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SUMARIO

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ON OPERATIONS OF CONVOLUTION TYPE AND ORTHONORMAL SYSTEMS ON COMPACT ABELIAN GROUPS

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INTRODUCTION. This paper is divided into three sections: the present one (which contains the motivation of the others) and the following parts I and II. Part II is devoted to the study of certain Banach algebras, to which one is naturally led when trying to solve the problem of introducing a convolution in a general finite measure space. Part I deals with necessary and sufficient conditions on an orthonormal system of measurable functions on a compact abelian group G, to be the image of the character group $G^{\wedge}$, under measurable transformations on the original group G.

Preliminary results. 1. Given two finite measure spaces (X_i, Σ_i, μ_i), $i = 1, 2$, $(\mu_i\ (X_i) < \infty)$, we say that they are B-isomorphic if there exists a σ-isomorphism between the Boolean algebras Σ_1/N_1 and Σ_2/N_2, where N_i denotes the sets of measure zero of Σ_i. If besides, the σ-isomorphism between the measure algebras Σ_i/N_i preserves measure we say that the spaces are m-isomorphic. Following D. Maharam's paper ([8]) we call a finite measure algebra *homogeneous* if any two principal ideals admit minimal σ-basis of the same power. The result of Maharam which interests us is the following: a) A finite measure space (X, Σ, μ), is m-isomorphic to the measure theoretic union of a denumerable set of spaces of the form $P_n = (\prod_{1\leq i<\gamma_n} [0,1)_i, \Sigma_n, k_n \cdot \mu_n)$ and a purely atomic space. The γ_n are infinite ordinal numbers (and the leading ordinals of their cardinal classes) and verify $\gamma_n <_{n+1}$; Σ_n represents the σ-field of Baire sets of the compact groups $\prod_{1\leq i<\gamma_n} [0,1)_i$; μ_n is the normalized Haar measure and k_n a real number such that $0 \leq k_n \leq 1$. Given

(X, Σ, μ) the γ_n and k_n are uniquely determined. b) If (X, Σ, μ) is homogeneous (and non atomic) the Maharam's representation is reduced to only one product of unit circles.

2. If we try to introduce a convolution type operation on a finite measure space (X, Σ, μ) we can do it as follows. Supposing that our space has no atoms, it is m-isomorphic to $\bigcup P_n$, and therefore there is induced a natural isomorphism τ between both L^1-spaces. Since each P_n is a compact group it has a convolution defined in the ordinary way. Representing by χ characteristic functions, we define $f^* g$, for $f, g \in L^1(X, \Sigma, \mu)$ as:

$$f\,(^*)\;g = \tau^{-1}\left(\sum_n \left(\tau(f)\,\chi_{P_n} * \tau(g)\,\chi_{P_n}\right)\right) \tag{1}$$

If (X, Σ, μ) also has atoms no problem arises because they are easier to manage.

This suggests to study locally compact spaces obtained as union of locally compact abelian groups and to define convolution type operations in analogous fashion as (1). We do this in part II and we see that, as it might be expected, these spaces admit a formal treatment like a common locally compact abelian group. However the Bochner theorem splits into two parts and is the main difference with the theory developed in [1] or [7].

3. Since an infinite product of unit circles, from the measure theoretic point of view, can be replaced by product of a set with the same power of copies of the two-element group, we see that there are several ways of introducing a convolution operation like (1). To see how they are related we can restrict ourselves to study the same problem for a fixed compact abelian group.

The first observation we need is contained in the next lemma.

Lemma 1. Any compact non-finite group is m-isomorphic to a product of unit circles.

Proof. It is necessary to prove that a compact group G is homogeneous. Given two homogeneous sets A and B of positive measure, contained in G, there exists a point x such that $x\,A \cap B$ has positive measure ([2], p. 261). Therefore, A, B and $x\,A \cap B$ have the same type of homogeneity, QED.

(As it is well-known (cf. [9]) a compact group is isomorphic to a product of unit circles, $[0,1)_i$, $1 \leqq i < \gamma$, if and only if, its dual group is isomorphic to the direct sum of L_i, $1 \leqq i < \gamma$, where each L_i represents the set of integers. Therefore, a product of unit circles is characterized as the dual group of a free group. The real line with the discret topology is not a free group, and therefore its dual group provides us of an example of a non-finite compact group not a product of circles).

Lemma 1 and Maharam's theorem show that if we have a measure space (which, for the sake of brevity, we shall suppose without atoms) (X, Σ, μ), which is m-isomorphic to the spaces $\bigcup_n P_n$, $\bigcup_m Q_m$, P_n, Q_m, compact abelian groups, then the study of the relationship between the convolution operation defined by (1) and

$$f\ (*')\ g = \tau'^{-1}\ (\sum_m\ (\tau'(f)\ \chi_{Q_m} * \tau'(g)\ \chi_{Q_m}\ )) \qquad (2)$$

is reduced to the study of the relationship between the convolutions of two compact spaces of the same homogeneity. (We have supposed that two different Q_m have different homogeneity types).

4. *Lemma 2. Two compact (non-finite) abelian groups* [1] *are m-isomorphic if and only if their duals groups are of the same power.*

Proof. By lemma 1 it is sufficient to prove that any compact non-finite commutative group G, is m-isomorphic to a product P of as many copies of the unit circle as is the power of $G^\wedge$.

(It is immediate that the power of the set of unit circles taken into consideration is the same as that of the character group $P^\wedge$). Since a m-isomorphism preserves orthonormal complete systems of functions in L^2, the power of $P^\wedge$ is the same as that of $G^\wedge$, QED.

The same argument proves also the following extension:

Two compact (commutative or not) groups are m-isomorphic if and only if their families of sets of equivalent, irreducible, unitary matrix representations have the same power, (cf. § 32 and § 33 of [9]).

5. Given a compact abelian group G, let (e_i) be its character system. For two functions f and g of $L^2(G)$ with Fourier series: $f = \sum c_i\, e_i$, $g = \sum d_i\, e_i$, we have:

$$f\ (*)\ g = \sum c_i\, d_i\, e_i \qquad (3)$$

Formula (3) permits to define an operation of convolution type with any complete orthonormal system of functions of L^2. Among

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(1) For compact groups we always suppose that they have been provided with the normalized Haar measure.

these systems there are some which can be considered as the image under an m-isomorphism of the character system of certain compact groups. An orthonormal system (η_i) of this type must verify certain conditions, for example, its functions must be uniformly bounded and constitute a multiplicative group, *i. e.*, for any j and k, $|\eta_j| \leq 1$ *a. e.*, $\eta_j \cdot \eta_k = \eta_e$ *a. e.*, and so on.

The interest of these particular systems may be justified as follows. Let f, $g \in L^2(G)$ and $f = \sum c_i \eta_i$, $g = \sum d_i \eta_i$ and consider the convolution defined by (3):

$$f\ (*) = \sum c_i\, d_i\, \eta_i$$

and that defined by (1):

$$f\ (*)\ g = \tau^{-1}(\tau(f) * \tau\ (g)),$$

where (η_i) is the image of (e_i) under the m-isomorphism τ^{-1} from the group F onto the group G, and where the convolution in the second member of the last equality must be understood in the usual sense. It is easy to see that both definitions provide the same function $(a.e.)$. In other words, with these particular systems, (3) defines a convolution which is "essential" in the sense that, except by a m-isomorphism, it is the convolution on a certain commutative compact group.

6. It arises naturally the question, what are the conditions which must satisfy an orthonormal complete system to be the m-isomorphic image of the character system of a certain group. This question receives several answers in Part I. Now we consider an example of this situation. Let G be the unit interval $[0,1)$, with the operation of sum (mod. 1), and F the product of countable many copies of the two-element group. It is well-known that there exists a measure preserving transformation of F onto G constructed with the dyadic intervals. But this is exactly the transformation which sends the family of characters of F onto the Walsh system of the interval, (cf. [10], p. 34, Ex. 6). We leave the easy verification to the reader.

