Multiscale Expansion and Integrability of Dispersive Wave Equations

A. Degasperis[1]

Dipartimento di Fisica, Universitá di Roma “La Sapienza”, P.le A.Moro 2, 00185 Roma, Italy and Istituto Nazionale di Fisica Nucleare, Sezione di Roma
antonio.degasperis@roma1.infn.it

1 Introduction

The propagation of *nonlinear dispersive* waves is of great interest and relevance in a variety of physical situations for which model equations, as infinite-dimensional dynamical systems, have been investigated from various perspectives and to different purposes. In the ideal case in which waves propagate in a one-dimensional medium (no diffraction) without losses and sources, quite a number of special models, so called *integrable* models, have been discovered together with the mathematical tools to investigate them. This important progress has provided important contributions to such matters as dispersionless propagation (solitons), wave collisions, wave decay, long-time asymptotics among others. On the mathematical side, such progress on integrable models has considerably contributed also to our present (admittedly not concise) answer to the question “What is intergrability?”, which can be found in [1], and a partial guide to the vaste literature on the theory of solitons is given in Ref. [2].

It is plain that integrable models, though both useful and fascinating, remain exceptional: nonlinear partial differential equations (PDEs) in 1+1 variables (space+time) are generically not integrable. The aim of these notes is to show how an algorithmic technique, based on *multiscale* analysis and *perturbation* theory, may be devised as a tool to establish how " far " is a given PDE from being integrable. This method basically associates to a given PDE one or more, generally simpler, PDEs with respect to rescaled space and time variables.This approach [3] has been known in applicative contexts [4] since several decades as it provides approximate solutions when only one, or a few, monochromatic “carrier waves” propagate in a strongly dispersive and weakly nonlinear medium. More recently [5] it has proved to be also a simple way to obtain *necessary conditions* which a given PDE has to satisfy in order to be integrable, and to discover integrable PDEs as well [6].

The basic philosophy of this approach is to derive from a nonlinear PDE one or more PDEs whose integrability properties are either already known or easily found. In this respect, a general remark on this method is the following. Integrability is not a precise notion, and different degrees of integrability can be attributed to a PDE within a certain class of solutions and boundary conditions, according to the technique of solving it. For instance, C-integrable are termed those nonlinear equations which can be transformed into linear equations via a change of variables [6], and S-integrable are those equations which can be linearized (within a certain class of solutions) by the method of the spectral (or scattering) transform (see, f.i., [7]). Examples of C-integrability are the equations ($u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$ etc.)

$$u_t + a_1 u_x - a_3 u_{xxx} = a_3(3uu_x + u^3)_x, \qquad u = u(x,t), \quad (1.1a)$$

$$u_t + a_1 u_x - a_3 u_{xxx} = 3a_3 c(u^2 u_{xx} + 3uu_x^2) + 3a_3 c^2 u^4 u_x, \qquad u = u(x,t), \quad (1.2a)$$

which are both mapped to their linearized version (a_1, a_3, c are constant coefficients)

$$v_t + a_1 v_x - a_3 v_{xxx} = 0, \qquad v = v(x,t), \quad (1.3)$$

the first one, (1.1a), by the (Cole-Hopf) transformation

$$u = v_x/v \quad (1.1b)$$

and the second one, (1.2a), by the transformation [6]

$$u = v/(1+2cw)^{1/2}, \qquad w_x = v^2. \quad (1.2b)$$

Well-known examples of S-integrable equations are the modified Korteweg-de Vries (mKdV) equation

$$u_t + a_1 u_x - a_3 u_{xxx} = 6a_3 c u^2 u_x\,, \qquad u = u(x,t), \quad (1.4a)$$

and the nonlinear Schroedinger (NLS) equation (a_1, a_2, a_3, c are real constant coefficients)

$$u_t - i a_2 u_{xx} = 2 i a_2 c |u|^2 u\,, \qquad u = u(x,t), \quad (1.5a)$$

whose method of solution is based on the eigenvalue problem

$$\psi_x + ik\sigma\psi = Q\psi\,, \qquad \psi = \psi(x,k,t), \quad (1.6)$$

where ψ is a 2-dim vector, σ is the diagonal matrix diag(1, -1) and $Q(x,t)$ is the off-diagonal matrix

$$Q = \begin{pmatrix} 0 & u \\ -cu & 0 \end{pmatrix} \quad (1.4b)$$

where u is real for the mKdV equation (1.4a) and (the asterisk indicates complex conjugation)

$$Q = \begin{pmatrix} 0 & u \\ -cu^* & 0 \end{pmatrix} \tag{1.5b}$$

where u is complex for the NLS equation (1.5a). Here k is the spectral variable. In any case, whatever type of integrability is involved, we adopt in our treatment the "first principle" (*axiom*) that integrability is preserved by the multiscale method. Though in some specific cases, where integrability can be formulated as a precise mathematical property, one can give this principle a rigorous status, we prefer to mantain it throughout our treatment as a robust assumption. Its use, according to contexts, may lead to interesting consequences. One is that it provides a way to obtain other (possibly new) integrable equations. On the other hand, if a PDE, which has been obtained by this method from a given PDE, is proved to be nonintegrable, then from our first principle it there follows that that given PDE cannot be integrable, and this implication leads to *necessary* conditions of integrability. Some of these conditions are found simple and, therefore, of ready practical use. Others conditions are instead the results of lengthy algebraic manipulations which require a rather heavy computer assistance. Finally, this way of reasoning leads to the following observation, which has been pointed out in [6]. Suppose the same PDE is obtained by multiscale reduction from any member of a fairly large family of PDEs; so we can call it a "model PDE". Then the principle stated above explains why a model PDE may be at the same time widely applicable (because it derives from a large class of different PDEs) and integrable (because it suffices that just one member equation of that large family of PDEs be integrable). The most widely known example of such case is the NLS equation (1.5a) which is certainly a model equation (as shown below) with many applications (f.i. nonlinear optics and fluid dynamics [4]), and whose integrability has been discovered in 1971 [8] but it could have been found even earlier by multiscale reduction from the KdV equation $u_t + u_{xxx} = 6uu_x$ (the way to infer the S-integrability of the NLS equation from the S-integrability of the KdV equation has been first pointed out in [9]), whose integrability has been unveiled in 1967 [10].

The method of multiscale reduction which we now introduce is a perturbation technique based on three main ingredients : i) Fourier expansion in harmonics, ii) power expansion in a small parameter ϵ, iii) dependence on a (finite or infinite) number of "slow" space and time variables, which are first introduced via an ϵ-dependent rescaling of x and t and are then treated as independent variables. This last feature explains why this approach is also referred to as multiscale perturbation method or multiscale reduction.

In order to briefly illustrate how these basic ingradients naturally come into play in the simpler context of ordinary differential equations (ODEs), let us consider the well-known Poincaré-Lindstedt perturbation scheme to construct small amplitude oscillations of an anharmonic oscillator around a stable equilibrium position. Let our one-degree dynamical system be given by the nonlinear equation ($\dot{q} \equiv dq/dt$)

$$\ddot{q} + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \dots \ , \qquad q = q(t, \epsilon) \tag{1.7a}$$

where the small perturbative parameter ϵ is here introduced as the initial amplitude,

$$q(0, \epsilon) = \epsilon \ , \qquad \dot{q}(0, \epsilon) = 0 \ . \tag{1.7b}$$

The equation of motion (1.7a) is autonomous as all coefficients $\omega_0, c_2, c_3, \dots,$ are time-independent, and it has been written with its linear part in the lhs and its nonlinear (polynomial or, more generally, analytic) part in the rhs. In this elementary context, the model equation which is associated with this family of dynamical systems, is of course the harmonic oscillator equation, $\ddot{q} + \omega_0^2 q = 0$, which obtains when the amplitude ϵ is so small that all nonlinear terms can be neglected. In fact, the purpose of the Poincaré- Lindstedt approach is to capture the deviations from the harmonic motion which are due to the nonlinear terms in the rhs of (1.7a). Since, for sufficiently small ϵ, the motion is periodic, namely

$$q(t, \epsilon) = q(t + \frac{2\pi}{\omega(\epsilon)}, \epsilon) \ , \tag{1.8}$$

it is natural to change the time variable t into the phase variable θ,

$$\theta = \omega(\epsilon) t \ , \qquad q(t, \epsilon) = f(\theta, \epsilon) \ , \tag{1.9}$$

even if the frequency $\omega(\epsilon)$ is not known as it is expected to depend on the initial amplitude ϵ. Then the equations (1.7) now read ($f' \equiv df/d\theta$)

$$\omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \dots , \quad f(0, \epsilon) = \epsilon, \ f'(0, \epsilon) = 0, \tag{1.10}$$

and we look for approximate solutions via the power expansions

$$\omega^2(\epsilon) = \omega_0^2 + \gamma_1 \epsilon + \gamma_2 \epsilon^2 + \dots \quad , \tag{1.11}$$

$$f(\theta, \epsilon) = \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \dots \quad . \tag{1.12}$$

We note that the periodicity condition $f(\theta) = f(\theta + 2\pi)$ implies that $\omega(0) = \omega_0$; inserting the expansions (1.11) and (1.12) in the differential equation (1.10) and equating the lhs coefficients with the rhs coefficients of each power of ϵ, yields an infinite system of differential equations, the first one, at $O(\epsilon)$, is homogeneous, while all others, at $O(\epsilon^n)$ with $n > 1$, are nonhomogeneous, i.e.

$$O(\epsilon): \quad f_1^{''} + f_1 = 0, \quad f_1(0) = 1, \ f_1^{'}(0) = 0, \tag{1.13}$$

$$O(\epsilon^n): \; f_n^{''}+f_n = \{-n,-n+1,\ldots,-1,0,1,\ldots,n-1,n\}, f_n(0)=0, f_n'(0)=0. \tag{1.14}$$

The notation in this last equation refers to harmonic expansion with the following meaning. Since the functions $f_n(\theta)$ are periodic in the interval $(0,2\pi)$, one can Fourier-expand them; however, because of the differential equaion they satisfy, only a finite number of the Fourier exponentials $\exp(i\alpha\theta)$, α being an integer, enters in their representation. This is easily seen by recursion: $f_1(\theta)=\frac{1}{2}(\exp(i\theta)+\exp(-i\theta))$, and since $f_n(\theta)$, for $n>1$, satisfies the forced harmonic oscillator equation where the forcing term in the rhs of (1.14) is an appropriate polynomial of $f_1, f_2, \ldots, f_{n-1}$, its expansion can only contain the harmonics $\exp(i\alpha\theta)$ with $|\alpha|\le n$. Thus, the integers in the curly bracket in the rhs of (1.14) indicate the harmonics which enter in the Fourier expansion of the forcing term, and this implies that $f_n(\theta)$ itself has the Fourier expansion

$$f_n(\theta)=\sum_{\alpha=-n}^{n} f_n^{(\alpha)}\exp(i\alpha\theta), \quad n\ge 1, \tag{1.15}$$

where the complex numbers $f_n^{(\alpha)}$ have to be recursively computed. To this aim, it is required that also the coefficents γ_n in the expansion (1.11) be computed, and the way to do it is to use the periodicity condition $f_n(\theta)=f_n(\theta+2\pi)$, or, equivalently, the condition that the ϵ-expansion (1.12) be uniformly asymptotic (note that we do not address here the problem of convergence of the series (1.12) but we limit ourselves to establish uniform asymptoticity). The point is that, for each $n\ge 2$, the forcing term in (1.14) contains the fundamental harmonics $\exp(i\theta)$ and $\exp(-i\theta)$ which are solutions of the lhs equation (i.e. of the homogeneous equation), and are therefore secular, namely at resonance.

