NONLINEAR WAVES AND SOLITONS

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CONTENTS

CONTENTS

CONTENTS

CONTENTS

CONTENTS

NOTATION: t = time , x = space , $u_t = \partial u/\partial_t$, $u_x = \partial u/\partial_x$, etc.
***LINEAR* wave equations in 1+1 dimensions:**

TRANSLATION WAVES

$$u_t + v\, u_x = 0 \ , \ \ u = u(x,t) = u_0(x - vt)$$

DISPERSIVE WAVES

$$u_t + i\omega(-i\frac{\partial}{\partial_x})u = 0 \, , \ u(x,0) = u_0(x)$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \hat{u}_0(k) e^{i[kx-\omega(k)t]}$$

$$u_0(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, \hat{u}_0(k) e^{ikx}$$

$\omega = \omega(k) =$ dispersion relation

INTRODUCTION: EXAMPLES OF LINEAR WAVE EQUATIONS

- Schrödinger equation: $\omega(k) = k^2$

$$iu_t + u_{xx} = 0$$

- Linearized Korteweg-de Vries equation: $\omega(k) = k^3$

$$u_t - u_{xxx} = 0$$

- Linearized Benjamin-Ono equation: $\omega(k) = sign(k)k^2$

$$u_t - Hu_{xx} = 0 \;\; , \;\; Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} dy \frac{f(y)}{y - x}$$

LINEAR WAVE EQUATIONS HAVE INFINITELY MANY CONSERVATION LAWS

LOCAL :

$$\rho_t(x,t) + j_x(x,t) = 0$$

GLOBAL :

$$C = \int_{-\infty}^{+\infty} dx\, \rho(x,t) \quad , \quad \frac{dC}{dt} = 0$$

EXERCISE: FIND 3 OR 4 LOCAL COSERVATION LAWS OF THE WAVE EQUATIONS DISPLAYED ABOVE

INTRODUCTION: NONLINEAR WAVE EQUATIONS

$$Lu = N(u)$$

GENERICALLY A NONLINEAR WAVE EQUATION POSSESSES ONLY FEW CONSERVATION LAWS

EXCEPTIONALLY A WAVE EQUATION HAS INFINETELY MANY CONSERVATION LAWS AND IT IS INTEGRABLE

EXAMPLES:

$$u_t + u_{xxx} = P(u)u_x \quad , \quad P(u) = \sum_{n=0}^{N} c_n u^n$$

THEOREM: THIS EQUATION IS INTEGRABLE IF AND ONLY IF

$N \leq 2$

$$P(u) = c_0 + c_1 u + c_2 u^2$$

INTRODUCTION: NONLINEAR WAVE EQUATIONS CONT.

C-INTEGRABILITY: LINEARISATION VIA A CHANGE OF VARIABLES

ECKHAUS EQUATION: $iu_t + u_{xx} = [2(|u|^2)_x + |u|^4]\, u$

$$u(x,t) \to v(x,t) = u(x,t)\, \exp[\int_{x_0}^{x} dy\, |u(y,t)|^2\,]$$

$$iv_t + v_{xx} = 0$$

EXERCISES:

1. FIND THE INVERSE TRANSFORMATION $v(x,t) \to u(x,t)$
2. FIND EXPLICIT SOLUTIONS OF THE ECKHAUS EQUATION
3. DISCUSS THE STABILITY OF THE PLANE WAVE SOLUTION

INTRODUCTION: NONLINEAR WAVE EQUATIONS CONT.