Part I

Almost everywhere multiplicative systems. 1. Let G and F be compact abelian groups and $G^\wedge$, $F^\wedge$, their dual groups. It is well-known that:

Theorem 1. G and F are isomorphic if and only if $G^\wedge$ is isomorphic to $F^\wedge$.

Our purpose in this section is to extend theorem 1 to other situations. What theorem 1 says is that if there exist an isomorphism between the character systems $G^\wedge = (\epsilon_i)$ and $F^\wedge = (\eta_i)$, then there exists an isomorphism $T: F \rightarrow G$, of F onto G, such that: $\eta_i(y) = \epsilon_i(Ty)$.

Theorem 2. Let $F^\sim = (\eta_i(y))$ be a complete o.n. system of functions of $L^2(F)$, which is under the multiplication a.e. (²) *an isomorphic group to $G^\wedge = (\epsilon_i(x))$. Then, there exists an m-isomorphism between F and G such that $F^\sim$ is the image of $G^\wedge$. The converse is obviously true* (³).

Proof. Consider the unitary operator U defined by the correspondence $\epsilon_i \rightarrow \eta_i$, given by hypothesis. We see next that for any $f \in L^2(G)$ and $g \in L^\infty(G)$, *it holds*

$$U(fg) = U(f).U(g) \qquad a.e., \tag{1}$$

Since, $U(\epsilon_i\,\epsilon_j) = U(\epsilon_{ji}) = \eta_{ij} = \eta_i . \eta_j = U(\epsilon_i).U(\epsilon_j)$, we have:

$$U((\sum_1^N (f,\epsilon_i)\,\epsilon_i)\,(\sum_1^M (g,\epsilon_j)\,\epsilon_j)) = U(\sum_1^N (f,\epsilon_i)\,\epsilon_i)\;.\;U(\sum_1^M (g,\epsilon_j)\,\epsilon_j). \tag{2}$$

If $M \rightarrow \infty$, we obtain:

$$U((\sum_1^N (f,\epsilon_i)\,\epsilon_i)\,g) = U(\sum_1^N (f,\epsilon_i)\,\epsilon_i)\;.\;U(g), \tag{3}$$

If in (3) we make $N \rightarrow \infty$, the first member tends in $L^2(F)$ to $U(fg)$, and the second in $L^1(F)$ to $U(f).\,U(g)$.

Therefore: $U(fg) = U(f)\,U(g)$, a.e. .

The theorem follows now from the next theorem 3, which is essentially Von Neumann's multiplication theorem, (cf. [3], [4] and the crossed references there mentioned).

Theorem 3. Let (X, Σ, μ) and (Y, Φ, ν) be probability spaces. If U is a unitary operator from $L^2(X)$ onto $L^2(Y)$ which verifies

(²) Given η_1 and η_2 of $F^\sim$, $\eta_1 . \eta_2$ is equal a.e. to an element of $F^\sim$ and there exists η_3 such that $\eta_1\,\eta_3 = 1$ a.e.

(³) An m-isomorphism gives a correspondence between classes of functions and - without mentioning it every time - we pick out a representative function when it is necessary.

$$U(fg) = U(f) \,.\, U(g) \quad a.\,e.,$$

for any $f \in L^2(X)$, $g \in L^\infty(X)$, *then* U *is induced by an m-isomorphism.*

Proof. If χ is a characteristic function, since $U(\chi)$ is finite a. e., we have: $(U(\chi))^2 = U(\chi)$, and therefore $U(\chi)$ is a characteristic function (a. e.). Besides, χ and $U(\chi)$ define sets of the same measure, and U determines a measure preserving mapping of Σ/N_Σ into Φ/N_Φ. From:

$$U\,(\chi_1 + \chi_2 - 2\,\chi_1\,\chi_2) = U\,(\chi_1) + U\,(\chi_2) - 2\,U\,(\chi_1)\,.\,U\,(\chi_2),$$

we see that, $U\,(\chi_1) = U\,(\chi_2)$ a. e. if and only if $\chi_1 = \chi_2$ a. e., and the mapping is one-to-one. The continuity of the operator U implies that it is a σ-isomorphism. It is also onto. In fact, it is necessary to prove that if $U(h) = \chi$, then h is a characteristic function. Let $h_n(x) = h(x)$ if $|\,h(x)\,| \leqq n$, and $= 0$ if $|\,h(x)\,| > n$. Then,

$$U(h)\,.\,U(h_n) = U(hh_n) = \chi\,.\,U\,(h_n), \text{ and}$$
$$U^{-1}\,(\chi\,.\,U(h_n)) = h\,.\,h_n\,.$$

Since $U(h_n) \to \chi$ and $h_n \to h$ we have: $\lim U^{-1}\,(\chi\,.\,U(h_n)) = U^{-1}\,(\chi)$. Besides, hh_n tends (pointwise) to h^2. Then,

$$U^{-1}\,(\chi) = h = \lim U^{-1}\,(\chi\,.\,U\,(h_n)) = h^2,$$

and h is a characteristic function.

This proves theorem 3 (and 2).

2. This paragraph deals with several generalizations of theorem 2. To find the right conditions to be imposed to the η-system, we make some observations. a) If T is a measure preserving transformation from F into G, (both compact abelian groups), it may happen that $T(F)$ is not a mesasurable set, however, it is a thick subset of G, i.e., $\mu * (G - T\,(F)) = 0$. Besides the functions $\eta_i(y) = \epsilon_i(Ty)$, $(\epsilon_i) = G^\wedge$, are measurable functions and form a multiplicative (a.e.) group isomorphic to the ϵ-system (because $T\,(F)$ being thick, is dense in G). Since:

$$\int \eta_i\,(y)\,\overline{\eta_j(y)}\;d\,\nu = \int \epsilon_i(Ty)\;\overline{\epsilon_j(Ty)}\;d\,\nu = \int \epsilon_i\,(x)\,\overline{\epsilon_j\,(x)}\;d\,\nu\,T^{-1} =$$
$$= \int \epsilon_i(x)\;\overline{\epsilon_j(x)}\;d\,\mu = \delta_{ij}\,,$$

then η-system is orthonormal.

b) For any i and y, $\eta_i(y) \in \epsilon_i(G)$. And also the functions η_i are measure preserving transformations from F into the compact subgroup $\epsilon_i(G)$ of the unit circle (always with the normalized Haar measure). This follows from the next lemma.

Lemma 1. Any character e of a compact abelian group G, is a measure preserving transformation from G onto the compact group $e(G)$.

Proof. Cf. [2], § 63.

c) If we ask $T(F)$ to be dense on G, then the family $\epsilon_i(Ty) = \eta_i(y)$ is isomorphic to (ϵ_i) without asking T to be measure preserving.

d) If T is also continuous, the η_i are continuous functions. Now we pass to the converses of the preceding observations.

Theorem 4. Let $F^\sim = (\eta_i(y))$ be a system of measurable functions on the compact abelian group F isomorphic as a multiplicative (a. e.) group to the character group $G^\wedge = (\epsilon_i)$ of the compact abelian group G. Suppose that for any i, $\eta_i(F) \subset \epsilon_i(G)$. Then there exists a measurable transformation from F into G such that $T(F)$ generates ([4]) *G and for any i, $\epsilon_i(Ty) = \eta_i(y)$ a.e.y.* (The measurability of η_i and T is with respect to the Baire σ-rings. If the measurability of the η_i's is assumed to be with respect to a σ-field containing the Baire σ-ring the same result holds).

Theorem 5. If besides of the hypotesis of theorem 4 we require the η_i to be continuous functions then T is also continuous.

Theorem 6. If besides of the hypothesis of theorem 4 we re quire every function $\eta_i(y)$ to be a measure preserving mapping from F into $\epsilon_i(G)$, then T is measure preserving.

Theorem 7. If besides of the hypothesis of theorem 4 we require the system (η_i) to be orthonormal, then T is measure preserving.