At this point, and for future use, we observe that, in a more general setting, if

$$v'(\theta)-Av(\theta)=w(\theta)+u(\theta) \tag{1.16}$$

is the equation of the motion of a vector $v(\theta)$ in a linear (finite or infinite dimensional) space and A is a linear operator, then, if the vector $w(\theta)$ solves the homogeneous equation,

$$w'(\theta)-Aw(\theta)=0, \tag{1.17}$$

then the forcing term $w(\theta)$ in (1.16) is secular. This is apparent from the θ-dependence of the general solution of (1.16), which reads

$$v(\theta)=\tilde{v}(\theta)+\theta w(\theta) \tag{1.18}$$

where $\tilde{v}(\theta)$ is the general solution of the equation $\tilde{v}'(\theta)-A\tilde{v}(\theta)=u(\theta)$.

In our present case, the occurence of the harmonics $\exp(i\theta)$ and $\exp(-i\theta)$ in the forcing term in the rhs of (1.14) forces the solution $f_n(\theta)$ to have a non-periodic dependence on θ, and therefore the condition that the coefficients of $\exp(i\theta)$ and $\exp(-i\theta)$ must vanish is a crucial ingredient of our computational scheme. In fact, this condition fixes the value of the coefficient γ_{n-1} and this completes the recurrent procedure of computing, at each order in ϵ, both the frequency

$$\omega(\epsilon) = \epsilon_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \ldots , \tag{1.19}$$

and the solution $f(\theta, \epsilon)$, see (1.12). As an instructive exercise, we suggest the reader to compute the frequency $\omega(\epsilon)$ up to $O(\epsilon^2)$ (answer: $\omega_1 = 0, \omega_2 = -(10c_2^2 + 9\omega_0^2 c_3)/24\omega_0^3$).

This approach has been often used in applications with the aim of computing approximate solutions; in that context the properties of the series (1.11) and (1.12) of being convergent, or asymptotic, and also uniformly so in t, is of crucial importance (see, f.i., [11] and the references quoted there), particularly when one is interested also in the large time behaviour. Our emphasis here is instead in the formal use of the double expansion (see (1.12) and (1.15))

$$q(t, \epsilon) = \sum_{n=1} \sum_{\alpha=-n}^{n} \epsilon^n \exp(i\alpha\theta) f_n^{(\alpha)} \tag{1.20}$$

where $\theta = \omega_0 t + \omega_1 \epsilon t + \omega_2 \epsilon^2 t + \ldots$ and therefore here and in the following we drop any question related to convergence and approximation.

Let us consider now the propagation of nonlinear waves, and let us apply the Poincaré-Lindstedt method to PDEs. For the sake of simplicity, here and also below throughout these notes, we focus our attention on the following family of equations which are first order in the variable time

$$Du = F[u, u_x, u_{xx}, \ldots] \quad , \quad u = u(x,t), \tag{1.21}$$

with the assumptions that this equation be real, that the linear differential operator D in the lhs have the expression

$$D = \partial/\partial t + i\omega(-i\partial/\partial x) \quad , \tag{1.22}$$

where $\omega(k)$ is a real odd analytic function,

$$\omega(k) = \sum_{m=0} a_{2m+1} k^{2m+1} \quad , \tag{1.23}$$

and that F in the rhs be a nonlinear real analytic function of u and its x-derivatives. For instance, the subfamily

$$\omega(k) = a_1 k + a_3 k^3 \quad , \; F = c u_x^3 + (c_2 u^2 + c_3 u^3 + \ldots)_x \quad , \tag{1.24}$$

contains three S-integrable equations, i.e. the KdV equation ($c = 0, c_n = 0$ for $n \geq 3$), the mKdV equation (1.4a) and the equation [12]

$$u_t + a_1 u_x - a_3 u_{xxx} = -a_3[\alpha \mathrm{sinh} u + \beta(\mathrm{cosh} u - 1) + u_x^2/8]u_x. \qquad (1.25)$$

Since the linearized version of the PDE (1.21), $Du = 0$, has the harmonic wave solution

$$u = \exp[i(k_0 x - \tilde{\omega}_0 t)] \ \ , \ \ \tilde{\omega}_0 = \omega(k_0) \ \ , \qquad (1.26)$$

one way to extend the Poincaré-Lindstedt approach to the PDE (1.21) is to look for solutions, if they exist, which are periodic plane waves,

$$u(x,t) = f(\theta, \epsilon), \ \ \theta = k(\epsilon)x - \tilde{\omega}(\epsilon)t, \ \ f(\theta, \epsilon) = f(\theta + 2\pi, \epsilon) \ \ , \qquad (1.26)$$

together with the power expansions

$$f(\theta, \epsilon) = \epsilon f_1(\theta)) + \epsilon^2 f_2(\theta) + \ldots,$$

$$k(\epsilon) = k_0 + k_1\epsilon + k_2\epsilon^2 + \ldots \ \ , \ \ \tilde{\omega}(\epsilon) = \tilde{\omega}_0 + \tilde{\omega}_1\epsilon^2 + \tilde{\omega}_2\epsilon^2 + \ldots \qquad (1.27)$$

This approach can be easily carried out as for the anharmonic oscillator since the function $f(\theta, \epsilon)$ does now satisfies the real ODE

$$-\tilde{\omega}(\epsilon) f^{(1)}(\theta, \epsilon) + i\omega(-ikd/d\theta) f(\theta, \epsilon) = F[f, kf^{(1)}, k^2 f^{(2)}, \ldots], k = k(\epsilon), \qquad (1.28)$$

where $f^{(j)} \equiv d^j f(\theta, \epsilon)/d\theta^j$. Periodic plane waves in fluid dynamics have been investigated along these lines and, though exact solutions are known for instance for water waves models (such as the KdV equation) in terms of Jacobian elliptic functions (cnoidal waves), approximate expressions have been found more than a century ago (Stokes approximation) [13].

The class of periodic plane-wave solutions (if they exists) is too restrictive to our purpose. In fact their construction requires going from the PDE (1.21) to the ODE (1.28), a step which implies loss of information about the PDE itself. Therefore we now turn our attention to the class of solutions of the wave equation (1.21) whose leading term in the perturbative expansion is a quasi-monochromatic wave, namely a wave-packet whose Fourier spectrum is not one point but is well localized in a small interval of the wave number axis, $(k - \Delta k, k + \Delta k)$, where k is a fixed real number and $\Delta k/k$ is small,

$$u(x,t) \simeq \Delta k \int_{-\infty}^{+\infty} d\eta A(\eta) \exp\{i[x(k + \eta\Delta k) - t\omega(k + \eta\Delta k)]\} + c.c.; \qquad (1.29)$$

here the amplitude $A(\eta)$ is sharply peaked at $\eta = 0$, and the additional complex conjugated term is required by the condition (which we mantain here and in the following) that $u(x,t)$ is real, $u = u^*$.

The perturbation formalism which is suited to deal with this class of solutions is still close to the Poincaré-Lindstedt approach to the anharmonic oscillator. In fact, let us go back to the two-index series (1.20) and substitute θ with the expansion $\theta = \omega_0 t + \omega_1 t_1 + \omega_2 t_2 + \ldots$, where we have formally introduced the rescaled "slow" times $t_n = \epsilon^n t$; then the formal expansion (1.20) reads

$$q(t,\epsilon) = \sum_{n=1} \sum_{\alpha=-n}^{n} \epsilon^n E^\alpha q_n^{(\alpha)}(t_1, t_2, \ldots) \quad , E \equiv \exp(i\omega_0 t) \quad , \tag{1.30}$$

where the functions $q_n^{(\alpha)}$ depend only on the slow-time variables t_n. The scheme of computation based on the expansion (1.30) is equivalent to that shown above, and it goes with inserting the expansion (1.30) into the equation (1.7a), and by treating the time variables t_n as independent variables. In particular the derivative operator d/dt takes the ϵ - expansion

$$d(E^\alpha q_n^{(\alpha)})/dt = E^\alpha (i\alpha\omega_0 + \epsilon\partial/\partial t_1 + \epsilon^2 \partial/\partial t_2 + \ldots) q_n^{(\alpha)}, \tag{1.31}$$

and similarly expanding the lhs and rhs of (1.7a) in powers of ϵ and of E finally yields a system of PDEs whose solution (after eliminating secular terms) gives the same result as the (much simpler) frequency-renormalization method based on (1.9) and (1.11). In this case the service of the multiscale technique is merely to display the three ingredients of the approach we use below for PDEs, i.e. the power expansion in a small parameter ϵ, the expansion in harmonics and the dependence on slow variables.

Let us now proceed with applying the multiscale perturbation approach to solutions of the PDE (1.21) along the line discussed above. As a preliminary observation, in the case the PDE (1.21) is linear, i.e. $F = 0$, the expression (1.29) is exact as it yields the Fourier representation of the solution. If we introduce the harmonic solution

$$E(x,t) \equiv \exp[i(kx - \omega t)], \;\; \omega = \omega(k), \tag{1.32}$$

the small parameter $\epsilon \equiv \Delta k/k$ and the slow variables $\xi \equiv \epsilon x, t_n \equiv \epsilon^n t$ for $n \geq 1$, the Fourier integral takes the expression of a "carrier wave" whose small amplitude is modulated by a slowly varying envelope (no higher harmonics are generated in the linear case)

$$u(x,t) = \epsilon E(x,t) u^{(1)}(\xi, t_1, t_2, \ldots) + c.c.. \tag{1.33}$$

Since the envelope function is (see (1.29))

$$u^{(1)}(\xi, t_1, t_2, \ldots) = k \int_{-\infty}^{+\infty} d\eta A(\eta) \exp[i(k\eta\xi - k\omega_1\eta t_1 - k^2\omega_2\eta^2 t_2 - \ldots)], \tag{1.34}$$

it satisfies the set of PDEs

$$\partial_{t_n} u^{(1)} = (-i)^{n+1}\omega_n \partial_\xi^n u^{(1)} \ , \ n = 1, 2, \ldots \tag{1.35}$$

