICAL SCHEMES

Owing to a known property of polynomial interpolation the smallest inherent error occurs at the middle point t_2. Therefore it is more convenient to calculate the perturbation corresponding to the middle point by the simple formula

$$P_2 = [1.5,-0.5,-0.5,1.5]^T.R \tag{31}$$

thus skipping the calculation of P_1 and P_3.

The following set of three points may be overlapped on two points of the previous set and again calculate the perturbation in the middle point of the new set and so forth. In this way the effect in the inherent error can be reduced, although at the cost of increasing the computational effort.

With a different scheme but similar to the process described above, it is possible to arrive to an algorithm of the form

$$\begin{aligned} P_2 = \tfrac{1}{24}[&R_{21} + R_{32} + R_{12} + R_{23} - 6(\Delta f_{21} + \Delta f_{23}) \\ &+ 2(\Delta f_{12} + \Delta f_{32}) - (\Delta f_{31} + \Delta f_{13})] \end{aligned} \tag{32}$$

where

$$R_{jk} = \frac{12}{h^2}(y_k - y_{jk}) \tag{33}$$

and

$$\Delta f_{jk} = f(y_k,t_k) - f(y_{jk,t_k}). \tag{34}$$

Besides, it is possible to show that an upper bound of the error of P_2, due to measurement errors, in this case is given by the expression

$$|\delta P_2| < \frac{\dot{\varepsilon}}{h} + h^2\frac{|P_2^{(2)}|}{6} \tag{35}$$

which means that this algorithm eliminates the effect of measurement errors in position and remains sensitive only to errors in velocity. More refined discussions may be found in [4, 7, 8].

V. NUMERICAL EXPERIMENTS

The main object of our experiments is to show the strength of our method when applied to simulated motions of an artificial satellite where the answers are known thus allowing to evaluate the accuracy of our results. For such experiments we have introduced some simplifications consisting only in the quantitatively most important perturbations reduced to simple terms. The change is not negligible in real cases but in our simulations the essential nature of the problem is slightly altered.

1) *Basic Data*: The initial geocentric orbit of the satellite is defined by the following set of coordinates and velocities referred to an equatorial geocentric Cartesian system.

Position:

$$x_0 = 412.197 \text{ K}$$
$$y_0 = -2711.161 \text{ Km}$$
$$z_0 = 6492.853 \text{ Km}$$

Velocity:

$$\dot{x}_0 = -1.786 \text{ Km/s}$$
$$\dot{y}_0 = -6.791 \text{ Km/s}$$
$$\dot{z}_0 = -2.715 \text{ Km/s.}$$

These data correspond to the following set of parameters of an elliptic orbit

$$a = 7064.7 \text{ Km} \quad \text{(semiaxis)}$$
$$e = 0.0025 \quad \text{(eccentricity)}$$
$$i = 98^\circ.28 \quad \text{(inclination)}$$
$$\Omega = 78^\circ.48 \quad \text{(ascending node)}$$
$$\omega = 90^\circ.28 \quad \text{(argument of perigee)}$$
$$M = 21^\circ.05 \quad \text{(initial mean anomaly).}$$

2) *Atmospheric Drag*: Usually it is assumed that a satellite moving inside the Earth's atmosphere with a velocity V experiences an acceleration opposed to the direction of V and of a magnitude

$$\gamma = -\tfrac{1}{2}\text{sign}\{V\}C_d \frac{S}{m} V^2 \rho \qquad (36)$$

where C_d is a dynamic coefficient, S is the effective transversal section of the satellite, m its mass, and ρ the density of the atmosphere. In our simulations we adopted (for this particular satellite) the approximation

$$\gamma = -\text{sign}\{V\}.7.V^2 E\text{-}13 \qquad (37)$$

assuming implicitly that V is tangential to the orbit and that ρ has a basic magnitude E-13, in sudden and shortlived increments that will be introduced in our experiments multiplying γ by a convenient factor F.