Proof of theorem 4. First of all, we want to show that we can replace the system $F^\sim = (\eta_i)$ by another one with the same properties and which is everywhere multiplicative, e., if $\eta_i \leftrightarrow \epsilon_i$ and $\epsilon_i \epsilon_j = \epsilon_k$, then $\eta_i(x)\, \eta_j(x) = \eta_k(x)$ for every $x \in F$. We give a proof by induction. Suppose that a certain subgroup $\triangle$ of $F^\sim$ has been replaced by a family $\overline{\triangle}$ in such a way that: a) if $\tilde{\eta}$ $(\in \overline{\triangle})$ replaces η $(\in \triangle)$, then $\eta = \tilde{\eta}$ a. e., b) the elements of $\overline{\triangle}$ form an everywhere multiplicative group isomorphic to $\triangle$. Let α be an element of $F^\sim$,

([4]) G is the least closed subgroup containing $T(F)$.

$\alpha \bar{\epsilon} \triangle$, an $[\alpha]$ the subgroup generated by α. If α^n does not coincide (almost everywhere) with a function of $\triangle$ whatever be $n \neq 0$, then we define $\overline{[\triangle, \alpha]} = [\overline{\triangle}, \alpha]$, where [.] indicates the subgroup generated by the set of elements contained between the brackets. If for some $n \neq 0$, α^n coincides a. e. whith a function of $\triangle$, then let m be the least positive integer with such a property. Then, $\alpha^m = \tilde{\eta}\ (\epsilon \overline{\triangle})$ a.e. Let us define $\bar{\alpha} = \tilde{\eta}^{1\setminus m}$ for every x where $\alpha^m(x) \neq \tilde{\eta}\ (x)$, and $\bar{\alpha} = \alpha$ where $\alpha^m\ (x) = \tilde{\eta}\ (x)$. Then, $[\overline{\triangle}, \bar{\alpha}]$ is an everywhere multiplicative group, and $[\overline{\triangle}, \bar{\alpha}] \sim [\triangle, \alpha]$, and we define $\overline{[\triangle, \alpha]} = [\overline{\triangle}, \bar{\alpha}]$. It only remains to prove that $\bar{\alpha}\ (F) \subset \epsilon\ (G)$, where ϵ is the image of α in the assumed isomorphism between $F^{\sim}$ and $G^{\wedge}$. It is obvious if $\epsilon\ (G)$ coincides with the unit circle. If not, $\epsilon\ (G)$ is the set of all *k-th* roots of the unity, for some k. Since $\eta\ (F) \subset \epsilon^m\ (G)$, by the inductive hyphotesis we have, $\tilde{\eta}\ (F) \subset \epsilon^m\ (G)$. From the very definition of $\bar{\alpha}$ we get $\bar{\alpha}\ (F) \subset\ \epsilon\ (G)$. Therefore, we can suppose that our system $F^{\sim}$ is an everywhere multiplicative group such that $\eta_i\ (F) \subset \epsilon_i\ (G)$.

Let $P = \Pi\, \epsilon_i\ (G)$ be the cartesian product of the image groups $\epsilon_i\ (G)$ when ϵ_i runs through $G^{\wedge}$. It is a compact abelian group. Let S be an application from F into P defined by $(Sy)_i = \eta_i(y)$. Let G' be the compact subgroup of P generated by $S(F)$. Then $S(F) \subset G'$ and $pr_i(S(y)) = \eta_i(y)$.

It is well-known that the family of proyections pr_i of P onto $\epsilon_i(G)$ is a set of generators of the free group $P^{\wedge}$. Then, the functions pr_i restricted to G' form a set of generators of the character group $G'^{\wedge}$, (cf. [9], II).

Let $c = pr_1^{a1}\ pr_2^{a2}\ \ldots\ pr_n^{an}$, ($a_i$ integers) be a character of G'. Then, $c(Sy) = \eta_1^{a1}\ \ldots\ \eta_n^{an}$ will be, by hypothesis, equal to some η-function, say η_k : $c(Sy) = \eta_k(y)$. This means that c coincides with pr_k on $S(F)$. Since G' is the least compact subgroup of P containing $S(F)$ and since c and pr_k are characters of G' *which* coincide on $S(F)$, we have: $c = pr_k$ on G'. Then $G'^{\wedge}$ is isomorphic to (pr_i), and therefore to (η_i) and to $(\epsilon_i) = G^{\wedge}$. By the Pontriagin duality theorem G and G' are isomorphic. Let ψ be the isomorphism, $\psi : G' \rightarrow G$, for which $pr_i \rightarrow \epsilon_i$, and T be defined by $T(y) = \psi\ (S(y))$. T is therefore an application from F into G. We have also: $pr_i/G' = \epsilon_i \circ \psi$, and $(\epsilon_i \circ \psi \circ S)\ (y) = \epsilon_i(Ty) = pr_i(Sy) = \eta_i(y)$. For any set M of the Borel field in the unit circle, $T^{-1}(\epsilon_i^{-1}(M)) = \eta_i^{-1}(M)$ belongs to the Baire σ-field of F, and since the family of sets $\{ \epsilon_i^{-1}(\mathrm{M}) ; \epsilon_i \varepsilon G^\wedge, M$ a Borel set $\}$ generates the σ-ring of Baire sets, we conclude that for any Baire set of G, $T^{-1}(G)$ belongs to the Baire σ-ring of F. This concludes the proof of theorem 4.

Proof of theorem 5. Since $(Sy)_i = \eta_i(y)$, S is a continuous mapping from F into G'. From $T = \psi \circ S$, we *get* the desired continuity of T.

Proof of theorem 6. This theorem reduced to theorem 7 in the following way. We have seen that $T^{-1}(\epsilon_i^{-1}(M)) = \eta_i^{-1}(M)$. From lemma 1 and the hypothesis we obtain:

$$\nu(T^{-1}(\epsilon_i^{-1}(M))) = \nu(\epsilon_i^{-1}(M)) = m_i(M \cap \epsilon_i(G)) = \\ = \mu(\epsilon_i^{-1}(M)), \tag{4}$$

where m_i is the normalized Haar measure on $\epsilon_i(G)$.

From:

$$\int \eta_i(y) \overline{\eta_j(y)}\, d\nu = \int \eta_k(y)\, d\nu = \int \epsilon_k(Ty)\, d\nu = \int \epsilon_k(x)\, d\nu T^{-1} = \\ = by\ (4) = \int \epsilon_k(x)\, d\mu = \int \epsilon_i(x) \overline{\epsilon_j(x)}\, d\mu = \delta_{ij}, \tag{5}$$

we see that the family $F^\sim$ is orthonormal. Then, the preservation of measure follows from theorem 7.

Proof of theorem 7. From (5), we get

$$\left(\sum_1^n c_i \eta_i(y), \sum_1^m d_j \eta_j(y)\right) = \sum_{i=1, j=1}^{n, m} c_i d_j \,(\epsilon_i(Ty), \epsilon_j(Ty)) = \\ = \sum c_i d_j\,(\epsilon_i(x), \epsilon_j(x)) = \left(\sum_1^n c_i \epsilon_i(x), \sum_1^m d_j \epsilon_j(x)\right) \tag{6}$$

Therefore finite linear combinations of functions of $G^\wedge$ have an image of equal norm. We want to show that $\nu(T^{-1}M) = \mu(M)$ for any Baire set M. We shall prove it for M open.

From this it follows that the preservation of measure holds for any null set. An easy L^2 approximation argument together with the last observation and (6), conclude the proof.

If M is an open Baire set, it is σ-compact, and therefore it can be constructed a sequence (g_n) of linear combinations of functions of $G^\wedge$ such that $g_n(x) \to \chi_M(x)$ everywhere and boundedly. The same happens to $h_n(y) = g_n(Ty)$ and $\chi_{T^{-1}M}(y)$. Since $\|g_n - g_m\|_2 = \|h_n - h_m\|_2$, it follows that (h_n) converges in $L^2(F)$ to a certain function which must coincide with $\chi_{T^{-1}M}(y)$. Moreover, $\|\chi_{T^{-1}M}\|_2 = \lim \|h_n\|_2 = \lim \|g_n\|_2 = \|\chi_M\|_2$.