In order to write down these equations, we have assumed that the dispersion function $\omega(k)$ is analytic at k, so that its Taylor series

$$\omega(k + \epsilon\eta k) = \sum_{n=0}^{\infty} \omega_n \eta^n k^n \epsilon^n, \ \ \omega_n(k) = \frac{1}{n!}\frac{d^n}{dk^n}\omega(k) \ \ , \tag{1.36}$$

is convergent. This shows that one has to ask that $u^{(1)}$ depends on as many rescaled times t_n as the number of nonvanishing coefficients ω_n in the expansion (1.36); f.i. if $\omega(k)$ is a polynomial of degree N, the multiscale method requires the introduction of at most N new independent time variables, this being a rule which holds also in the nonlinear case. More interestingly, we note that in the *linear* case, because of the hierarchy of compatible evolution equations (1.35) with respect to the slow times, the commutativity property $[\partial_{t_n}, \partial_{t_m}] = 0$ is trivially satisfied, whereas, in the *nonlinear* case this commutativity condition is of paramount importance and is strictly related to integrability in more than one way. Indeed, the purpose of section 3 is to show that the picture we have outlined in the linear case can be extended to the nonlinear case under appropriate conditions. The main consequence of nonlinearity is the generation of harmonics which are different from the fundamental one (1.32), together with the occurrence of undesired secular terms which force the amplitudes to grow with time. Killing the secular terms to keep the amplitudes bounded for all times is the basic way to derive a number of evolution equations. An old result in this direction, first derived in nonlinear optics and in fluid dynamics [4], is the dependence of the leading order amplitude $u_1^{(1)}(\xi, t_1, t_2)$ of the fundamental harmonic on the first two slow times t_1 and t_2, namely $u_1^{(1)}$ traslates with respect to t_1 with the group velocity ω_1 and evolves with respect to t_2 according to the NLS equation. Thus, at this order, the solution $u(x, t)$ of the PDE (1.21) is approximated by the expression

$$u(x, t) = \epsilon v(\xi - \omega_1 t_1, t_2)E(x, t) + c.c. + O(\epsilon^2) \ \ , \tag{1.37}$$

where

$$v_{t_2} = i\omega_2(v_{\xi\xi} - 2c|v|^2 v) \equiv K_2(v) \ \ . \tag{1.38}$$

In order to proceed further, the natural point to start from is the harmonic expansion of the solution $u(x, t)$,

$$u(x, t) = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)}(\xi, t_1, t_2, \ldots)E^\alpha(x, t) \ \ , \tag{1.39}$$

where $E(x,t)$ si defined by (1.32) and, since u is real, $u=u^*$, the coefficients $u^{(\alpha)}$ satisfy the reality condition

$$u^{(\alpha)*} = u^{(-\alpha)} \quad . \tag{1.40}$$

As for the slow variables, and guided by the approximate expression (1.29) where we set $\Delta k = \epsilon^p k$, with $p>0$, we define

$$\xi = \epsilon^p x,\ \ t_n = \epsilon^{np} t\ ,\ p>0,\ n=1,2,\dots . \tag{1.41}$$

As a consequence, the differential operators ∂_t and ∂_x, as acting on the expansion (1.39), are replaced by the power expansions

$$\partial_x \to \partial_x + \epsilon^p \partial_\xi\ ,\ \partial_t \to \partial_t + \epsilon^p \partial_{t_1} + \epsilon^{2p} \partial_{t_2} + \dots . \tag{1.42}$$

Inserting these expansions in the linear operator D, see (1.22), yields the formula

$$D[u^{(\alpha)}E^\alpha] = E^\alpha D^{(\alpha)} u^{(\alpha)}, \tag{1.43}$$

which defines the differential operator $D^{(\alpha)}$ acting *only* on the slow variables (1.41). Moreover, like the operators (1.42), also the differential operator $D^{(\alpha)}$ has a power expansion in ϵ,

$$D^{(\alpha)} = D_0^{(\alpha)} + \epsilon^p D_1^{(\alpha)} + \epsilon^{2p} D_2^{(\alpha)} + \dots , \tag{1.44}$$

the first term being just the multiplication by the constant

$$D_0^{(\alpha)} = i[\omega(\alpha k) - \alpha\omega(k)], \tag{1.45}$$

since $DE^\alpha = D_0^{(\alpha)} E^\alpha$.

Let us consider now the nonlinear part, namely the rhs of the PDE (1.21). Since F is supposed to be an analytic function, its decomposition in harmonics,

$$F[u, u_x, u_{xx}, \dots] = \sum_{\alpha=-\infty}^{+\infty} F^{(\alpha)}[u^{(\beta)}, u_\xi^{(\beta)}, u_{\xi\xi}^{(\beta)}, \dots] E^\alpha, \tag{1.46}$$

which is implied by the expansion (1.39), defines the functions $F^{(\alpha)}$ of the amplitudes $u^{(0)}, u^{(\pm 1)}, u^{(\pm 2)}, \dots$ and their derivatives with respect to ξ. For future reference, we note that the functions $F^{(\alpha)}$ have the gauge property of transformation

$$F^{(\alpha)} \to \exp(i\alpha\theta) F^{(\alpha)}$$

when the amplitude $u^{(\alpha)}$ in its arguments is replaced by $\exp(i\alpha\theta)u^{(\alpha)}$, where θ is an arbitrary constant.

Combining now the expansion (1.39), and the definition (1.43), with the expansion (1.46) shows that the PDE (1.21) is equivalent to the (infinite) set of equations

$$D^{(\alpha)}u^{(\alpha)} = F^{(\alpha)}, \tag{1.48}$$

which, since also $F^{(\alpha)}$ obviously satisfies the reality condition

$$F^{(\alpha)*} = F^{(-\alpha)}, \tag{1.49}$$

needs to be considered only for nonnegative α, i.e. for $\alpha \geq 0$.

In the following sections, the equations (1.48) will be investigated after expanding the amplitudes $u^{(\alpha)}$ in power of ϵ. In this respect, it should be pointed out that the approximate expression (1.29) of the solution $u(x,t)$ clearly shows that the smallness of u may originates in two ways, one from $\Delta k/k$ and the other from the amplitude A. In fact, we find it convenient to define ϵ by requiring that u itself be $O(\epsilon)$, and this explains why we have introduced the so far arbitrary parameter p in the rescaling (1.41) which define the slow variables.

In section 2, since we will look at the equations (1.48) at the lowest order in ϵ, only few harmonics will be considered. This analysis, when carried out in a systematic way, eventually yields a certain number of model PDEs in the slow variables, whose integrability properties, if known, lead to formulate necessary conditions of integrability for the original PDE (1.21).

In the third section we tackle instead the problem of pushing the investigation of (1.48) to higher orders in the ϵ- expansion. This analysis displays interesting connections with integrability and it gives a way to set up an entire hierarchy of necessary conditions of integrability.

We end this introduction with few remarks. First, for pedagogical reasons, we have constrained the family of PDEs considered here to satisfy appropriate conditions in order to simplify the formalism. These limitations are mainly technical and do not play an essential role. For instance, extensions of the family of PDEs (1.21) may include differential equations of higher order in t for complex vector, or matrix, solutions in higher spacial dimensions.

Second, we have confined our interest to the multiscale technique which yields model equations of nonlinear Schroedinger type. Similar arguments, however, do apply also to the weakly dispersive regime where the prototypical model equation is instead the KdV equation [14], or to the resonant, or nonresonant, interaction of N waves [6].

Finally, a different approach which similarly yields necessary conditions for integrability, and has common features with the one described in Section 3, has been introduced by Kodama and Mikhailov [28]. There the perturbation expansion is combined with the property of integrable systems of possessing symmetries, and the order-by-order construction of such symmetries is the core of the method. Other ways to relate integrability to perturbative expansions in a small parameter have been investigated within different mathematical settings. The interested reader may refer to Zakharov and Schulman [25] for the Hamiltonian formalism. Also the use of normal form theory has been designed to this purpose in various contexts, see f.i. [26], [27] and [28].

2 Nonlinear Schroedinger type model equations and integrability

In this section we investigate the basic equations (1.48) which have been obtained via the harmonic expansion (1.39) of a quasi - monochromatic solution of the PDE (1.21). Here we consider only the lowest significant order in the small parameter ϵ, but before illustrating our computational scheme, which is mainly based on Refs. [16], [17] that the interested reader should consult for details and generalizations, we point out first the main ideas and aims of our approach.

Consider first that, once the ϵ–expansion is introduced into the equation (1.48), the linear operator $D^{(\alpha)}$ takes the expression (1.44) whose coefficients, in addition to the first one (1.45), are easily found to be

$$D_n^{(\alpha)} = \partial_{t_n} - (-i)^{n+1}\omega_n(\alpha k)\partial_\xi^n \quad , \qquad n \geq 1 \quad , \tag{2.1}$$

where the function $\omega_n(k)$ is defined by (1.36). Then, at the lowest order in ϵ, the operator $D^{(\alpha)}$ in (1.48) should be replaced by the coefficient $D_0^{(\alpha)} = i[\omega(\alpha k) - \alpha\omega(k)]$; therefore, if $D_0^{(\alpha)}$ is not vanishing, the equation (1.48) for $u^{(\alpha)}$ becomes merely an algebraic equation whose solution is readily obtained. Because of this simple property, we term "slave harmonics" those harmonics such that, for their corresponding integer α, the quantity $D_0^{(\alpha)}$ does not vanishes, i.e.

$$\omega(\alpha k) - \alpha\omega(k) \neq 0. \tag{2.2}$$

If instead α is such that $D_0^{(\alpha)} = 0$, then we say that its corresponding harmonic is at resonance or, shortly, that α is a "resonance". The important feature of resonant harmonics is that their amplitude satisfies a differential equation in the slow variables (see (2.1)) rather than an algebraic equation as for slave harmonics. Of course, the harmonics $\alpha = 0, \pm 1$ are always (i.e. for any wavenumber k) at resonance (recall that $\omega(k)$ is on odd function, $\omega(-k) = -\omega(k)$). However it may well happen that $D_0^{(\alpha)} = 0$ for $|\alpha| \neq 0, 1$ for a particular value of k; in this case also their corresponding harmonics are *accidentally* (i.e. not for all values of k) at resonance and their amplitudes are expected to satisfy differential equations which may be coupled to the equations for the fundamental harmonics amplitude.

The repeated application of this argument to the next term of the expansion of $D^{(\alpha)}$ will be shown below to lead to the introduction of weak and strong resonances, and the systematic investigation of all resonant cases does finally produce a list of *ten* model PDEs of nonlinear Schroedinger type. These evolution equations are reported and discussed below in this secion, together with the implication of these findings with respect to integrability.

The starting ansatz is the ϵ-dependence at the leading order of the amplitude $u^{(\alpha)}$ in (1.39),

$$u^{(\alpha)} = \epsilon^{1+\gamma_\alpha} \psi_\alpha \quad \alpha = 0, \pm 1, \pm 2, \dots , \tag{2.3}$$

where the parameters γ_α are nonnegative, $\gamma_\alpha \geq 0$, and, of course, even, $\gamma_{-\alpha} = \gamma_\alpha$, with the condition

$$\gamma_1 = 0, \tag{2.4}$$

which fixes the small parameter ϵ.

Looking only at the lowest order in ϵ greatly simplifies our analysis in two ways: it restricts our attention only to the first harmonics $|\alpha| = 0, 1, 2$ and, secondly, it allows the amplitudes ψ_α, see (2.3), to be considered as functions only of the slow variables ξ, t_1 and t_2. Moreover, since ξ and t_1 are of the same order in ϵ (see (1.41)), it turns out convenient to replace the slow space coordinate ξ with the new coordinate

$$\xi = \epsilon^p (x - Vt) \tag{2.5}$$

in the frame moving with the group velocity,

$$V = d\omega(k)/dk = \omega_1(k), \tag{2.6}$$

of the fundamental harmonics ($|\alpha| = 1$), so that the amplitudes ψ_α depend throughout this section only on two variables,

$$\psi_\alpha = \psi_\alpha(\xi, \tau) \;\; , \;\; \tau \equiv t_2 = \epsilon^{2p} t. \tag{2.7}$$

As an additional remark, the following treatment suggests that it is convenient to take advantage of the fact that the nonlinear function in the rhs of the PDE (1.21) under investigation could be an x-derivative of a (polynomial or analytic) function, namely that it could be written as $\partial_x^h F(u, u_x, u_{xx}, \dots)$, where it is advisable to choose for the integer h its highest possible value. This is only a technical point as the final results can be also derived, though more painfully, by starting with a lower value of h or by setting tout court $h = 0$, as in (1.21). Thus we rewrite the PDE (1.21)

$$Du = (\partial/\partial x)^h F[u, u_x, u_{xx}, \dots], \tag{2.8}$$

where

$$F[u, u_x, u_{xx}, \dots] = \sum_{m=2}^{\infty} \sum_{j_1=0}^{\infty} \sum_{j_2=j_1}^{\infty} \dots \sum_{j_m=j_{m-1}}^{\infty} c^{(m)}_{j_1,\dots,j_m} u^{(j_1)} u^{(j_2)} \dots u^{(j_m)}, \tag{2.9}$$

with $u^{(j)} \equiv (\partial/\partial x)^j u(x,t)$. Thus the family of PDEs we consider below is fully characterized by the following parameters: the real coefficients a_{2m+1} which define the dispersion function $\omega(k)$, see (1.22) and (1.23), the integer h (see (2.8)) and the real coefficients $c^{(m)}_{j_1,\dots,j_m}$, see (2.9). The method described here provides necessary conditions which these parameters have to satisfy in order that the PDE (2.8) be integrable.