3) *Gravitational Potential of the Earth*: For practical applications, assuming axial symmetry for the Earth's body, the adopted form of the external potential as a function of polar equatorial coordinates $\{r, \phi\}$ is

$$V(r,\phi) = \frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty} J_k \left(\frac{r_e}{r}\right)^k \mathfrak{P}_k(\cos\phi)\right] \qquad (38)$$

where

$\mu = 398600.5 \text{ Km}^3/\text{s}^2$

(gravitational constant of the Earth),

$r_e = 6378.388$ Km (equatorial radius of the Earth),

$\mathfrak{P}_k$ = Legendre polynomials,

J_k = coefficients whose most important values are:

$J_2 = 0.00108263, \qquad J_3 = -0.00000253$

$J_4 = 0.00000161.$

For our simulations we adopted the limited expression

$$V(r,\phi) = \frac{\mu}{r}\left(\frac{r_e}{r}\right)^2 \frac{J_2}{2}(2 - 3\sin^2\phi) \qquad (39)$$

from which there result the correspondent Cartesian perturbations

$$R_x = 3\mu\left(\frac{r_e}{r}\right)^2\left(\frac{J_2}{2r}\right)\left[\left(\frac{x}{r^2}\right)(3\sin^2\phi - 2) - \phi_x \sin(2\phi)\right]$$
$$R_y = 3\mu\left(\frac{r_e}{r}\right)^2\left(\frac{J_2}{2r}\right)\left[\left(\frac{y}{r^2}\right)(3\sin^2\phi - 2) - \phi_y \sin(2\phi)\right]$$
$$R_z = 3\mu\left(\frac{r_e}{r}\right)^2\left(\frac{J_2}{2r}\right)\left[\left(\frac{z}{r^2}\right)(3\sin^2\phi - 2) - \phi_z \sin(2\phi)\right] \qquad (40)$$

where

$$\phi = \arctan\left(\frac{z}{x^2 + y^2}\right)^{1/2} \qquad (41)$$

and

$$\phi_x = \frac{z\cos^2\phi + x}{(x^2 + y^2)^{3/2}}$$
$$\phi_y = \frac{z\cos^2\phi + y}{(x^2 + y^2)^{3/2}} \qquad (42)$$
$$\phi_z = \frac{\cos^2\phi}{(x^2 + y^2)^{1/2}}.$$

A. Experiments

Experiment 1: This experiment has a theoretical character because it is assumed that the only perturbation is due to the atmospheric drag. Our object is to show how our method may work in cases of sudden increases of the magnitude of the

TABLE I
Magnitude of Perturbations Due to Atmospheric Drag in Km/s^2, Including an Interval with Increasing Factor $F = 100$

Time t (min)	Simulated $\gamma(t)$	Estimated $P(t)$
10	.298064D-10	.295935D-10
40	.365122D-08	.363093E-08*
70	.381316D-08	.197564E-08*
100	.305461D-10	.306355D-10

Note: *Ends of interval.

TABLE II
Magnitude of Perturbations Due to J_2 Term of Gravitational Potential of Geoid Plus Atmospheric Drag Augmented by Increasing Factor F

Time t (min)	F	Simulated $p(t)$	Estimated $P(t)$ $\sigma_1 = \sigma_2 = 0$	Estimated $P(t)$ $\sigma_1 = 30$ m $\sigma_2 = .3$ m/s
10	1.0	.112118D-04	.207851D-04	.233214D-04
40	1.2	.170896D-04	.217511D-04	.229470D-04
70	10	.109580D-04	.070344D-04	.103784D-04
100	100	.841497D-05	1.58177D-05	1.510525D-05

Note: σ_1,σ_2: standard deviations of random errors in measured positions and velocities.

perturbation that might be due to changes in the solar activity that affect the atmospheric density.

To simulate the motion of the satellite we integrated numerically the equations (1) and calculated the perturbation from formula (37) and collected our basic results of positions, velocities, and perturbations at regular intervals of 4 min covering one geocentric revolution of an approximate period of 100 min. In order to apply the procedures described in Section II and illustrated synthetically in Fig. 1 we considered our basic results subdivided in successive groups of data corresponding to three successive instants t_1, t_2, and t_3.

Then, to estimate the perturbation $P(t_2)$, we used systematically the formula (32) considering the inherent and measurement errors as negligible (in some cases we tested formula (31) obtaining approximately similar results). In such a manner we estimated the value of the perturbation corresponding to the middle point of each successive group of simulated data.

It is worthwhile to remark that the reference orbit of our procedures consists of a series of Keplerian orbits osculating to successive points of the simulated perturbed orbit.

To check the errors in our results we compared the estimated perturbations and those simulated with formula (37).