Independece. 3. In this paragraph we want to see how the concepts of free group and independence in the sense of probability are related. We call *almost free* an abelian group G which is isomorphic to the direct product ΠZ_i of a family (Z_i), $i \in I$, of cyclic groups. We also say that a subset Γ of G is an *almost free family of generators* of G if in the isomorphism between G and ΠZ_i, the image of Γ is exactly a family of generators of the cyclic groups Z^i. Finally, we say that a set Γ of functions on a compact group G is *p-independent* if it is a set of generators of $G^\wedge$ independent in the sense of probability.

Proposition 1. Let G be a compact abelian group and $G^\wedge$ its character group. Let $\triangle$ be a subgroup of $G^\wedge$. Then Γ is a p-independent set of $\triangle$ if and only if it is an almost free family of generators of $\triangle$.

Proof. Let us see the "only if" part. First of all we observe that the results mentioned in [2], pp. 191-193 remain true, even for not necessarily real functions. We need only to prove that:

$$\prod_{j=1}^{N-1} e_{i_j}^{n_j} = e_{i_N}^{n_N} \quad \text{implies} \quad e_{i_N}^{n_N} \equiv 1 .$$

We have:

$$1 = \int \Big(\prod_{j=1}^{N-1} e_{i_j}^{n_j} \Big) e_{i_N}^{-n_N} \, d\mu = \text{(by the hypothesis of independence)} =$$

$$= \Big(\prod_{j=1}^{N-1} \int e_{i_j}^{n_j} \, d\mu \Big) . \int \overline{e_{i_N}^{n_N}} \, d\mu = \Big| \int e_{i_N}^{n_N} \, d\mu \Big|^2 , \text{ and therefore:}$$

$$1 = \Big| \int e_{i_N}^{n_N} \, d\mu \Big| .$$

From the last equality we obtain that $e_{i_N}^{n_N}$ must be identically one.

We pass now to the "if" part. Suppose first that $\triangle = G^\wedge$. Then G is isomorphic to a product of compact groups G_i and such that $(\Pi\, G_i)^\wedge = \Pi\, Z_i$. From this follows immediately the p-independence of Γ. If $\triangle$ is not all of $G^\wedge$, let us consider the subgroup $H = \bigcap \{ x \in G;\ g(x) = 1,\ g \in \triangle \}$. Observe now that the product measure of the (normalized) Haar measures on H and G/H is the Haar measure on G. Besides $\triangle$ is the character group of G/H.

These two observations reduce the case $\triangle \neq G$ to the case $\triangle = G$, Q.E.D..

To finish this paragraph we shall observe how these concepts are invariant under measure preserving transformations.

Proposition 2. Let T be a measurable transformation from F into G, F and G compact abelian groups. Let $(\epsilon_i(x)) = G^\wedge$ and $\eta_i(y) = \epsilon_i(Ty)$. a) *If $T(G)$ is dense in G then $\Gamma = (\epsilon_{i_s}\ (x))$* is an *almost free family of $G^\wedge$ if and only if $(\eta_{i_s}\ (y))$ is almost free* (5). b) *If T is measure preserving, then $(\epsilon_{i_s}\ (x))$ is p-independent if and only if $(\eta_{i_s}\ (y))$ is p-independent* (6).

Proof. a) follows from the definitions. b) is a consequence of

$$\nu\ \left(\bigcap_{i=1}^{n} \eta_i^{-1}\,(M_i)\right) = \nu\ (\bigcap_{i=1}^{n} T^{-1}\,(\epsilon_i^{-1}\,(M_i))) =$$

$$= \nu\ \left(T^{-1}\left[\bigcap_{i=1}^{n} \epsilon_i^{-1}\,(M_i)\right]\right) =$$

$$= \mu\ \left(\bigcap_{i=1}^{n} \epsilon_i^{-1}\,(M_i)\right) = \prod_{i=1}^{n} \mu\,(\epsilon_i^{-1}\,(M_i)) = \prod_{i=1}^{n} \nu\,(\eta_i^{-1}\,(M_i))\ .$$

Part II

1. Let $\{ G_i \}$ be a family of locally compact groups, pairwise disjoint and all of them commutative. We denote by G the union $\bigcup_{i \in I} G_i$ with the supremum topology, i.e., a set is open if and only if it intersects in an open set every G_i. Therefore, G is a locally compact space and every Baire (Borel) set in G is of the form $\sum_{i \in J} M_i$, $M_i \subset G_i$, where J is a denumerable subset of I and M_i is a Baire (Borel) set of G_i. For any function f on G, f_i will

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(5) In the following sense $\eta_{i_1}^{m_1} \cdots \eta_{i_n}^{m_n} = \eta^{k^l}$ implies $\eta^{k^l} \equiv 1$.

(6) In an obvious sense.

denote its restriction to the clopen subset G_i. *Evidently,* $f \in L^1(G)$ when and only when $f = \Sigma_{i \in J} f_i$, J is finite or countably infinite, $f_i \in L^1(G_i)$ *and* $\| f \|_1 = \Sigma_{i \in J} \int_{G_i} | f_i | d\mu_i < \infty$. Briefly, $L^1(G) = \Sigma L^1(G_i)$. We define now the convolution between two functions f, g of $L^1(G)$ *by*:

$$f * g = \Sigma f_i * g_i \quad . \tag{1}$$

From, $\| f * g \|_p = \| \Sigma f_i * g_i \|_p \leqslant \Sigma \| f_i \|_1 \| g_i \|_p \leqslant \leqslant \| g \|_p . \Sigma \| f_i \| = \| f \|_1 . \| g \|_p$, we see that $L^1(G)$ is a Banach algebra with this operation as multiplication. Of course, convolution is commutative, associative and bilinear. To avoid long proofs and to reduce the repetitions we stick to Loomis' book for the nomenclature and references on Banach and group algebras.

Let M be a regular maximal ideal of $L^1(G)$. From the very definitions follow that: 1) the restrictions M_i to G_i of the functions of M constitute an ideal; 2) the restriction to G_i of an identity of $L^1(G)$ modulo M, is an identity of $L^1(G_i)$ mod. M_i; 3) every M_i is maximal or equal to $L^1(G_i)$, and there is one and only one different from $L^1(G_i)$. Then,

Lemma 1. The space of regular maximal ideals of $L^1(G)$ coin*cides with the set theoretic union of the spaces of maximal regular ideals of the* $L^1(G_i)$.

Each maximal regular ideal is the kernel of a multiplicative linear functional, and conversely. Since, the space G is such that $(L^1(G))^* = L^\infty(G)$, (cf. [7], p. 43), any maximal regular ideal is associated to a function of $L^\infty(G)$. This function will be called a *character* of G. Let k be the index for which $M_k \neq L^1(G_k)$, and $a_M(x)$ the character associated to M. *For* any function f such that $f_k = 0$ a.e., it holds: $f(M) = \int f(x) \overline{a_M(x)} d\mu = 0$, since $f \in M$. Therefore, $a_M(x)$ is zero except on G_k, and obviously, coincides there with a character of G_k. Then,

Lemma 2. The characters of G define a set in one-to-one correspondence with the union of the character groups G_i; each character of G is zero on every G_i except on one of them and there coincides with a character of that group.

The topology of the family of characters $G^\wedge$ of G, is by definition the weak topology induced by the functions $\hat{f}(M) = \hat{f}(a_M) = \int f(x) \overline{a_M}(x) dx$, $f \in L^1(G)$. It follows easily that, *Lemma 3. The topology of $G^\wedge$ is equal to the supremum of the topologies of the spaces G_i. It coincides with the topology of the uniform convergence on compact sets of G.*

The last assertion of lemma 3 is very easy, and also the Pontrjagin's theorem. The space of maximal regular ideals of $G^\wedge$ is homeomorphic to G.