By taking into account the x-derivative in the rhs of (2.8) together with the ansatz (2.3), we first rewrite the equation (1.48) in the form

$$\epsilon^{1+\gamma_\alpha} D^{(\alpha)} \psi_\alpha = (i\alpha k + \epsilon^p \partial_\xi)^h F^{(\alpha)}. \tag{2.10}$$

We obtain thereby nontrivial evolution equations for the quantities $\psi_\alpha(\xi,\tau)$ by first taking the limit $\epsilon \to 0$ (after having made an appropriate choice for the exponents γ_α and p) and then by performing some algebraic calculations and also some "cosmetic rescalings" on the dependent and independent variables, so as to present the results in neater form.

Let us first treat the linear part, namely the left-hand-side of (2.8). Clearly we get

$$D^{(\alpha)} = \epsilon^{2p}\partial/\partial\tau + i\sum_{m=0}^{M} \epsilon^{pm} A_\alpha^{(m)}(k)(-i\partial/\partial\xi)^m \tag{2.11}$$

and

$$A_\alpha^{(0)}(k) = \omega(\alpha k) - \alpha\omega(k) \quad , \tag{2.12a}$$

$$A_\alpha^{(1)}(k) = \omega_1(\alpha k) - \omega_1(k) \quad , \tag{2.12b}$$

$$A_\alpha^{(s)}(k) = \frac{1}{s!}\frac{d^s}{dq^s}\omega(q)|_{q=\alpha k}, \quad s \geq 2. \tag{2.12c}$$

Here the coefficients $A_\alpha^{(s)}(k)$ with $s = 0, 1$ have been singled out because of the special role they play in the following. Note that by definition

$$A_1^{(0)} = A_1^{(1)} = 0; \tag{2.13}$$

this corresponds to the pivotal role of the component $\psi_1(\xi,\tau)$ which is the amplitude of the fundamental harmonic. It is indeed clear from (2.10) and (2.11) that the value of γ_α which is determined by the requirement to match the dominant terms as $\epsilon \to 0$ of the quantities in the right-hand-side of (2.10), tends to be smaller if $A_\alpha^{(0)}$ vanishes and even smaller if in addition also $A_\alpha^{(1)}$ vanishes and so on. Of course the smaller is the value of γ_α, the larger is the role that the component $\psi_\alpha(\xi,\tau)$ plays in the regime of weak nonlinearity (small ϵ). This qualitative notion is given quantitative substance below; but already at this stage it indicates that the different possibilities discussed below emerge from various different assumptions about the vanishing of some of the quantities $A_\alpha^{(s)}(k)$; a vanishing which might occur for all values of k, as it were for *structural reasons*, or it might happen only for some special value of k, on which attention may then be focussed.

For these reasons, in the following the harmonic α is called *weak resonance* if $A_\alpha^{(0)}(k)$, but not $A_\alpha^{(1)}(k)$, vanishes,

$$A_\alpha^{(0)}(k) = 0 \;\; , \;\; A_\alpha^{(1)}(k) \neq 0, \tag{2.14}$$

while we say that the harmonic α is a *strong resonance* if, in addition to $A_\alpha^{(0)}(k)$, also $A_\alpha^{(1)}\ (k)$ vanishes,

$$A_\alpha^{(0)}(k) = A_\alpha^{(1)}(k) = 0. \tag{2.15}$$

Of course, one could consider also the case of even stronger resonances by requiring that, in addition to (2.15), also the condition $A_\alpha^{(2)}(k) = 0$ be satisfied. However these cases are obviously less generic, and they will not be treated here.

Let us now consider the nonlinear rhs of (2.10). Inserting the ansatz (2.3) in the rhs of (2.9) yields the expression

$$F^{(\alpha)} = \sum_{m=2}^{\mu} \epsilon^{m-1} f_\alpha^{(m)} + O(\epsilon^\mu), \tag{2.16}$$

with

$$f_\alpha^{(m)} = \sum_{\{\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m ; \sum_{j=1}^m \alpha_j = \alpha\}} \epsilon^\Gamma \{ g(\alpha_1, \alpha_2, \ldots, \alpha_m) \psi_{\alpha_1} \ldots \psi_{\alpha_m} + O(\epsilon^p) \}; \tag{2.17}$$

here

$$\Gamma \equiv \gamma_{\alpha_1} + \gamma_{\alpha_2} + \ldots + \gamma_{\alpha_m} \quad , \tag{2.18}$$

and for the constants g we get

$$g(\alpha_1, \ldots, \alpha_m) = \sum_{\{0 \leq j_1 \leq \ldots \leq j_m\}} (ik)^J c_{j_1, \ldots, j_m}^{(m)} [\sum_{P(\alpha_1, \ldots, \alpha_m)} \Pi_{\rho=1}^m (\alpha_\rho)^{j_\rho}] \quad , \tag{2.19}$$

where $J = j_1 + j_2 + .. + j_m$, and the notation $\sum_{P(\alpha_1, \ldots, \alpha_m)}$ indicates the sum over all permutations of the indices $\alpha_1, \ldots, \alpha_m$ having different values.

Additional, drastic simplifications occur when further steps are taken towards implementing the $\epsilon \to 0$ limit; indeed in this context we shall generally need to consider only the quadratic and cubic terms of F in (2.8), because the contribution of all other terms turn out to be negligible. Hence (2.10) can now be written, in more explicit form, as follows:

$$\epsilon^{2p} [\psi_{1\tau} - i A_1^{(2)} \psi_{1\xi\xi}] = (ik)^h$$

.

$$\cdot [\varepsilon^{1+\gamma_0} g(0,1) \psi_0 \psi_1 + \varepsilon^{1+\gamma_2} g(-1,2) \psi_1^* \psi_2 + \varepsilon^2 g(1,1,-1) |\psi_1|^2 \psi_1] \;, \tag{2.20a}$$

$$\varepsilon^{\gamma_0+p}[A_0^{(1)}\psi_{0\xi}+\varepsilon^p\psi_{0\tau}]=(\partial/\partial\xi)^h.$$

$$\cdot\varepsilon^{hp}[\varepsilon^{1+2\gamma_0}g(0,0)\psi_0^2+\varepsilon g(-1,1)|\psi_1|^2+\varepsilon^{1+2\gamma_2}g(-2,2)|\psi_2|^2]\ , \qquad (2.20b)$$

$$\varepsilon^{\gamma_2}\{\mathrm{i}A_2^{(0)}\psi_2+\varepsilon^p A_2^{(1)}\psi_{2\xi}+\varepsilon^{2p}[\psi_{2\tau}-\mathrm{i}A_2^{(2)}\psi_{2\xi\xi}]\}=(2\mathrm{i}k)^h.$$

$$\cdot[\varepsilon g(1,1)\psi_1^2+\varepsilon^{1+\gamma_0+\gamma_2}g(0,2)\psi_0\psi_2]. \qquad (2.20c)$$

The coefficients g which appear in these PDEs are found, via the formula (2.19), to have the expressions

$$g(0,0)=c_{0,0}^{(2)}\ \ , \qquad (2.21a)$$

$$g(0,n)=2c_{0,0}^{(2)}+\sum_{j=1}^{\infty}(-1)^j(nk)^{2j}c_{0,2j}^{(2)}+\mathrm{i}\sum_{j=0}^{\infty}(-1)^j(nk)^{2j+1}c_{0,2j+1}^{(2)},\ n\neq 0\ , \qquad (2.21b)$$

$$g(n_1,n_2)=(1-\frac{1}{2}\delta_{n_1n_2})[\sum_{j=0}^{\infty}(-1)^jk^{2j}\sum_{j'=0}^{j}c_{j',2j-j'}^{(2)}(n_1^{j'}n_2^{2j-j'}+n_1^{2j-j'}n_2^{j'})$$

$$+\mathrm{i}\sum_{j=0}^{\infty}(-1)^jk^{2j+1}\sum_{j'=0}^{j}c_{j',2j+1-j'}^{(2)}(n_1^{j'}n_2^{2j+1-j'}+n_1^{2j+1-j'}n_2^{j'})],\ n_1\neq 0,\ n_2\neq 0, \qquad (2.21c)$$

The equations (2.20) contain terms of different order in the small parameter ε, and this requires some explaning.

In the first place, many other terms which might have been present have been omitted because they are of higher order in ε than terms which are present. This is for instance the case for *cubic* terms in the right-hand- side of (2.20a) involving $\psi_0,\ \psi_2,$ which are of higher order than *quadratic* terms which are present. Of course this argument, and analogous ones below, are applicable only if the relevant dominant terms are indeed present, namely provided they are not absent. Note that such an absence might happen for some "accidental" reason (possibly only for some special value of k) or for a "structural" reason, for instance if the original equation (2.8) contains nonlinear terms only of cubic order and higher, but no quadratic terms.

The second point that must be emphasized about (2.20) is that these equations generally contain contributions of different orders in ε, and only those of lowest order are relevant. The identification of these depends of course on the assignments of specific numerical values to p (of course $p > 0$) and to the paramenters γ_α (of course $\gamma_\alpha \geq 0, \alpha = 0, 1, 2$). These assignments are dictated by the structure of these equations (2.20), and by assumptions which have to be made about the vanishing or nonvanishing of the quantities $A_\alpha^{(m)}(k)$, $m = 0, 1, 2, \alpha = 0, 1, 2$, appearing in the left-hand-side of (2.20 b,c); hence one must consider many subcases, according to which resonance are present. Let us reemphasize that, in this treatment which yields the results reported here, the assumption is made that *all* nonlinear terms which might be present at the lowest order in ε are indeed present, namely that no nonlinear terms are missing due to "accidental" cancellations or "structural" causes. Whenever this hypothesis turns out not to hold, the analysis leading to the assignment of the exponents p and γ_α must be performed anew by taking into account higher order terms in ε. This analysis can be based on the equations (2.20) only if all the relevant higher order terms are already present in the r.h.s. of these equations, otherwise account of additional terms in the ε-expansion is necessary. Explicit instances of this phenomenon are reported in [16].