In relation with the last remark of Section III we have found that in the examples that follow the regular intervals of size $h = 4$ minutes gave the best results. With $h = 10$ min the results were still good though somehow less precise.

Our results are summed up in Table I. In it is included the factor F that multiplies the atmospheric drag in order to simulate the effects of a sudden increment in the solar activity, in the interval indicated in Table I.

To finish this experiment we used the estimated perturbations of Table I, to simulate a maneuver to counterbalance their effects. In this way we could restore the perturbed orbital elements to their initial values, especially the parameters a and e.

In the case of a satellite mission where its trajectory is prescribed as a set of successive osculating Keplerian orbits the positions and velocities exact values $y(t)$ and $\dot{y}(t)$ may be calculated and from the differences with the corresponding measured values $\tilde{y}(t)$ and $\dot{\tilde{y}}(t)$ the perturbation $P(t)$ may be obtained and used to restore the perturbed trajectory to the prescribed one in the manner described above.

Experiment 2: This case is closer to a real one because the perturbations to be estimated consists of the summation of both the atmospheric drag and the second term of the gravitational potential of the Earth as described in Section V. As in the Experiment 1, the reference orbit consists again of a series of Keplerian orbits osculating to successive points of the simulated perturbed orbit.

In this case we include the effect of measurement errors by adding to the simulated data random numbers corresponding to a Gaussian distribution with standard deviations σ_1 and σ_2 for coordinates of positions and velocities, respectively, as indicated in Table II.

The explanations are entirely similar to those of the Experiment 1 and the results obtained are given in Table II.

VI. CONCLUSIONS

The method presented in this paper can help in the determination of the perturbing forces affecting the motion of artificial satellites of the Earth. Atmospheric drag deserves a special attention due to its rapid changes depending on the variable solar energy absorbed by the atmosphere.

We want to emphasize that the method presented here is essentially deterministic. The same kind of problem could be approached by a least squares recursive process or a filtering technique used for parameter identifications; however, such a process is known to be rather sensible to any inadequacy of the mathematical model to a real problem of a dynamical system or to errors in the basic data. Our deterministic method is based only in the simple assumptions that the differential equations of the problem can be expressed in short intervals by a Taylor valid expansion and that the unknown perturbations can also be approximated piecewise by polynomials or a combination of other elementary functions. This method has also the advantage of its applicability in a short interval of one satellite revolution as it was shown in our examples.

In a real case the results of our method should allow to design some appropriate maneuvers to control the orbital motion of the satellite. Otherwise our method may be used as a first approach to a complicated problem in order to build up an adequate model for a later refined analysis of the perturbed motion.

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In the United States he has developed the following activities. In 1957, as a Fellow of the Guggenheim Foundation, in the IBM Watson Laboratory at Columbia University he did studies of computational methods in celestial mechanics. In 1958 he continued his work in the Computer Center of Princeton University, Princeton, NJ. From 1959 to 1961 he was appointed a senior astronomer in the staff of the Smithsonian Astrophysical Observatory, in Cambridge, MA to do research in space science and tracking of artificial satellites. His work was published in several numbers of the series of "Special Reports" and he also authored a monograph entitled "*A Guide to Celestial Mechanics*", both edited by the Observatory. In 1967/1968, he spent a year as a fellow of the National Science Foundation, at the Laboratory of Theoretical Studies of Goddard Space Flight Center of NASA to do work on theory of orbits. At the same time he gave a graduate course on celestial mechanics at the University of Georgetown in Washington, D.C. In 1977, again as a fellow of the Guggenheim Foundation, he worked at the Laboratory of Theoretical Mechanics of Texas University in Austin, on numerical methods of dynamical systems. His research interests are in numerical and computational methods of dynamical systems and he has published over 40 papers in these areas.

In 1998/1999 he authored a book, composed of three volumes, entitled "*An Introduction to Astrodynamics, Theory and Numerical Methods*", written in Spanish. It has been edited by the National Commission of Space Activities of Argentina (CONAE). Since 2000 a new asteroid discovered at the Astronomical Observatory of the National University of San Juan, Argentina, has been catalogued as 4617=1976, "Zadunaisky" after approval by the International Astronomical Union. In 2003, he received from the Argentine National Academy of Sciences their prize on "Astronomy" for his contributions on the calculus and analysis of orbits of asteroids and artificial satellites.