Moreover, two locally compact spaces like G, G^1 and G^2, such that there exists a homeomorphism which is a group isomorphism from every group contained in G^1 onto a group in G^2, will be called an *isomorphism* between G_1 and G_2. Therefore, *$G^{\wedge\wedge}$ is isomorphic to G.*

2. $L^1(G)$ has a symmetric involution defined by $f^\# = \Sigma f^\#_i$ where $f_i^\# = f_i(x^{-1})$. Obviously, $f^{\#\wedge} = f^{\wedge -}$. This involution is an isometry on $L^1(G)$.

Let $L^0(G)$ be a dense ideal in $L^1(G)$ defined as: $f \in L^0(G)$ iff $f \in L^1(G)$, f is a continuous function and $\Sigma \, || f_i ||_\infty < \infty$. Let $\phi(f)$ be the positive linear functional (i.e., $\phi\,(f * f^\#) \geqslant 0$ for any $f \in L^1$) on L^0, defined by:

$$\phi\,(f) = \Sigma f_i\,(e_i)\,,$$

where e_i is the identity of G_i.

An element p of $L^\infty(G)$ will be called *positive definite* if $\phi\,(p * f) = \Theta_p(f)$ is a positive functional on $L^1(G)$.

Auxiliary the orem. If $p \in L^0$ is positive definite and extendible (7) *then there exists a unique Baire measure m on $G^\wedge$ such that $\phi(p * f) = \int_{G^\wedge} \hat{f}(a)\,\hat{p}(a)\,dm(a) = \Sigma\,(p_i * f_i)\,(e_i)$, and $\hat{p} \in L^1\,(G^\wedge, m)$.*

Proof. Notice that $L^1\,(G)$ is semi-simple and self-adjoint. Then, the theorem follows from theorem 26. J of [7].

The restriction of p to G_i, p_i, verifies:

$$(p_i * f_i)\,(e_i) = \int_{G_i^\wedge} \hat{f}_i(a)\,\hat{p}_i(a)\,dm(a)$$

and the proof in 36. B, [7], shows that the restriction of m *to* $G_i^\wedge$ coincides with its Haar measure there.

(7) $\Theta_p(f)$ can be extended so as to remain positive when an identity is added to L_1

3. *First Bochner Theorem. The formula,*

$$p(x) = \int a(x)\, d\,\mu\,(a),$$

establishes an isomorphism between the functions $p(x) \in L^\infty(G)$ *which define positive linear functionals and the positive Baire measure* μ *such that:*

$$\int_{G_i^\wedge} d\,\mu\,(a) \leqq k < \infty, \text{ for any } i.$$

Proof. If $p \in L^\infty(G)$ defines a positive functional (that is, $(f * f^\#, p) \geqslant 0$) then p_i defines a positive functional and

$$\| p_i \|_\infty \leqq \| p \|_\infty .$$

From the Bochner theorem (36.A, [7]) it follows that: $p_i(x) = \int_{G_i^\wedge} a\,(x)\, d\,\mu_i\,(a)$, where μ_i is a Baire measure, positive and such that $\int_{G_i^\wedge} d\,\mu_i\,(a) = \| p_i \|_\infty \leqq \| p \|_\infty = k.$

Then, $p\,(x) = \int_{G^\wedge} a\,(x)\, d\,\mu\,(a)$, where $\mu\,(a)$ coincides with μ_i on G_i.

Conversely, given a $\mu\,(a)$ with the mentioned properties, $p_i\,(x) = \int_{G_i^\wedge} a\,(x)\, d\,\mu\,(a)$, is a function of $L^\infty(G_i)$ which defines a positive functional on $L^1(G_i)$. Besides, $\| p_i \|_\infty = \int_{G_i^\wedge} d\,\mu\,(a) \leqq$ $\leqq k$, and $(f * f^\#, p) = \Sigma\, (f_i * f^\#{}_i, p_i) \geqq 0$.
(As in the corollary to 36.A, [7], every p_i is essentially uniformly continuous on G_i).

Second Bochner theorem. Let $p \in L^\infty(G)$. p *defines a positive and extendible linear functional iff* $\int_{G^\wedge} d\,\mu\,(a) < \infty$.

Proof. It is a direct application of the Herglotz-Bochner-Weyl-Raikov theorem (cf. [7]).

Corollary. If $p(x)$ *defines an extendible positive linear functional then* $\Sigma \| p_i \|_\infty < \infty$. *If besides* $p(x) \in L^1(G)$, *then* $p(x) \in L^0(G)$.

Proof. It follows from the preceding theorem, observing that $\int_{G_i^\wedge} d\,\mu_i\,(a) = \| p_i \|_\infty$ and that p_i is esentially uniformly continuous.

Lemma 4. $p \in L^\infty\,(G)$ defines an extendible positive linear functional if and only if p defines an extendible positive definite linear functional.

Proof. If p defines an extendible positive functional, then $(f, p) =$ (cf. p.96, [7]) $= \overline{(f^\#, p)} = (p, f^\#) = (f, p^\#)$, and therefore, $p = p^\#$. Since $\phi\,(p * f) = (f, p^\#) = (f, p)$, p is positive definite.

If p is positive definite, the functional $(f, p^\#) = \phi\,(p * f)$ is positive and extendible, and therefore, $p^\# = (p^\#)^\# = p$ defines a positive functional.

For the group algebra $L^1\,(G)$ of a locally compact abelian group G, a linear positive functional is continuous if and only if it is extendible, (cf. [7], p. 126). However, for locally compact spaces of the type defined in the first paragraph, the continuity of a linear positive functional is not equivalent to its extendibility as first and second Bochner theorem show. For G a group, $p \in L^\infty\,(G)$ defines a positive functional iff it is definite positive ($\in L^0\,(G)$). An essential role is played by the extendibility, but this is not showed up because of its equivalence with continuity. This can be seen from lemma 4. If in that lemma we drop the condition on extendibility on the positive linear functional defined by $p\,(x) \in L^\infty\,(G)$, from Bochner theorems it follows that, in general, is not true that $p \in L^0(G)$. This different behaviour is a consequence of the lack of an approximate identity on $L^1\,(\bigcup_{i \in I} G_i)$ when I is not finite.

4. *Inversion theorem. If $p \in L^1\,(G) \cap L^\infty\,(G)$ and defines an extendible positive linear functional, then $\hat{p} \in L^1\,(G^\wedge)$ and*

$$p(x) = \int a(x)\,\hat{p}(a)\,d\,a\,,$$

where $d\,a$ is a certain measure on $G^\wedge$ which coincides with a Haar measure on every $G_i^\wedge$.

Proof. From lemma 4, it follows that $p\,(x)$ is positive definite. Since p_i is positive definite on G_i, p may be assumed to be continuous, and therefore $p \in L^0\,(G)$.

Then by the auxiliary theorem,

$$(f, p) = \phi\,(p * f) = \int_{G^\wedge} \hat{f}\,(a)\,\hat{p}\,(a)\,d\,a, \text{ and } \hat{p} \in L^1\,(G^\wedge).$$

On the other hand we have:

$$(p, f) = (f^{\#}, p^{\#}) = (f^{\#}, p) = \int \hat{f}^{\#} \hat{p} \, d\alpha = \int \hat{p} \overline{\hat{f}} \, d\alpha.$$

From the last formula we obtain:

$(p, f) = (\int \hat{p}(\alpha)\, \alpha(x)\, d\alpha, f)$ for any $f \in L^1$.

We shall denote by P ($\subset L^0$) the family of functions of $L^1 \cap L^\infty$ which define positive definite and extendible linear functionals on $L^1(G)$ and by $P^\wedge$, the analogous family on $G^\wedge$. By $[P]$ we designe the subspace generated algebraically by P.

Plancherel Theorem. The Fourier transformation $f \to \hat{f}$ preserves scalar products when confined to $[P]$. Its L^2-closure is a unitary mapping from $L^2(G)$ onto $L^2(\hat{G})$.