We finally display the model equations which obtain from (2.20) in the notation $\psi_0 = \theta, \psi_1 = \varphi, \psi_2 = \chi, \xi = x$ and $\tau = t$. There are 10 such equations:

$$\mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda|\varphi|^2\varphi \ ; \tag{2.22}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\theta\varphi \ , \\ \theta_x = \lambda^{(2)}|\varphi|^2 \ \ ; \end{cases} \tag{2.23}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\chi\varphi^* \ , \\ \chi_x = \lambda^{(2)}\varphi^2 \ \ ; \end{cases} \tag{2.24}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\theta\varphi + \lambda^{(2)}\chi\varphi^* \ , \\ \theta_x = \lambda^{(3)}|\varphi|^2 \ , \\ \chi_x = \lambda^{(4)}\varphi^2 \ ; \end{cases} \tag{2.25}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\theta\varphi \ , \\ \theta_t = \lambda^{(2)}\theta^2 + \lambda^{(3)}|\varphi|^2 \ \ ; \end{cases} \tag{2.26}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\theta\varphi \ , \\ \theta_t = \lambda^{(2)}(|\varphi|^2)_x \ \ ; \end{cases} \tag{2.27}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}|\varphi|^2\varphi + \lambda^{(2)}\theta\varphi \ , \\ \theta_t = \lambda^{(3)}(|\varphi|^2)_{xx} \ \ ; \end{cases} \tag{2.28}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu\varphi_{xx} = \lambda^{(1)}\theta\varphi + \lambda^{(2)}\chi\varphi^* \ , \\ \theta_t = \lambda^{(3)}(|\varphi|^2)_x \ , \\ \chi_x = \lambda^{(4)}\varphi^2 \ \ ; \end{cases} \tag{2.29}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu^{(1)}\varphi_{xx} = \lambda^{(1)}\chi\varphi^* \ , \\ \mathrm{i}\chi_t + \nu^{(2)}\chi_{xx} = \lambda^{(2)}\varphi^2 \ \ ; \end{cases} \tag{2.30}$$

$$\begin{cases} \mathrm{i}\varphi_t + \nu^{(1)}\varphi_{xx} = \lambda^{(1)}\theta\varphi + \lambda^{(2)}\chi\varphi^* \ , \\ \theta_t = \lambda^{(3)}\theta^2 + \lambda^{(4)}|\varphi|^2 + \lambda^{(5)}|\chi|^2 \ , \\ \mathrm{i}\chi_t + \nu^{(2)}\chi_{xx} = \lambda^{(6)}\theta\chi + \lambda^{(7)}\varphi^2 \ \ . \end{cases} \tag{2.31}$$

Let us emphasize that the coefficients ν and λ appearing in different equations are *different* quantities, even if they have the same symbol. Note moreover that the equations featuring in the left-hand-side the zeroth harmonic $\psi_0 = \theta$ are *real* , hence *all* coefficients (both ν and λ) appearing in them are real; while for the other equations the coefficients ν are *real*, the coefficients λ are generally *complex*. It should be also clear that the structure of these equations reflects the existence of structural and/or accidental resonances. In fact, since the fundamental harmonic $\alpha = 1$ is, by definition, strongly at resonance, its amplitude φ always satisfies a PDE which is firs-order in time and second-order in space; on the other hand, the zeroth harmonic is always weakly resonating and either it does not appear at all when $h \geq 1$ (because the first-order differential equation it satisfies can be explicity integrated) or, when $h = 0$, it couples to the other resonating harmonics through a first-order differential equation which can be either in x or in t depending on whether it is weakly or, respectively, strongly resonanting. Similarly for the amplitude χ of the second harmonic: if this harmonic is slave, it does not appear in the model equation, otherwise it satisfies a coupled differential equation which is first-order in x if it is only weakly resonanting, and is first-order in t and second order in x if it is also strongly at resonance.

The derivation by reduction of these ten nonlinear Schroedinger type model equations is the starting point to make contact with integrability. Indeed, from the knowledge that a model equation is not integrable we deduce that that particular original PDE in the class (2.8), from which the model equation follows by reduction, cannot be integrable. To the aim of illustrating the way to convert this general statement in concrete results we select out of the ten equations (2.22-31) the following four PDEs, whose integrability properties are already known (for more details and examples, see [17]).

Equation (2.22): this is the NLS equation which obains if $A_0^{(1)}(k) \neq 0, A_1^{(2)}(k) \neq 0\, , A_2^{(0)}(k) \neq 0$ and $h \geq 1$, with $\nu = A_1^{(2)}(k)$ and, if $h = 1$,

$$\begin{aligned} \lambda = -k[A_0^{(2)}(k)g(0,1)g(-1,1) + 2kA_0^{(1)}(k)g(-1,2)g(1,1) \\ +A_0^{(1)}(k)A_0^{(2)}(k)g(-1,1,1)]/A_0^{(1)}(k)A_0^{(2)}(k); \end{aligned} \tag{2.32}$$

this equation is known to be S-integrable if

$$\mathrm{Im}(\lambda) = 0. \tag{2.33}$$

Equation (2.23): it corresponds to $h = 0$, and $A_{(0)}^{(1)}(k) \neq 0$ and $A_1^{(2)}(k) \neq 0$; in this case $\nu = A_1^{(2)}(k)$, and

$$\lambda^{(1)} = g(0,1), \;\; \lambda^{(2)} = g(-1,1)/A_0^{(1)}(k); \tag{2.34}$$

this system of equations has been found [18] to pass the Painlevé type test only if

$$\lambda^{(1)}\lambda^{(2)} = 0, \tag{2.35}$$

namely, if it effectively linearizes.
Equation (2.24): this obtains if $h \geq 1$ and if, for some real nonvanishing value $k = \tilde{k}$, $A_2^{(0)}(\tilde{k}) = 0$, $A_1^{(2)}(\tilde{k}) \neq 0$ and $A_2^{(1)}(\tilde{k}) \neq 0$. In this case $\nu = A_1^{(2)}(\tilde{k})$ and, if $h = 1$,

$$\lambda^{(1)} = -\tilde{k}g(-1,2), \;\; \lambda^{(2)} = 2i\tilde{k}g(1,1)/A_2^{(1)}(\tilde{k}), \tag{2.36}$$

where, of course, the coefficients $g(-1,2)$ and $g(1,1)$ are valued here at $k = \tilde{k}$. Also this equation has been found [19] to pass the Painlevé-type test only if (2.35) holds.
Equation (2.27): this is the case if $h = 1$, and if, for some real nonvanishing value $k = \tilde{k}$, $A_0^{(1)}(\tilde{k}) = 0$ and $A_1^{(2)}(\tilde{k}) \neq 0$. Then $\nu = A_1^{(2)}(\tilde{k})$ and

$$\lambda^{(1)} = i\tilde{k}g(0,1), \;\; \lambda^{(2)} = g(-1,1), \tag{2.37}$$

where $g(0,1)$ and $g(-1,1)$ are evaluated at $k = \tilde{k}$. This system has been proved to be S-integrable [20] only if

$$\mathrm{Im}\lambda^{(1)} = \mathrm{Im}\lambda^{(2)} = 0. \tag{2.38}$$

With this information in our hands we are now in the position to formulate necessary conditions of integrability. For a systematic exploration of the various cases in which such conditions arise and apply, the reader is refereed to [17], while we limit ourselves to give here only few instances of our method, and of its potentialities.

We first observe that the integrability conditions for the four equations we have selected, i.e. (2.22), (2.23), (2.24) and (2.27), involve both the linear part (through the coefficients $A_\alpha^{(n)}$, see (2.12)) and the nonlinear part (through the coefficients g, see (2.21) and (2.9)) of the PDE (2.8) we wish to test, and that both the coefficients $A_\alpha^{(n)}$ and g are functions of the real parameter k. It is then clear that the integrability conditions (such as (2.33) and (2.35)) which hold for an arbitrary value of k produce a number of necessary conditions for the PDE (2.8) which is larger than the number of necessary conditions which originates from expressions such as (2.36) and (2.37) since these hold only for special values (if any) of k (say $\tilde{k}$).

Let us first assume that the PDE (2.8) we are going to test by our method is in the class with $h = 0$, namely its nonlinear term is not a derivative. Then, if the appropriate reduced equation is (2.23), the requirement that $g(0,1)$ or $g(-1,1)$ vanish for *all* real values of k entails, via (2.21b) and (2.21c), quite explicit restrictions only on the nonlinear part of (2.8). This is made explicit by the following:

Lemma 1. A necessary condition for the integrability of a nonlinear evolution PDE of type (2.8) with $h = 0$ is that either

$$c_{0n}^{(2)} = 0 \ , \ n = 0, 1, 2, \ldots , \tag{2.39}$$

or

$$\sum_{j=0}^{n} (-1)^j c_{j2n-j}^{(2)} = 0, \ \ n = 0, 1, 2, \ldots , \tag{2.40a}$$

namely

$$c_{00}^{(2)} = 0, \ \ c_{02}^{(2)} - c_{11}^{(2)} = 0, \ \ c_{04}^{(2)} - c_{13}^{(2)} + c_{22}^{(2)} = 0, \tag{2.40b}$$

and so on. Clearly the condition (2.39) comes from the requirement that $g(0, 1)$ vanish, while (2.4) comes from the requirement that $g(-1, 1)$ vanish, see (2.35) and (2.34). Since they both require that $c_{00}^{(2)}$ vanish we obtain the following remarkably neat result:

Lemma 2. Every nonlinear PDE of type (2.8) with $h = 0$ featuring in its nonlinear part a term $c_{00}^{(2)} u^2$ is *not integrable.*

Consider now the class of PDEs (2.8) with $h = 1$, and assume that the appropriate reduced model equation is the NLS equation (2.22). The requirement (2.33) with (2.32) for S-integrability involves both quantities related to the linear and nonlinear parts of the original equation (2.8), but in many cases it amounts to the requirements that (i) the quantity $g(0, 1)$ be real (note that $g(-1, 1)$ is always real, see (2.21c)); (ii) the quantities $g(-1, 2)$ and $g(1, 1)$ be both real or both imaginary; (iii) the quantity $g(-1, 1, 1)$ be real. Given the arbitrariness of k, the first of these three conditions clearly entails the vanishing of all the coefficients $c_{0n}^{(2)}$ with n odd; the second condition entails the vanishing of $c_{12}^{(2)}, c_{14}^{(2)}$ and $c_{23}^{(2)}$ and many other relations for the coefficients $c_{nm}^{(2)}$ with $n + m$ odd; the third condition entails the vanishing of $c_{001}^{(3)}$ and many other relations for the coefficients $c_{nmj}^{(3)}$ with $n + m + j$ odd. These are very stringent, and quite explicit, conditions on the nonlinear part of (2.8) (the case in which $h > 1$ can be similarly treated [17]).

Assume now that the original PDE (2.8), with $h = 1$, has passed the test based on the conditions specified above, namely that all conditions entailed by the requirement (2.33), with (2.32), are satisfied. Since these conditions are only necessary, no much information is gained, a part from a definite hint that our PDE may indeed turn out to be integrable. However, we can still push our method to look for additional conditions to be satisfied. This is in fact the case if a special value of $k, k = \tilde{k}$, exists such that either the condition $A_2^{(0)}(\tilde{k}) = 0$ holds, this being appropriate to obtain the model equation (2.24), or the condition $A_0^{(1)}(\tilde{k}) = 0$ holds, this being the case for the model equation

(2.27). In the first case, a necessary condition for the integrability of a PDE of type (2.8) with $h = 1$ is that, for such special value fo $k, k = \tilde{k}$, at least one of the two quantities $g(-1,2), g(1,1)$ vanish, see (2.35) with (2.36). The applicability and potency of this result is of course somewhat reduced relative to the conditions previously found, due to the requirement to restrict consideration to only those real values $\tilde{k}$ of k (if any) which satisfy the appropriate equality and inequalities specified above. Yet there clearly is a large class of nonlinear evolution PDEs to which these necessary conditions are applicable [17].