S-INTEGRABILITY: SPECTRAL ANALYSIS (NONLINEAR FOURIER TRANSFORM)

KORTEWEG- DE VRIES EQUATION (KdV): $u_t + u_{xxx} = c_0 u_x + c_1 u u_x$

MODIFIED KORTEWEG- DE VRIES EQUATION (mKdV):
$u_t + u_{xxx} = c_0 u_x + c_2 u^2 u_x$

NONLINEAR SCHROEDINGER EQUATION (NLS):
$iu_t \; + \; u_{xx} = 2\eta |u|^2 \, u$

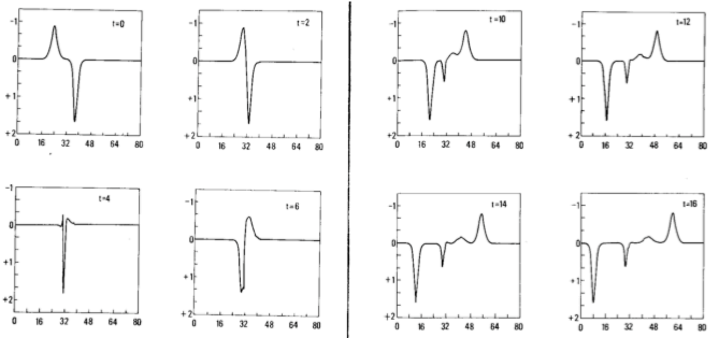
$$\Psi_x = \begin{pmatrix} -ik & \eta u^* \\ u & ik \end{pmatrix} \Psi \ , \quad \Psi = \begin{pmatrix} \psi_1(x,t,k) \\ \psi_2(x,t,k) \end{pmatrix} \ , \quad u = u(x,t)$$

$$\Psi_t = \begin{pmatrix} 2ik^2 + i\eta|u|^2 & -2k\eta u^* - i\eta u_x^* \\ -2ku + iu_x & -2ik^2 - i\eta|u|^2 \end{pmatrix} \Psi$$

$$\Psi_{xt} = \Psi_{tx}$$

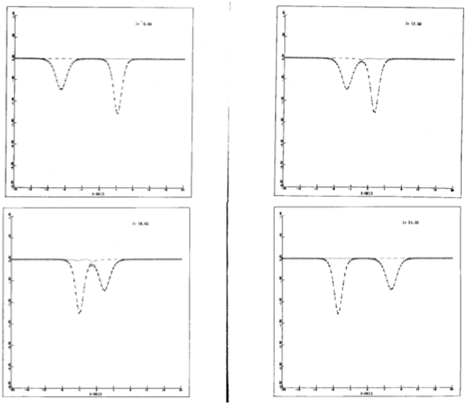
INTRODUCTION: NONLINEAR WAVE EQUATIONS CONT.

REGULARIZED LONG WAVE EQUATION $u_t + u_x - u_{xxt} = 6uu_x$



INTRODUCTION: NONLINEAR WAVE EQUATIONS CONT.

KDV EQUATION: $u_t + u_{xxx} = 6uu_x$



INTRODUCTION: NONLINEAR WAVE EQUATIONS CONT.

HISTORICAL NOTES

1845............SCOTT-RUSSELL

1895............KORTEWEG-DE VRIES

1955............FERMI-PASTA-ULAM

1965............ZABUSKY-KRUSKAL

1967............GARDNER-GREENE-KRUSKAL-MIURA

1971............ZAKHAROV-SHABAT

POINCARE'-LINDSTEDT

$$\ddot{q} + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \ldots\ , \qquad q = q(t, \epsilon)$$

$$q(0, \epsilon) = \epsilon\ , \qquad \dot{q}(0, \epsilon) = 0$$

$$q(t, \epsilon) = q(t + \frac{2\pi}{\omega(\epsilon)}, \epsilon)\ ,\ \theta = \omega(\epsilon)t\ , \qquad q(t, \epsilon) = f(\theta, \epsilon)$$

$$\omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \ldots\ , \quad f(0, \epsilon) = \epsilon,\ f'(0, \epsilon) = 0$$

$$\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \ldots\ ,\quad f(\theta, \epsilon) = \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \ldots$$