Proof. For $p \in P$, we have $p = p^{\#}$, and therefore $\hat{p} = \overline{\hat{p}}$. Then $\phi(p_1 * p_2 = (p_1, p_2^{\#}) = (p_1, p_2)$, equals by the auxiliary theorem do $(\hat{p}_1, \overline{\hat{p}_2}) = (\hat{p}_1, \hat{p}_2)$. Then, $(p_1, p_2) = (\hat{p}_1, \hat{p}_2)$. This equality can be extended to $[P]$ and to the L^2-clousure of $[P]$, i.e., to $L^2(G)$. The Fourier transformation is onto because it is so for $L^2(G_i)$ and $L^2(G^\wedge_i)$.

We want to prove now that $[P]^\wedge = [P^\wedge]$. Given $p \in P$, let us take the positive part q of its real component. Then, $\hat{q}$ defines a positive definite functional on $L^1(G^\wedge)$. Besides,

$$\Sigma \| \hat{q}_i \|_\infty \leqq \Sigma \| q_i \|_1 < \infty .$$

From second Bochner theorem it follows that it is extendible, and from the inversion theorem, that $\hat{q} \in L^1(G)$. Therefore, $\hat{p} \in [P^\wedge]$.

The inclusion in the other sense follows from the inversion and Pontrjagin theorems.

5. Finally, we observe that the regularity of $L(G)$, the tauberian theorem, the theorem on invariant subspaces, and the condition D for $L(G)$, admit the same statement for G a locally compact abelian group or G a locally compact space as defined in paragraph 1. The proofs are trivial or follow the same lines as given in [7]. (Under a translate of $f(x)$ in $\{y_i\}$, $y_i \in G_i$, the function equal to $f(xy_k)$ on G_k, $k \in I$, is to be understood). The generalized Wiener tauberian theorem can be translated in almost the same way as it is very easy to verify. It has no content if every G_i is compact.

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Renato Caccioppoli, *Opere,* en dos volúmenes, Cremonese, Roma 1963.

La Unión Matemática Italiana, con la contribución del Consiglio Nazionale delle Ricerche, encomendó a una comisión presidida por el Prof. Mauro Picone e integrada por ocho profesores, entre los cuales se cuentan quienes fueron discípulos avanzados y, posteriormente, estrechos amigos del singular matemático napolitano, la realización de esta obra en la que se ha reunido, prácticamente, la totalidad de las publicaciones que, desde la primera de 1926 (resumen de su tesis de doctorado) hasta la última de 1955 traducen el pensamiento científico de Renato Caccioppoli.

Las mismas se han distribuído en dos volúmenes siguiendo el criterio, según se aclara en el prefacio, de incluir en el primero los trabajos sobre argumentos de la teoría propiamente dicha de funciones de variable real: integración, totalización, funciones de conjunto, investigaciones vinculadas al análisis funcional, a las ecuaciones diferenciales ordinarias y en derivadas parciales, a las funciones de una o varias variables complejas y las referentes a las funciones pseudo-analíticas. También en el prefacio se incluye un útil comentario, a modo de orientación, de las publicaciones contenidas en el texto, señalándose en él, fundamentalmente, las ideas centrales, varias de ellas originales del propio Caccioppoli, que le sirvieron de guía en sus trabajos de investigación. Se incluye, por último, una lista en orden cronológico de la totalidad de las publicaciones por él realizadas.

La obra satisface una necesidad evidente. Los trabajos en ella reproducidos traducen (*) "una personalidad científica de un vigor y de una originalidad

(*) Giuseppe Scorza Dragoni, *Renato Caccioppoli,* Appendice necrol. ai Rend. dei Lincei, Fasc. III, Roma 1963. (Esta necrología contribuye en muexcepcional"; y, dada la naturaleza de los mismos, en varios de los cuales, sin preocuparse mucho en los detalles (como el propio autor lo señala) llega a resultados que dejan abierto un gran campo para posteriores investigaciones, es lógico compartir la esperanza de quienes inspiraron y se ocuparon de la aparición de la presente obra, cual es que la misma sea amplio motivo de inspiración para la actual y posteriores generaciones de estudiosos.

Edmundo Rofman

LORCH, EDGARD R., *Spectral theory*, Oxford University Press, New York, 1962.

Es un libro más bien pequeño (158 pgs.), de clara impresión, dividido en seis capítulos con los siguientes contenidos:

I. *Espacios de Banach*: t. de Hahn - Banach, topologías w y w', t. de Alaoglu, ejemplos y ejercicios;

II. *Transformaciones lineales*: t. de la transformación inversa, transformaciones cerradas, principio de acotación uniforme, proyecciones, t. del espacio nulo y rango de una transformación lineal, t. ergódico medio;

III. *Espacio de Hilbert*: t. de representación de F. Riesz, sistemas ortonormales, transformaciones no acotadas y adjuntas, t. de Hellinger - Toeplitz (transformaciones autoadjuntas), proyecciones, descomposición de la unidad, transformaciones unitarias, ejemplos y ejercicios;

IV. *Teoría espectral de las transformaciones lineales*: espectros, procedimientos de integración, las proyecciones fundamentales, radio espectral, funciones analíticas de operadores;

V. *Estructura de las transformaciones autoadjuntas*: operadores definido positivos, espectro puntual, descomposición en tipos puros, el espectro continuo;

VI. *Algebras de Banach conmutativas*: la representación regular, reducibilidad e idempotentes, álgebras que son cuerpos, ideales, álgebras cocientes, homomorfismos e ideales maximales, el radical, representación, ejemplos y aplicaciones.

En un estilo ágil y claro el autor presenta los resultados fundamentales en la teoría de espacios de Banach y Hilbert, descomposición espectral de operadores autoadjuntos y álgebras de Banach conmutativas con unidad, pudiendo recomendarse este libro a todo aquel que desee iniciarse en esas disciplinas y que no cuente entre sus conocimientos más que elementos de topología general, de funciones analíticas y teoría de la integral.

Muchos de los ejemplos y ejercicios, por otra parte muy bien seleccionados, son teoremas que amplían la teoría general. La representación integral, $A = \int \lambda \, dE \, \lambda$, para operadores autoadjuntos es tratada como fue publicada por el autor en el Acta Szeged (1950), tratamiento que posee la ventaja de ser válido para operadores acotados y no acotados.

R. Panzone

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cho a definir las notables características de la figura de Caccioppoli, dado que destaca aspectos de la misma no mencionados en el prefacio de la obra que comentamos).

CORRELACIONES ANGULARES EN Hg^{198}

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RESUMEN: Con el objeto de establecer la relación de mezcla en la transición de $2' \rightarrow 2$ (*) en el Hg^{198}, se realizó una experiencia de correlaciones angulares. La relación de mezcla obtenida es $\delta = -1.10 \pm 0.35$.

A diferencia de dos experiencias realizadas anteriormente, se utilizó una fuente líquida de Au^{198}. El acuerdo del presente resultado con los anteriores demostró que el estado de la fuente no influía en el resultado de la experiencia.

1. *Introducción*

Medidas de la relación de mezcla en la transición $2' \rightarrow 2$, de 0.680 Mev, en el Hg^{198}, fueron realizadas anteriormente por D. Schiff y F. Metzger (1), y C. Schrader (2). Estos autores obtuvieron independientemente la evidencia de que el carácter dipolar era comparable al carácter cuadripolar de la transición.

Por otra parte es sabido que, sistemáticamente, en los núcleos par-par, el carácter $M1$ en las transiciones $2' \rightarrow 2$ es despreciable frente a la componente $E2$ (3), siendo el Hg^{198} una excepción en este sentido.

Debido a que en las experiencias anteriores se utilizaron fuentes sólidas de Au^{198}, nuestro propósito fue investigar si esta circunstancia tenía influencia en los resultados a través de campos internos que afectaran la correlación angular.

2. *Preparación de la fuente*

La fuente se obtuvo bombardeando Au^{197} (n, γ) Au^{198}, en el reactor $RA1$ de la Comisión Nacional de Energía Atómica, con un flujo de 10^{11} neutrones/seg. cm^2.