In the second case, namely that in which the model equation is (2.27), a necessary condition for the integrability of a PDE (2.8) with $h = 1$ is that, for the appropriate special value of k, i.e. $k = \tilde{k}$ such that the zeroth harmonic is strongly resonating, the quantity $g(0,1)$ be imaginary (or vanish),

$$\mathrm{Re}[g(0,1)] = 0 \ , \ \ k = \tilde{k}. \tag{2.41}$$

This requirement follows from (2.38), (2.37) and from the property of $g(-1,1)$ to be always real. This result is analogous to the previous one inasmuch as it requires focussing on special values $\tilde{k}$ of k.

Let us state again that we have presented here only some of the necessary conditions which can be established by the multiscale reduction method and that more instances and applications are discussed in [17] where a distinction between necessary conditions for C-integrability and for S-integrability is also made. We also observe that various extensions are possible and worth of further research; for examples, different classes of PDEs, other than (2.8) can be investigated, say for vector or matrix solutions as well as with more spacial variables; and/or different model equations, other than the four equations considered here, can be taken as starting points for the derivation of other necessary conditions for integrability.

3 Higher order terms and integrability

In this section our perturbative analysis of the original PDE (1.21) is extended to terms of higher order in ϵ. This extension is based on the expansion in powers of ϵ of the amplitude $u^{(\alpha)}$ in the equation (1.48), with the implication that computations become rather heavy. To the aim of simplifying the formalism by avoiding unessential complications, we add two assumptions which we mantain throughout this section. First we ask that the nonlinear part of our equation (1.21), namely its rhs F, be an odd function of u,

$$F \rightarrow -F \ \text{ if } \ u \rightarrow -u. \tag{3.1}$$

As it is easily verified, this parity property allows us to consistenly assume that the amplitudes of all even harmonics be vanishing,

$$u^{(2\alpha)} = 0 \ , \ \ |\alpha| \geq 0. \tag{3.2}$$

Therefore, from now on, we will have to deal only with the amplitudes $u^{(2\alpha+1)}$ of the odd harmonics . For instance, this condition on F is satisfied by the mKdV equation (1.4a), the C-integrable equation (1.2a) and by the class of PDEs (1.21) with (1.24) if $c_{2n} = 0$.

Our second assumption is that, in contrast with the analysis carried out in the previous section, no resonance occurs besides the fundamental harmonics $\alpha = \pm 1$. In other words, the resonance condition $D_0^{(\alpha)} = 0$, see (1.45), should hold only in the trivial case $|\alpha| = 1$.

These assumptions imply that all harmonics $\pm(2\alpha + 1)$ with $\alpha > 0$ are slave and that the coefficients $u^{(\alpha)}(n)$ of their ϵ-expansion,

$$u^{(\alpha)} = \sum_{n=1} \epsilon^n u^{(\alpha)}(n), \ \ |\alpha| > \ 1, \tag{3.3}$$

are therefore expressed as differential polynomials of the coefficients $u(n)$ of the expansion of the fundamental harmonic ($\alpha = 1$)

$$u^{(1)} \equiv u = \epsilon u(1) + \epsilon^2 u(2) + \ldots = \sum_{n=1} \epsilon^n u(n). \tag{3.4}$$

Here, and also in the following, we drop the harmonic upper index in the coefficients of this expansion because of the very special role played by the function $u^{(1)}$ in this scheme (it is the only amplitude which satisfies a differential equation). Moreover, as additional implication which can be easily retrieved from the basic equation (1.48), the leading order of each harmonic amplitude comes from the rule

$$u^{(\alpha)}(n) = 0 \ , \ \ \text{for} \ \ n < |\alpha|, \tag{3.5}$$

which is equivalent to setting $\gamma_{2\alpha+1} = 2\alpha$ for $\alpha \geq 0$ in the notation (2.3); the slow variables ξ and t_n are here defined as in (1.41) with $p = 1$, i.e.

$$\xi = \epsilon x, \ \ t_n = \epsilon^n t \ \ , n = 1, 2, \ldots \tag{3.6}$$

In order to perform all operations required by our approach the functions $u(n), n = 1, 2, \ldots,$ are required to be smooth in the real variable ξ, namely they are differentiable to any order in the whole ξ-axis.

The first step is inserting in the equation (1.48) with $\alpha = 1$ the appropriate ϵ-expansions, namely that of the linear opertor $D^{(1)} \equiv D$, see (1.44) with $\alpha = 1$ and $p = 1$,

$$D = \epsilon D_1 + \epsilon^2 D_2 + \ldots \ \ , \tag{3.7}$$

that of the amplitude $u^{(1)} \equiv u$, see (3.4), and finally the expansion of the nonlinear term,

$$F^{(1)} \equiv F = \epsilon^3 F_3 + \epsilon^4 F_4 + \ldots \quad ; \tag{3.8}$$

let us reemphasize here that, since the differential operators D_n, see (2.1) with $\alpha = 1$, have the expression

$$D_n = \partial_{t_n} - (-i)^{n+1}\omega_n(k)\partial_\xi^n, \ n \geq 1, \tag{3.9}$$

there is no need to introduce the slow time t_n if it happens that $\omega_n(k) = 0$. Thus, if the dispersion relation $\omega(k)$ is a polynomial of degree $N > 1$, the expansion (3.7) turns out to be a polynomial in ϵ of degree N with the implication that only N slow times enter into play. We also note that, because of the parity condition (3.1), the expansion (3.8) of the nonlinear term starts from the third order. In conclusion, the basic equation (1.48) with $\alpha = 1$, i.e. $D^{(1)}u^{(1)} = F^{(1)}$ or, in the present notation

$$Du = F, \tag{3.10}$$

obviously yields the triangular system of convolution type

$$D_1 u(n) + D_2 u(n-1) + \ldots + D_n u(1) = F_{n+1} \ . \tag{3.11}$$

Here, and in the following treatment, it is convenient to consider F_n as an element of the finite-dimensional vector space $\mathcal{P}_n$ defined as the set of all nonlinear differential polynomials in the functions $u(m)$ and $u^*(m)$ of order n and gauge 1. The meaning of this terminology is rather obvious: each monomial appearing in an element of $\mathcal{P}_n$ is a product of some $u(m), u^*(k)$ and their ξ-derivatives with the understanding that

$$\mathrm{order}(u_j(m)) = \mathrm{order}(u_j^*(m)) = m + j, \tag{3.12}$$

where we use the short-hand notation

$$u_j(m) \equiv \partial_\xi^j u(m). \tag{3.13}$$

On the other hand, by requiring that each polynomial in $\mathcal{P}_n$ be of gauge 1 we understand that such polynomials, say F_n, possess the transformation property

$$F_n \to e^{i\theta} F_n \quad \text{if} \quad u(m) \to e^{i\theta} u(m), \tag{3.14}$$

θ being an arbitrary real constant. By following these rules, the reader may easily verify that $\mathcal{P}_2$ is empty, dim $(\mathcal{P}_3) = 1$, the basis of $\mathcal{P}_1$ being the single monomial $|u(1)|^2 u(1)$, while dim $(\mathcal{P}_4) = 4$ where its basis may be given by the following four monomials: $|u(1)|^2 u(2), u(1)^2 u^*(2), |u(1)|^2 u_1(1), u(1)^2 u_1^*(1)$.

Therefore, each nonlinear term F_{n+1} in the rhs of (3.11) is a linear combination of the basis vectors (f.i. monomials) of the vector space $\mathcal{P}_{n+1}$, where the complex coefficients of such combination are determined by the nonlinear function in the rhs of our original PDE (1.21) (see the expansion (2.9) with m running only on the odd integers).

The next step aims to eliminating all secular terms which may enter in the system (3.11). Our analysis is briefly described below, and the reader who is interested in a detailed investigation of this point is referred to [21].

Consider first the equation (3.11) for $n = 1$, i.e. $D_1 u(1) = 0$ since $F_2 = 0$ (see (3.8)); because of the expression (3.9), $D_1 = \partial_{t_1} + \omega_1 \partial_\xi$, the function $u(1)$ depends on t_1 through the variable $\xi - \omega_1 t_1$. The next equation, say (3.11) with $n = 2$, reads (see (3.9))

$$D_1 u(2) = -[(\partial_{t_2} - i\omega_2 \partial_\xi^2) u(1) - F_3], \tag{3.15}$$

where its rhs plays the role of the nonhomogeneous (forcing) term with respect to the t_1-evolution. On the other hand, this term depends on t_1 through the variable $\xi - \omega_1 t_1$ (recall that $F_3 \epsilon \mathcal{P}_3$) and it satisfies therefore the homogeneous equation $D_1 f = 0$. This implies that the rhs of (3.15) is secular and its elimination requires that $u(1)$ satisfies, with respect to t_2, the evolution equation $(\partial_{t_2} - i\omega_2 \partial_\xi^2) u(1) = F_3$, namely just the NLS equation, which has been derived in the previous section. As a result of killing the secular term in (3.15), also $u(2)$ as $u(1)$ depends on t_1 through the variable $\xi - \omega_1 t_1$. This argument can be easily repeated for each integer n in (3.11) and,together with taking into account the structure of the differential polynomial F_{n+1}, it recursively leads to conclude that the coefficients $u(n)$ all satisfy with respect to the time t_1 the same (trivial) equation

$$D_1 u(n) = (\partial_{t_1} + \omega_1 \partial_\xi) u(n) = 0, \ n \geq 1. \tag{3.16}$$

The time t_1 plays no essential role and the system (3.11) reduces to

$$D_2 u(n-1) + D_3 u(n-2) + \ldots + D_n u(1) = F_{n+1}, \ n \geq 2 \ , \tag{3.17}$$

whose first equation (i.e. for $n = 2$) is the NLS equation

$$\partial_{t_2} u(1) = i\omega_2 (\partial_\xi^2 u(1) - 2c|u(1)|^2 u(1)) \equiv K_2[u(1)]; \tag{3.18}$$

the rhs of this equation defines the nonlinear operator K_2 and we have set $F_3 = -2i\omega_2 c|u(1)|^2 u(1)$.