POINCARE'-LINDSTEDT

$$\ddot{q} + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \dots \,, \qquad q = q(t, \epsilon)$$

$$q(0, \epsilon) = \epsilon \,, \qquad \dot{q}(0, \epsilon) = 0$$

$$q(t, \epsilon) = q(t + \frac{2\pi}{\omega(\epsilon)}, \epsilon) \,, \; \theta = \omega(\epsilon) t \,, \qquad q(t, \epsilon) = f(\theta, \epsilon)$$

$$\omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \dots \,, \quad f(0, \epsilon) = \epsilon, \; f'(0, \epsilon) = 0$$

$$\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \dots \,, \quad f(\theta, \epsilon) = \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \dots$$

MULTISCALE ODE 1

POINCARE'-LINDSTEDT

$$\ddot{q} + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \dots\dots \,, \qquad q = q(t, \epsilon)$$

$$q(0, \epsilon) = \epsilon \,, \qquad \dot{q}(0, \epsilon) = 0$$

$$q(t, \epsilon) = q(t + \frac{2\pi}{\omega(\epsilon)}, \epsilon) \,, \ \theta = \omega(\epsilon) t \,, \qquad q(t, \epsilon) = f(\theta, \epsilon)$$

$$\omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \dots\dots, \quad f(0, \epsilon) = \epsilon, \ f'(0, \epsilon) = 0$$

$$\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \dots \quad , \quad f(\theta, \epsilon) = \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \dots$$

POINCARE'-LINDSTEDT

$$\ddot{q} + \omega_o^2 q = c_2 q^2 + c_3 q^3 + \ldots. \ , \qquad q = q(t, \epsilon)$$

$$q(0, \epsilon) = \epsilon \ , \qquad \dot{q}(0, \epsilon) = 0$$

$$q(t, \epsilon) = q(t + \frac{2\pi}{\omega(\epsilon)}, \epsilon) \ , \ \theta = \omega(\epsilon) t \ , \qquad q(t, \epsilon) = f(\theta, \epsilon)$$

$$\omega^2(\epsilon) f'' + \omega_0^2 f = c_2 f^2 + c_3 f^3 + \ldots. , \quad f(0, \epsilon) = \epsilon, \ f'(0, \epsilon) = 0$$

$$\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \ldots \ , \ f(\theta, \epsilon) = \epsilon f_1(\theta) + \epsilon^2 f_2(\theta) + \ldots$$

MULTISCALE ODE 2

$$O(\epsilon):\quad f_1'' + f_1 = 0\ ,\ \ f_1(0) = 1\ ,\ \ f_1'(0) = 0$$

$$O(\epsilon^n):\ \begin{cases} f_n'' + f_n = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\} \\ f_n(0) = 0\ ,\ \ f_n'(0) = 0 \end{cases}$$

$$\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\} = \sum_{\alpha=-n}^{n} C_n^{\alpha} e^{i\alpha\theta}$$

SECULARITY EFFECT:
$\phi'' + \phi = Ce^{i\theta}\ ,\ \phi(\theta) = Ae^{i\theta} + Be^{-i\theta} + \frac{1}{2i}C\theta e^{i\theta}$

SECULARITY-FREE CONDITION: $C_n^{\pm 1} = 0$
FREQUENCY RENORMALIZATION EXERCISE: COMPUTE $\omega(\epsilon)$ UP TO $O(\epsilon^2)$ ANSWER : $\omega_1 = 0\ ,\ \omega_2 = -(10c_2^2 + 9\omega_0^2 c_3)/24\omega_0^3$

$$\theta = \omega(\epsilon)t = \omega_0 t + \omega_1 \epsilon t + \omega_2 \epsilon^2 t + \ldots 0 \quad , \quad t_n = \epsilon^n t$$

$$\theta = \omega_0 t + \omega_1 t_1 + \omega_2 t_2 + \ldots$$

$$q(t,\epsilon) = \sum_{n=1} \sum_{\alpha=-n}^{n} \epsilon^n \exp\left(i\alpha\theta\right) f_n^{(\alpha)}$$

$$q(t,\epsilon) = \sum_{n=1} \sum_{\alpha=-n}^{n} \epsilon^n E^\alpha q_n^{(\alpha)}(t_1, t_2, \ldots) \quad , E \equiv \exp(i\omega_0 t)$$

$$d(E^\alpha q_n^{(\alpha)})/dt = E^\alpha(i\alpha\omega_0 + \epsilon\partial/\partial t_1 + \epsilon^2\partial/\partial t_2 + \ldots)q_n^{(\alpha)}$$