El Au^{197} fue previamente disuelto en agua regia y se llevó a sequedad varias veces, agregando agua destilada para extraer el resi-

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(*) Transición del segundo nivel 2+ al primer 2+.

duo sólido. Con esta solución de ácido tricloroáurico se llenó la cápsula de lucite utilizada como soporte de la fuente durante la experiencia.

3. *Equipo y procedimiento*

Se utilizaron cristales de NaI(Tl) de 1.5 por 1.5 pulgadas colocados a 5 cm. de la fuente y montados sobre fototubos 6655A con blindaje magnético.

El cociente de coincidencias reales a coincidencias casuales es

$$I\ (0.680\gamma,\ 0.411\gamma)\ /\ I\ (\text{coinc. cas.}) \equiv \frac{1}{2\tau A}$$

FIG. 1

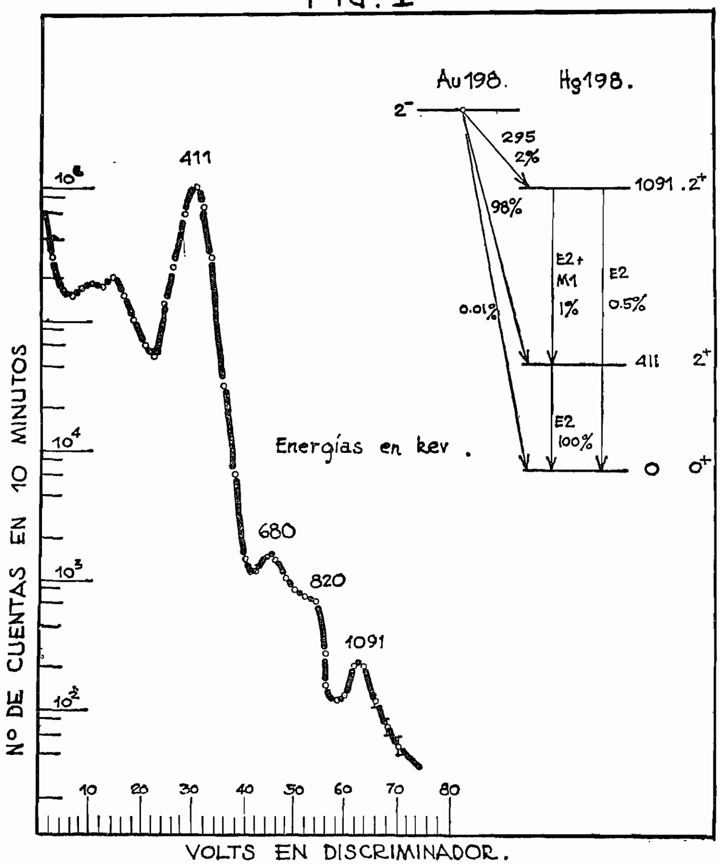

Fig. 1 — Espectro y esquema del decaimiento del Au^{198}

siendo τ el tiempo de resolución del circuito de coincidencias y A la actividad de la fuente (aproximadamente igual a la intensidad de la transición $2 \rightarrow 0$).

Para optimizar, entonces, las condiciones de la experiencia debe reducirse tanto como sea posible τ. El límite del tiempo de resolución τ es del orden del tiempo de decaimiento de los cristales de $NaI(Tl)$ si se dispone de preamplificadores y amplificadores adecuados.

El circuito de coincidencias rápidas utilizado, fig. 2, tiene una resolución de 3×10^{-8} seg, fig. 3, variable mediante el potenciómetro 1 de la fig. 2 hasta 6×10^{-8} seg. El pulso de salida conformado de 4.5 volts tiene una duración de 2 microseg.

FIG. 2.

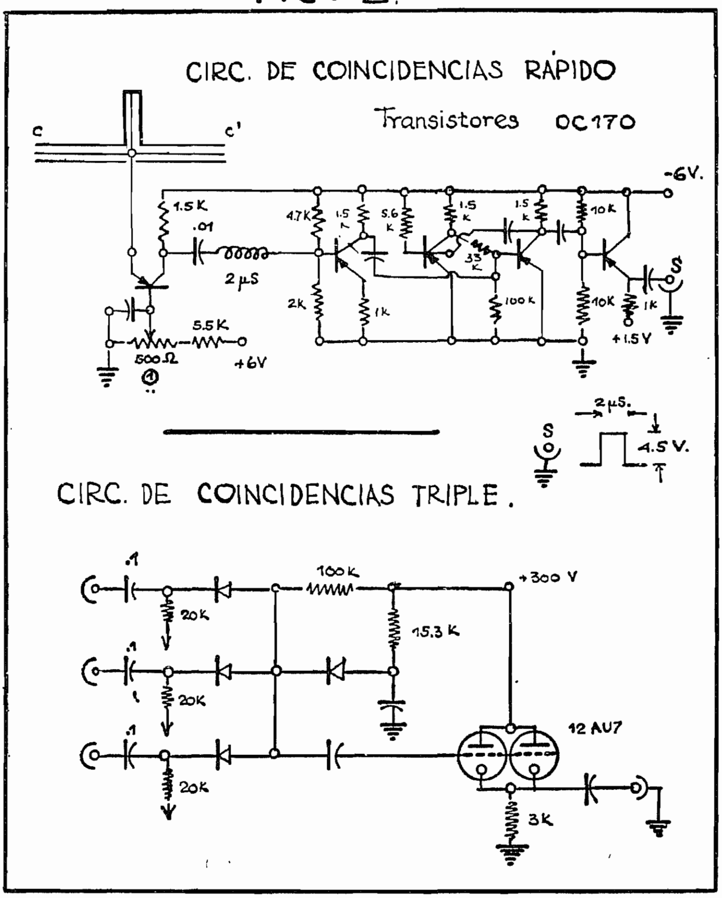

Fig. 2 — Esquema de los circuitos de coincidencias rápidas y lentas.

Para evitar que la radiación de frenamiento de las partículas beta emitidas por el Au^{198}, fondo cósmico y backscattering, contribuyan a la intensidad de las coincidencias casuales se emplearon equipos selectores.

El backscattering es particularmente importante para ángulos mayores de 140°. La energía de esta radiación es aproximadamente de 200 kev. Como, afortunadamente, ambos rayos de la cascada $2' \rightarrow 2 \rightarrow 0$ tienen una energía mayor que 400 kev, con los selectores se impidió totalmente la contribución del backscattering a las coincidencias casuales. Estos se incluyeron en el equipo junto a un circuito de coincidencias triple, de baja resolución, fig. 2, conectado como se indica en el diagrama en block de la fig. 4.

FIG. 3

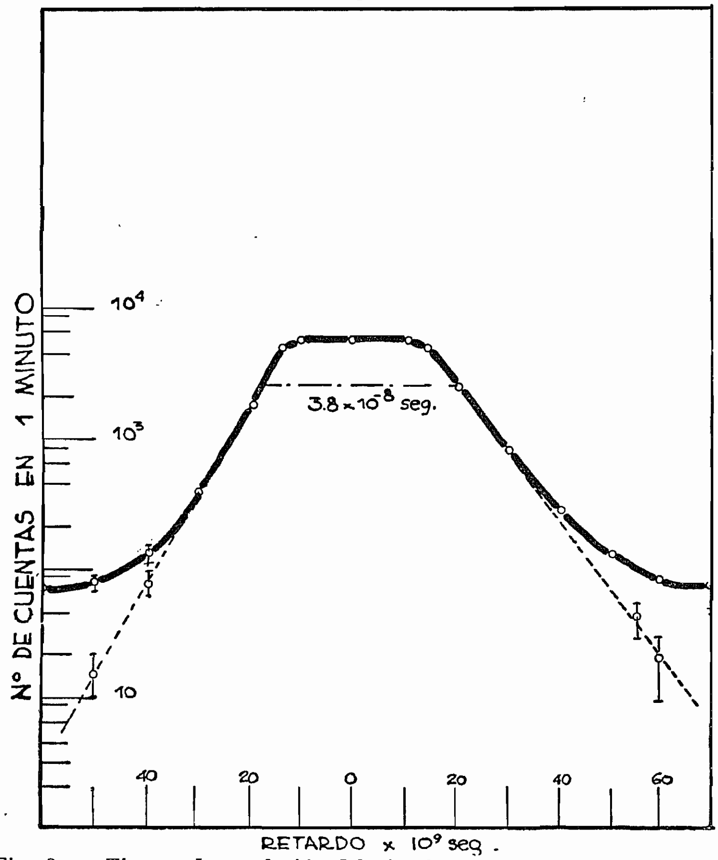

Fig. 3 — Tiempo de resolución del circuito de coincidencias rápidas.