Next we consider the equation (3.17) for $n = 3$, and we look at the evolution with respect to the time t_2. To this aim it is convenient to introduce the linear opeator

$$M_2 = \partial_{t_2} - K_2'[u(1)], \tag{3.19}$$

where $K_2'[u(1)]$ is the Frechet derivative of $K_2[u(1)]$, see (3.18), that is

$$\frac{d}{ds} K_2[u(1) + sv]|_{s=0} = K_2'[u(1)]v, \tag{3.20}$$

namely

$$M_2 v = v_{t_2} - i\omega_2 (v_{\xi\xi} - 4c|u(1)|^2 v - 2cu^2(1)v^*); \tag{3.21}$$

in fact, with this notation, the $n = 3$ equation (3.17) reads

$$M_2 u(2) + D_3 u(1) = \tilde{F}_4, \tag{3.22}$$

where $\tilde{F}_4 = F_4 + 2i\omega_2 c(2|u(1)|^2 u(2) + u^2(1)u^*(2))\epsilon\mathcal{P}_4$. Again one has to face the problem of secularities for this equation. First we observe that the term $\partial_{t_3} u(1)$ in $D_3 u(1)$ is secular since, obviously, $M_2(\partial_{t_3} u(1)) = 0$ as $M_2\sigma = 0$ is satisfied by any symmetry σ of the NLS equation (3.18). Second, we note that also the other term $\partial_\xi^3 u(1)$ in $D_3 u(1)$ is secular in the following sense. The requirement that the ϵ-expansion (3.4) of u is uniformly asymptotic in time implies that the coefficients $u(n)$ remain bounded as $t \to \infty$. In particular one should ask that the forcing term $\tilde{F}_4 - D_3 u(1)$ in (3.22) vanishes, as $t_2 \to \infty$, faster than $t_2^{-1/2}$ while the variable ξ/t_2 is kept fixed [21]. This restriction is equivalent to asking that $D_3 u(1)\varepsilon\mathcal{P}_4$, while, at the same time, the t_3-flow for $u(1)$ should also be compatible with the t_2-flow given by the NLS equation. The existence of such evolution of $u(1)$ with respect to t_3 is a fine consequence of the integrability of the NLS equation, provided the parameter c in (3.18) is real, $c = c^*$. Indeed, it is well-known that a whole hierarchy of flows,

$$\partial_{t_n} u(1) = K_n[u(1)], \;\; n = 1, 2, \ldots \tag{3.23}$$

exist which are all compatible (i.e. commuting) with each other; in the present context, these evolution equations may be conveniently rewritten as

$$D_n u(1) = (-i)^{n+1} \omega_n c V_{n+1}, \;\; n = 1, 2, \ldots \tag{3.24}$$

where V_n is a special elements of $\mathcal{P}_n$ which depends only on $u(1), u(1)^*$ and their ξ-derivatives. The expression of the first few of these polynomials are

$$\begin{gathered} V_2 = 0, \;\; V_3 \;\; = -2q_0 u(1) \;,\;\; V_4 = -6q_0 u_1(1), \\ V_5 = 2\,(3q_1 + 3cq_0^2 - q_{0\xi\xi})u(1) - 6(q_0 u_1(1))_\xi \;, \\ V_6 = 10\,(q_1 + 3cq_0^2 - q_{0\xi\xi})u_1(1) - 6(q_0 u_2(1))_\xi \;, \end{gathered} \tag{3.25}$$

where we use the notation (3.13) together with the definition

$$q_n = |u_n(1)|^2 \quad , \quad n = 0, 1, 2, \ldots \quad . \tag{3.26}$$

Thus, the requirement that the solution $u(2)$ of (3.22) remains bounded as $t_2 \to \infty$ is that $D_3 u(1) = \omega_3 c V_3$ or, equivalently (see (3.23)) that $u(1)$ satisfies the complex mKdV equation

$$\partial_{t_3} u(1) = K_3[u(1)] \quad , \tag{3.27}$$

with the implication that the equation (3.22) reread

$$M_2 u(2) = G_4 \quad , \tag{3.28}$$

with $G_4 = \tilde{F}_4 - \omega_3 c V_4 = F_4 + 2\mathrm{i}\omega_2 c(2|u(1)|^2 u(2) + u^2(1)u^*(2)) + 6\omega_3 c|u(1)|^2 u_1(1)\varepsilon\mathcal{P}_4$. The way to arrive at the equations (3.27) and (3.28) from (3.22) we have sketched here can be repeated for the equations (3.17) for all n, through a careful analysis of the asymptotic behaviour of the functions $u(n)$ as $t_2 \to \infty$ [21]. The upshot of this analysis is that the system of PDEs (3.17) splits into the NLS hierarchy (3.23) for the first coefficient $u(1)$ and the secularity-free system

$$M_2 u(n) + M_3 u(n-1) + \ldots + M_n u(2) = G_{n+2}\,, n = 2, 3, \ldots , \tag{3.29}$$

where G_n is an elemen of the vector space $\mathcal{P}_n$, and M_n is the linear operator

$$M_n = \partial_{t_n} - K_n'[u(1)], \tag{3.30}$$

where, again, $K_n'[u(1)]$ is the Frechet derivative of the nonlinear operator $K_n[u(1)]$ in the rhs of (3.23).

Let us point out here that the derivation of the triangular system of nonlinear PDEs (3.29) requires only that the lowest order nonlinear model equation (in this case the NLS equation) is integrable (i.e., in this case, the condition is that c be real, see (3.18)) so as to guarantee the existence of an infinite hierarchy of independent mutually commuting symmetries (such as (3.23)). However, if no further information on the original PDE (1.21) is at hand, one is left with the (hard) task of integrating the PDEs of the triangular system (3.29). Thus, at this point, the natural question to ask is whether the special property of the original PDE (1.21) of being (C- or S-) integrable reflects itself in a special property of the triangular system (3.29). Here below we briefly show that, indeed, the answer to this question leads to formulate a hierarchy of necessary conditions of integrability which lead to test a given PDE (see also [22]).

First we observe that in the obviously integrable case in which the PDE (1.21) is linear, say $F = 0$, the operator M_n (3.30) reduces to D_n, see (3.9) and the system (3.29) with $G_n = 0$ seperates into the hierarchy $D_n u(m) = 0, n = 1, 2, \ldots$, i.e. the same hierarchy for each function $u(m)$. In this case the consistency condition $[D_n, D_m] = 0$ is certainly plain but essential. The basic observation [22] now is that, if the original PDE (1.21) is C-integrable or S-integrable, then, similarly to the first coefficient $u(1)$ which satisfies the hierarchy of PDEs (3.23), each function $u(m)$, for $m \geq 2$, satisfies the hierarchy of PDEs

$$M_n u(m) = f_n(m) \quad , \quad n \geq 2, \; m \geq 2, \tag{3.31}$$

where the nonhomogeneous nonlinear term $f_n(m)$ in the rhs is a differential polynomial in $\mathcal{P}_{n+m}$. More precisely, one can show that

$$f_n(m) \in \mathcal{P}_{n+m}(m-1), \tag{3.32}$$

where $\mathcal{P}_n(j)$ is defined as the subspace of $\mathcal{P}_n$ whose elements are the differential polynomials of the functions $u(m)$ and $u^*(m)$ where the index m goes only up to j, say $1 \leq m \leq j$. Of course, since the functions $u(m)$ are also solutions of the system (3.29), the rhs terms of the hierarchy (3.31) has to be related to the rhs of (3.29) by the triangular condition

$$f_2(n) + f_3(n-1) + \ldots + f_n(2) = G_{n+2},\ n \geq 2. \tag{3.33}$$

In order for the system of PDEs (3.29) to split into separate PDEs, namely the equations (3.31), certain compatibility conditions must be met. In fact, since the linear operators M_n given by (3.30) commute with each other,

$$[M_n, M_m] = 0\ ,\ n \geq 1,\ m \geq 1, \tag{3.34}$$

as a straight consequence of the commutativity of the flows of the NLS hierarchy (3.23), then the hierarchy (3.31), for each $m \geq 2$, must satisfy the compatibility condition

$$M_j f_n(m) = M_n f_j(m). \tag{3.35}$$

Eliminating the time-derivatives by using the evolution equations (3.31) leads to rewrite the compatibility equation (3.35) as an algebraic condition which the differential polynomials $f_n(m)$ have to satisfy. In fact, this condition ultimately reads as a set of constraints on the components of $f_n(m)$ on the basis of the vector space $\mathcal{P}_{n+m}$.

The way to prove this interesting property of integrable PDEs is not reported here; it goes via the change of variable which linearizes the PDE (1.21) in the case of C-integrability (see, f.i., the transformation (1.2b)), or it makes use of the multiscale expansion of the spectral equation of the Lax-pair in the case of S-integrability (see, f.i., the ODE (1.6) with (1.4b)). We note here that this result opens the way to estabilish an integrability-test as it yields necessary conditions that the PDE (1.21) has to satisfy in order to be integrable. Indeed, if one can prove that no differential polynomials $f_n(m)$ exist such that (3.31) holds together with the relation (3.33), where G_n is given by the multiscale technique, see equations (3.29), then the original PDE (1.21) cannot be integrable.

The following two propositions are instrumental in setting up our test. *Proposition* 1: the homogeneous equation $M_n f = 0$ has no solution f in the vector space $\mathcal{P}_m$, namely

$$\mathrm{Ker}(M_n) \cap \mathcal{P}_m = \phi. \tag{3.36}$$

Proposition 2: if, for each $n \geq 2$, the equation

$$M_2 f_3(n) = M_3 f_2(n) \tag{3.37}$$

is satisfied with $f_2(n)$ and $f_3(n)$ given in the appropriate space, see (3.32), then differential polynomials $f_m(n)$, with $m \geq 4$ and (3.32), exist unique such that the flows $M_m u(n) = f_m(n)$ commute with each other for $m \geq 2$. Our method is then better illustrated by first showing the nonhomogeneous terms of the hierarchies (3.31) in the following Table (note that in $f_n(m)$ the index n labels the n-th member of the hierarchy of evolution equations, namely it refers to the time t_n, while m indicates the m-th coefficient of u in its ϵ-expansion):

$$\begin{array}{cccc}
f_2(2) \rightarrow & f_3(2)\ , & f_4(2)\ , & f_5(2)\ , \\
--- & + & + & + \\
G_4 & f_2(3)\ \rightarrow & f_3(3)\ , & f_4(3)\ , \\
 & --- & + & + \\
 & G_5 & f_2(4)\ \rightarrow & f_3(4)\ , \\
 & & --- & + \\
 & & G_6 & f_2(5)\ \rightarrow \\
 & & & --- \\
 & & & G_7
\end{array}$$

Table 1

This table is arranged in such a way that summing up the entries along the vertical lines reproduces the condition (3.33), while the arrow which connects $f_2(n)$ with $f_3(n)$ represents the compatibility equation (3.37). Note also that the pattern pictured in Table 1 looks like a ladder if only a finite number of slow times need to be introduced (the simplest picture obtains when only t_2 and t_3 are present as for the dispersion relation $\omega(k) = a_3 k^3$). Let us now proceed with our test. First one has to compute the differential polynomial G_4; this obtains from the cubic terms of the nonlinear part F of the PDE (1.21) to be tested (of course, the preliminary step of computing G_3 and, therefore, the real constant c which enters in the operators M_n has been already made). Then, because of (3.37) with $n = 2$ and the equality $f_2(2) = G_4$, one has to verify that the vector $M_3 G_4$ is in the image $M_2(\mathcal{P}_5(1))$ of the operator M_2. In order to envisage the actual computational task, one has to realize that the differential operator M_2 which maps vectors in $\mathcal{P}_n(m)$ into vectors in the bigger space $\mathcal{P}_{n+2}(m)$ is represented in such spaces as a rectangular matrix, with the implication that its image is a proper subspace of $\mathcal{P}_{n+2}(m)$. If it turns out that $M_3 G_4$ is not in $M_2(\mathcal{P}_5(1))$, then the original PDE (1.21) cannot be integrable and computations stop here. If, instead, $M_3 G_4$ belongs to the image of M_2, one can compute the vector $f_3(2)$ which solves the algebraic equation (3.37), and, because of Proposition 1, see (3.36), the solution $f_3(2)$ is unique. Proceeding to the next step requires first the computation of $f_2(3)$ by substraction (see Table 1),

$$f_2(3) = G_5 - f_3(2) \ , \tag{3.38}$$

where G_5 is obtained directly from the original PDE (1.21), and then the verification that $M_3 f_2(3)$ be in the image $M_2(\mathcal{P}_6(2))$. If this is not the case, this test leads to the conclusion that the original PDE (1.21) is not integrable, otherwise the test goes on with the next step in a similar way, namely one computes $f_3(3)$ by solving (3.37) for $n = 3$. Because of Proposition 2 (see above), the polynomial $f_4(2)$ can be computed and, by subtraction (see Table 1),

$$f_2(4) = G_6 - f_3(3) - f_4(2) \quad , \tag{3.39}$$

the polynomial $f_2(4)$ is obtained, this being the starting point for the next order.