INGREDIENTS :

- EXPANSION IN POWERS OF ϵ
- EXPANSION IN POWERS OF $\exp(i\omega_0 t)$
- INTRODUCTION OF SLOW TIMES t_1 , t_2 , t_3 , . . .

$$\begin{cases} Du & = & F[u, u_x, u_{xx}, \ldots] \ , \quad u = u(x,t) \\ D & = & \partial/\partial t + i\omega(-i\partial/\partial x) \\ \omega(k) & = & \sum_{m=0} a_{2m+1} k^{2m+1} \end{cases}$$

ONE QUASI-MONOCHROMATIC PLANE WAVE

$$u(x,t) \simeq \Delta k \int_{-\infty}^{+\infty} d\eta A(\eta) \exp\{i[x(k+\eta\Delta k) - t\omega(k+\eta\Delta k)]\} + c.c.$$

$$\omega(k+\epsilon\eta k) = \sum_{n=0}^{\infty} \omega_n \eta^n k^n \epsilon^n \ , \ \epsilon \equiv \Delta k/k$$

$$u(x,t) \simeq \epsilon E(x,t) u^{(1)}(\xi, t_1, t_2, \ldots) + c.c. \ , \ E(x,t) \equiv \exp[i(kx - \omega t)],$$

$$\xi \equiv \epsilon x \quad , \quad t_n \equiv \epsilon^n t$$

HARMONIC EXPANSION

$$u(x,t) = \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)}(\xi, t_1, t_2, \dots) E^{\alpha}(x,t) \quad , \quad u^{(\alpha)*} = u^{(-\alpha)}$$

$$\partial_x \longrightarrow \partial_x + \epsilon \partial_\xi \ , \ \partial_t \longrightarrow \partial_t + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + \dots$$

$$D[u^{(\alpha)} E^{\alpha}] = E^{\alpha} D^{(\alpha)} u^{(\alpha)}$$

$$F[u, u_x, u_{xx}, \dots] = \sum_{\alpha=-\infty}^{+\infty} F^{(\alpha)}[u^{(\beta)}, u^{(\beta)}_{\xi}, u^{(\beta)}_{\xi\xi}, \dots] E^{\alpha}$$

$$D^{(\alpha)} u^{(\alpha)} = F^{(\alpha)}$$

EPSILON EXPANSION

$$D^{(\alpha)} = -i\alpha\omega + \epsilon\partial_{t_1} + \epsilon^2\partial_{t_2} + \ldots + i\omega(k - i\epsilon\partial_\xi)$$

$$D^{(\alpha)} = D_0^{(\alpha)} + \epsilon D_1^{(\alpha)} + \epsilon^2 D_2^{(\alpha)} + \ldots$$

$$D_0^{(\alpha)} = i[\omega(\alpha k) - \alpha\omega(k)]\ ,\ D_n^{(\alpha)} = \partial_{t_n} - (-i)^{n+1}\omega_n(\alpha k)\partial_\xi^n \quad , \quad n \geq 1$$

$$u^{(\alpha)} = \sum_{n=1} \epsilon^n u^{(\alpha)}(n),\ |\alpha| > \ 1$$

$$\text{assumption} \quad : \quad F \to -F \ \text{if} \ u \to -u \quad \Longrightarrow u^{(2\alpha)} = 0$$

$$F^{(\alpha)} = \epsilon^3 F_3^{(\alpha)} + \epsilon^4 F_4^{(\alpha)} + \ldots$$

$$D_0^{(\alpha)} u^{(\alpha)}(n+1) + D_1^{(\alpha)} u^{(\alpha)}(n) + D_2^{(\alpha)} u^{(\alpha)}(n-1) + \ldots + D_n^{(\alpha)} u^{(\alpha)}(1) = F_{n+1}^{(\alpha)}$$