Thus this procedure may go on order by order, starting with G_4 at the order $n = 2$. Assume now that one has been able to iterate this computational scheme we have just illustrated up to the calculation of $f_2(n+1)$, and that the polynomial $M_3 f_2(n+1)$ turns out not to belong to the image $M_2(\mathcal{P}_{n+4}(n))$, then we have found an "obstruction" at order $n+1$ since this procedure cannot be carried on any further. Of course, the higher is n where the obstruction occurs, the *more integrable* is the original PDE (1.21). The specification of this property deserves a notation, so we say that the PDE (1.21) is A_n − *integrable*, meaning *asymptotically integrable up to order* n, if no *obstruction* occurs up to order n and if the *obstruction* (if any) occurs at order $m \geq n+1$. For instance, the PDE (1.21) is A_1 - integrable if the constant c in the NLS equation (3.18) is real. It is also A_2 - integrable if

$$M_3 f_2(2) = M_3(a|u(1)|^2 u_1(1) + b u(1)^2 u_1^*(1)) \tag{3.40}$$

is in the image of M_2, i.e. in $M_2(\mathcal{P}_5(1))$, and recall that the coefficients a and b are directly computed from the PDE (1.21) since $f_2(2) = G_4$. By a straight, but tedious, computation one obtains that $M_3 f_2(2)$ is in $M_2(\mathcal{P}_5(1))$ if and only if a and b are real, $a = a^*$ and $b = b^*$, otherwise one has the obstruction. If a and b are real, one can go further at $n = 3$. In this case $f_2(3)$ is a 12-dim complex vector and the condition that $M_3 f_2(3)$ be in $M_2(\mathcal{P}_2(6))$ turns out to yields 15 real conditions so that the general solution $f_2(3)$ depends on $2 \times 12 - 15 = 9$ real constants. As it is already clear from these first instances, the computational burden rapidly increases with n and a computer code is needed even for the first few orders. An idea on how easily a PC can run out of memory already at $n = 4$ or $n = 5$ is given by the dimensionality of the vector spaces involved. In the notation $\mathcal{P}_n(m) \rightarrow$ dim $(\mathcal{P}_n(m))$, we have: $\mathcal{P}_3(1) \rightarrow 1, \mathcal{P}_4(1) \rightarrow 2,\ \mathcal{P}_5(1) \rightarrow 5,\ \mathcal{P}_6(1) \rightarrow 8,\ \mathcal{P}_4(2) \rightarrow 4,\ \mathcal{P}_5(2) \rightarrow 12,\ \mathcal{P}_6(2) \rightarrow 26,\ \mathcal{P}_5(3) \rightarrow 14,\ \mathcal{P}_6(3) \rightarrow 34,\ \mathcal{P}_6(4) \rightarrow 36$.

In conclusion, this test is based on an infinite sequence of necessary conditions of integrability, one at each order of the ϵ - expansion of the amplitude of the fundamental harmonic. Formulated as it is here, several mathematical problems related to this method remain open for future investigations. Among others, natural generalizations of the family of PDEs (1.21) we have considered here are feasible. For instance, one can consider PDEs with more than one dispersion branch, as for PDEs of higher order of the time-derivative or systems of PDEs with vector or matrix solutions, and/or PDEs in more than $1+1$ independent variables. As an instance, we have applied [22] this test to the following family of third order PDEs

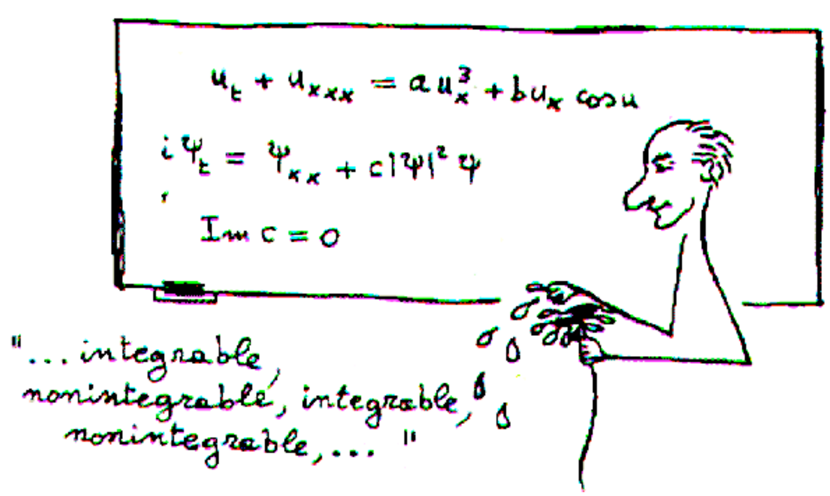
$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x \ , \tag{3.41}$$

which is not in the class (1.21). With the essistance of Mathematica, we have found that only three members of the family (3.41) are A_3 - integrable, namely the KdV equation ($\alpha = c_2 = c_3 = 0$), the Camassa-Holm [23] equation ($c_1 = -\frac{3}{2} c_3/\alpha^2 \ , \ \ c_2 = c_3/2$) and one new equation ($c_1 = -2c_3/\alpha^2 \ , \ \ c_2 = c_3$) which can be transformed, by a change of variables, to the form

$$m_t + m_x u + 3 m u_x = 0 \quad , \quad m = u - u_{xx} \quad . \tag{3.42}$$

Since the nonlinearity of this equation is quadratic and it passes our test up to order 3 (we could not push the test to higher order because of the heavy algebraic computations involved), we conjectured that this equation be integrable, but with no proof as our conditions are only necessary. Only in a subsequent investigation of (3.42), related in particular with the existence of special solutions known as *peakons*, it has been finally shown that the PDE (3.42) is S-integrable by explicitly displaying the associated Lax pair and conservation laws [24] together with multisoliton solutions [29].

Finally, since the conditions of integrability presented here are only necessary, once they are met, one may try *the daisy petals method*:



References

1. V.E.Zakharov (Ed.), *What is Integrability?*, Springer, Heidelberg, 1990.
2. A.Degasperis, "Resource Letter Sol-1: Solitons," Am. J. Phys. **66**, 486-497 (1998).
3. T.Taniuti and N.Yajima, "Perturbation method for a nonlinear wave modulation. I," J. Math. Phys. **10**, 1369-1372 (1969); Suppl. Prog. Theor. Phys. **55**, 1974 (special issue devoted to the *Reductive Perturbation Method for Nonlinear Wave Propagation*).
4. P.L.Kelley, "Self-Focusing of Optical Beams," Phys Rev. Lett. **15**, 1005 (1965); D.J. Benney and A.C.Newell, J.Math. and Phys. (now Stud. Appl. Math.) **46**, 133-139 (1967); V.E.Zakharov, "Instability of self-focusing of light" Soviet Phys. JETP **26**, 994-998 (1968); H.Hasimoto and H.Ono, "Nonlinear Modulation of Gravity Waves," J. Phys. Soc. Japan **33**,805 (1972); A.Hasegawa and F.Tappert, "Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion," Appl. Phys. Lett. **23**, 142 (1973).
5. F.Calogero and W.Eckhaus, "Necessary conditions for integrability," Inverse Probl. **3**, L27-L32 (1987).
6. F.Calogero, "Why are certain nonlinear PDEs both widely applicable and integrable?." In [1], pp. 1-62.
7. F.Calogero and A.Degasperis, *Spectral Transform and Solitons. Vol. I*, North Holland, Amsterdam, 1982; M.J. Ablowitz and P.A.Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge U.P., Cambridge, 1991.
8. V.E.Zakharov and A.B.Shabat, "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media," Soviet Phys. JETP **34**, 62-69 (1972) [Russian original: Zh. Eksp. Teor. Fiz. **61**, 118-134 (1971)].
9. V.E.Zalharov and E.A.Kuznetsov, "Multiscale Expansion in the Theory of Systems Integrable by the Inverse Scattering Transform," Physica **18D**, 455-463 (1986).
10. C.S.Gardner, J.M.Greene, M.D.Kruskal and R.M.Miura, "Method for solving the Korteweg-de Vries equation," Phys. Rev. Lett. **19**, 1095-1097 (1967).
11. J.A.Murdock, *Perturbations*, John Wiley & Sons, Inc., New York, 1991.
12. F.Calogero and A.Degasperis, "Reduction technique for matrix nonlinear evolution equations solvable by the spectral transform," J. Math.Phys. **22**, 23-31 (1981).
13. F.B.Whitham, *Linear and Nonlinear Waves*, John Wiley & Sons, Inc., New York, 1974.
14. R.A.Kraenkel, M.A.Manna and J.G.Pereira, "The Korteweg-de Vries hierarchy and long water-waves," J. Math. Phys. **36**, 307 (1995).
15. Y.Kodama and A.V.Mikhailov, "Obstacle to asymptotic integrability" In: *Algebraic Aspects of Integrable Systems: in memory of Irene Dorfman*, edited by A.S.Fokas and I.M. Gelfand, Birkhauser, Boston, 1996, pp. 173-204.
16. F.Calogero, A.Degasperis and Ji Xiaoda, "Nonlinear Schroedinger-type equations from multiscale reduction of PDEs. I. Systematic derivation," J. Math. Phys. **41**, 6399-6443 (2000).
17. F.Calogero, A.Desgasperis and Ji Xiaoda, "Nonlinear Schroedinger-type equations from multiscale reduction of PDEs. II. Necessary conditions of integrability for real PDEs," J. Math. Phys. **42**, 2635-2652 (2001).

18. A.Ramani (private communication).
19. R.Conte (private communication).
20. N.Yajima and M.Oikawa, "Formation and Interaction of Sonic-Langmuir Solitons," Prog. Theor. Phys. **56**, 1719-1739 (1976).
21. A.Degasperis, S.V.Manakov and P.M.Santini, "Multiple-scale perturbation beyond the Nonlinear Schroedinger Equation. I" Physica D**100**, 187-211 (1997).
22. A.Degasperis and M.Procesi, "Asymptotic Integrability." In: *Symmetry and Perturbation Theory*, edited by A.Degasperis and G.Gaeta, World Scientific, Singapore, 1999, pp. 23-37.
23. R.Camassa and D.D. Holm, " An integrable shallow water equation with peaked solitons," Phys. Rev. Lett. **71**, 1661-1664 (1993).
24. A.Degasperis, D.D. Holm and A.N.W. Hone, "A new integrable equation with peakon solutions," Theoretical and Mathematical Physics **133**, 1463-1474 (2002).
25. V.E. Zakharov and E.I. Schulman, " Integrability of Nonlinear Systems and Perturbation Theory." In [1], pp. 185-250.
26. D. Bambusi, " Birkhoff normal form for some nonlinear PDEs," Comm. Math. Phys. **234**, 253–285 (2003).
27. B. Dubrovin and Y. Zhang, " Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants, " arXiv:math/0108160v1 [math.DG].
28. Y. Hiraoka and Y. Kodama, " Normal form and solitons," Lecture notes, Proceedings of this Summer School and arXiv:nlin.SI/020602.
29. Y. Matsuno, " Multisoliton solutions of the Degasperis–Procesi equation and their peakon limit," Inverse Problems **21**, 1553–1570 (2005).