MULTISCALE PDE 4

SLAVE HARMONICS : $D_0^{(\alpha)} = i[\omega(\alpha k) - \alpha\omega(k)] \neq 0$

NO RESONANCE CONDITION : $[\omega(\alpha k) - \alpha\omega(k)] = 0$ IFF $\alpha = \pm 1$

FUNDAMENTAL HARMONIC : $\alpha = 1$

$$\mathrm{O}(\epsilon^2) : (\partial_{t_1} + \omega_1 \partial_\xi) u^{(1)}(1) = 0\,,\ u^{(1)}(1) = u^{(1)}(1)(\xi - \omega_1 t_1, t_2, t_3, \ldots)$$

$$\mathrm{O}(\epsilon^3) : (\partial_{t_1} + \omega_1 \partial_\xi) u^{(1)}(2) = -[(\partial_{t_2} - i\omega_2 \partial_\xi^2) u^{(1)}(1) - F_3^{(1)}]$$

SECULARITY OR RESONANCE PHENOMENON

$$(\partial_t - M)v(t) = w(t)\,,\ (\partial_t - M)w(t) = 0 \Longrightarrow v(t) = v_0(t) + tw(t)$$

SECULARITY FREE CONDITION : $w(t) = 0$

$$(\partial_{t_2} - i\omega_2 \partial_\xi^2) u^{(1)}(1) - F_3^{(1)} = 0$$

NLS EQUATION : $i\partial_{t_2} u^{(1)}(1) + \omega_2 \partial_\xi^2 u^{(1)}(1) = 2c|u^{(1)}(1)|^2 u^{(1)}(1)$

ONE RESONANCE : SECOND HARMONIC GENERATION

$$\omega(2k) - 2\omega(k) = 0$$

assumption : $F^{(\alpha)} = \epsilon^2 F_2^{(\alpha)} + \epsilon^3 F_3^{(\alpha)} + \epsilon^4 F_4^{(\alpha)} + \dots$
technical assumptions (check it!) : $F_2^{(\alpha)} = \partial_x \hat{F}_2^{(\alpha)}$, $\omega(0) = 0$

$$D_1^{(1)} u_1^{(1)} = c_1 u_1^{(2)} u_1^{(-1)} \; , \; D_1^{(2)} u_1^{(2)} = c_2 u_1^{(1)2}$$

$$\text{SHG} \begin{cases} \partial_t E_1 + v_1 \partial_x E_1 = c_1 E_2 E_1^* & , \quad v_1 = \omega_1(k_1) \\ \partial_t E_2 + v_2 \partial_x E_2 = c_2 E_1^2 & , \quad v_2 = \omega_1(k_2) \end{cases}$$

RED $\Longrightarrow$ VIOLET

TWO QUASI-MONOCHROMATIC PLANE WAVES

$$u(x,t) =$$

$$\sum_{\alpha_1=-\infty}^{+\infty} \sum_{\alpha_2=-\infty}^{+\infty} u^{(\alpha_1,\alpha_2)}(\xi, t_1, t_2, \ldots) \exp\{i[(\alpha_1 k_1 + \alpha_2 k_2)x - (\alpha_1 \omega_1 + \alpha_2 \omega_2)t]\}$$

$$\omega_1 = \omega(k_1)\,,\ \omega_2 = \omega(k_2)$$

NO RESONANCE :

$$\omega(\alpha_1 k_1 + \alpha_2 k_2) = \alpha_1 \omega(k_1) + \alpha_2 \omega(k_2)$$

$$\text{IFF} \quad \alpha_1 = \pm 1\,,\ \alpha_2 = 0 \ \text{ OR } \ \alpha_1 = 0\,,\ \alpha_2 = \pm 1$$

$$\omega_1(k_1) \neq \omega_1(k_2)$$

$$\text{NLS EQ.} : i\partial_{t_2} u^{(1,0)}(1) + \omega_2(k_1)\partial_\xi^2 u^{(1,0)}(1) = 2c_{10}|u^{(1,0)}(1)|^2 u^{(1,0)}(1)$$

$$\text{NLS EQ.} : i\partial_{t_2} u^{(0,1)}(1) + \omega_2(k_2)\partial_\xi^2 u^{(0,1)}(1) = 2c_{01}|u^{(0,1)}(1)|^2 u^{(0,1)}(1)$$

MULTISCALE PDE 7

STRONG RESONANCE : $\omega_1(k_1) = \omega_1(k_2)$

$$\text{VNLS}\begin{cases} i\partial_{t_2} u^{(1,0)}(1) + \omega_2(k_1)\partial_\xi^2 u^{(1,0)}(1) = & 2(c_{10}|u^{(1,0)}(1)|^2 \\ & +\gamma_{10}|u^{(0,1)}(1)|^2)u^{(1,0)}(1) \\ i\partial_{t_2} u^{(0,1)}(1) + \omega_2(k_2)\partial_\xi^2 u^{(0,1)}(1) = & 2(c_{01}|u^{(0,1)}(1)|^2 \\ & +\gamma_{01}|u^{(1,0)}(1)|^2)u^{(0,1)}(1) \end{cases}$$

ONE RESONANCE AT $\alpha_1 = 1\ ,\ \alpha_2 = 1\ :\ \omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$

THREE WAVE RESONANT INTERACTION

$$\begin{cases} k_3 &= k_1 + k_2 \\ \omega_3 &= \omega_1 + \omega_2 \end{cases}$$

technical assumptions (check it!) $: F_2 = \partial_x \hat{F}_2\ ,\ \omega(0) = 0$

$$\text{at}\ \ \mathrm{O}(\epsilon^2) \begin{cases} D_1^{(1,0)} u_1^{(1,0)} = c_1 u_1^{(1,1)} u_1^{(0,-1)} \\ D_1^{(0,1)} u_1^{(0,1)} = c_2 u_1^{(1,1)} u_1^{(-1,0)} \\ D_1^{(1,1)} u_1^{(1,1)} = c_3 u_1^{(1,0)} u_1^{(0,1)} \end{cases}$$

$$\mathrm{3WRI} \begin{cases} \partial_t E_1 + v_1 \partial_x E_1 = c_1 E_3 E_2^* \ , & v_1 = \omega_1(k_1) \\ \partial_t E_2 + v_2 \partial_x E_2 = c_2 E_3 E_1^* \ , & v_2 = \omega_1(k_2) \\ \partial_t E_3 + v_3 \partial_x E_3 = c_3 E_1 E_2 \ , & v_3 = \omega_1(k_3) \end{cases}$$

EXERCISES :

1. DERIVE THE NLS EQUATION FROM THE PDE

$$u_t + au_{xxx} + bu_x = f_1 uu_x + f_2 u^2 u_x$$

   BY MULTISCALE METHOD AND TELL IF IT IS " FOCUSING " OR " DEFOCUSING "
2. PROVE THAT THE " DEFOCUSING " NLS EQUATION $iu_t + u_{xx} - |u|^2 u = 0$ HAS NO SOLITARY WAVE SOLUTION ($i.e.$ $u(x,t) = \exp[i(ax+bt)]f(x-ct)$) IN $L_2(\mathcal{R})$ W.R.T. x.
3. FIND TWO CONSERVATION LAWS OF THE 3WRI EQUATION
4. SOLVE THE STATIONARY 3WRI EQUATION IN TERMS OF ELLIPTIC FUNCTIONS