Un espectro diferencial del Au^{198} medido con uno de los contadores, se muestra en la fig. 1. El pico que se insinúa a 0.820 Mev es debido a la coincidencia del rayo de 0.411 Mev consigo mismo dentro del cristal.

La magnitud que finalmente se mide es $W(\theta)$. Esto es el cociente entre las coincidencias reales obtenidas para cada ángulo θ, y la intensidad medida en el contador móvil, corrigiendo debidamente por el decaimiento de la fuente y pequeñas desviaciones en el descentrado de la misma.

Por otra parte, la teoría muestra que la función $W(\theta)$ se puede expresar como un desarrollo en polinomios de Legendre. Puesto que sólo consideramos probabilidades relativas es legítimo normalizar poniendo $A_0 = 1$

$$W(\theta) = 1 + \sum_{k-1}^{kmáx} A_k P_k \ (\cos \theta)$$

En este caso en donde se trata de una experiencia de correla-

FIG. 4

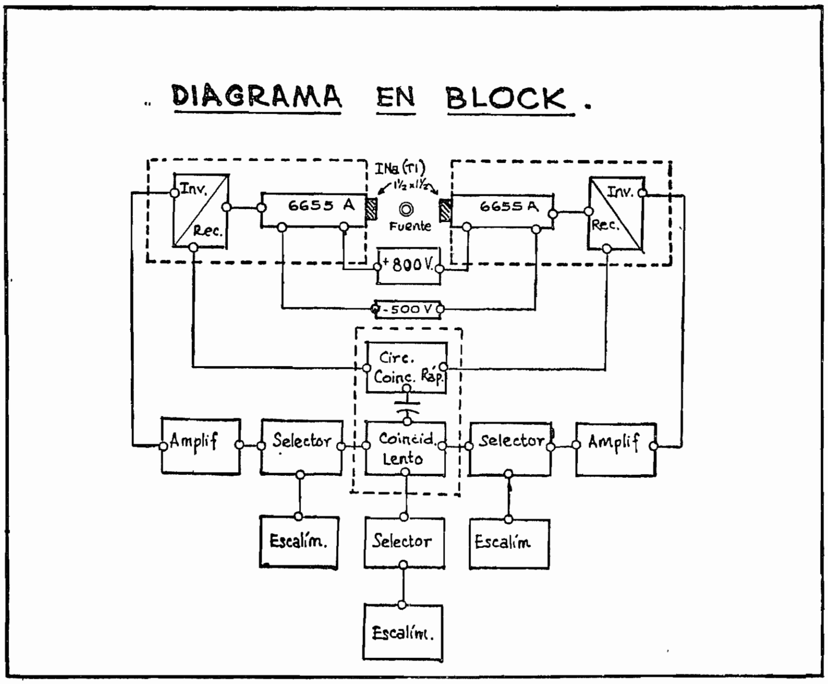

Fig. 4 — Diagrama en block del equipo utilizado.

ciones angulares sin considerar polarización alguna, A_k es nulo si k es impar. El k máximo está fijado por la condición anterior y por el spin I del nivel intermedio. Explícitamente

$$k\ (\text{máx.}) \leqq 2\,I$$

En nuestro caso $I = 2$ y por lo tanto

$$W(\theta) = 1 + A_2\, P_2 + P_4\, P_4$$

Utilizando los resultados de las mediciones realizadas a 90°, 135°, 180°, 225° y 270°, y mediante un ajuste por cuadrados mínimos, se obtuvo

$$A_2 = -\,0.258 \pm 0.064$$

$$A_4 = \quad 0.18 \ \ \pm 0.07$$

4. *Resultados y discusión*

Por sistemática de los núcleos par-par, el spin y la paridad del estado fundamental del Hg^{198} es 0^+. Medidas precisas del coeficien-

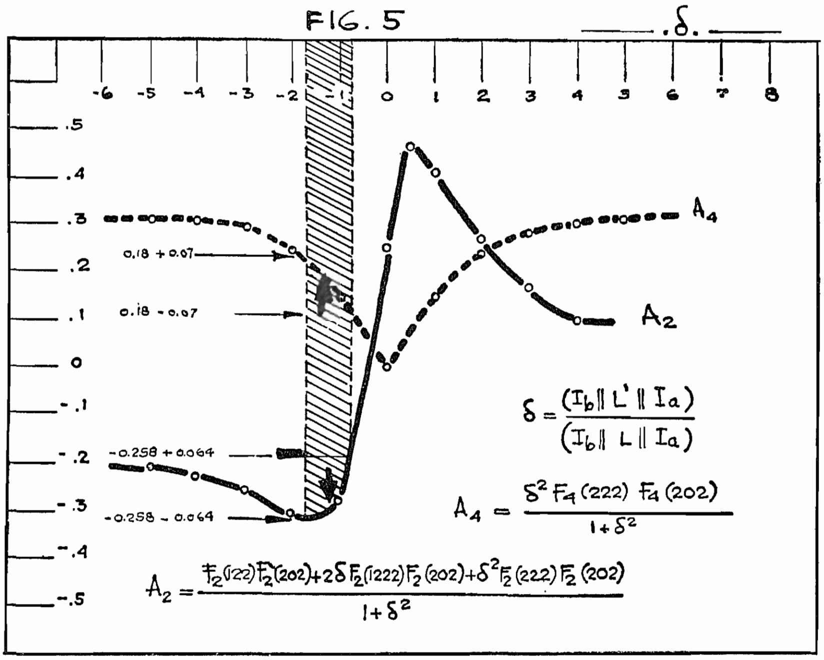

Fig. 5 — Gráfico de las funciones $A_2(\delta)$ y $A_4(\delta)$ y determinación de la relación de mezcla.

te de conversión interna (4) y (5) y excitación coulombiana (6) indican que la transición de 0.411 Mev, es cuadripolar eléctrica pura. Por tanto el spin y la paridad del nivel de 0.411 Mev es 2^+.

El spin del nivel de 1.091 Mev no puede ser 0 ni mayor que 3 porque existe la transición gama de 1.091 Mev y ésta tiene una intensidad comparable a la transición de 0.411 Mev.

Los coeficientes A_k teóricos para cascadas $X \rightarrow 2 \rightarrow 0$, $(X = 1$, 2 ó 3), para transiciones puras fueron comparadas con los obtenidos en la presente experiencia, una vez corregidos por la resolución angular de los contadores (7). En ningún caso se encontró acuerdo dentro de los errores experimentales.

Para el caso en que la primera transición sea una mezcla de carácter dipolar-cuadripolar, es conveniente definir una medida de la relación de mezcla δ tal como el cociente entre el elemento de matriz reducida de la transición cuadripolar y la transición dipolar, fig. 5. Los A_k resultan, entonces, ser funciones de δ.

Para las cascadas $1 \rightarrow 2 \rightarrow 0$ y $3 \rightarrow 2 \rightarrow 0$ se obtiene $A_4 \leqq 0$ para cualquier δ. Como el A_4 medido es positivo, el spin del nivel de 1.091 Mev es 2. La paridad es $+$ puesto que no es posible esperar una mezcla $E1$-$M2$.

La fig. 5 es el gráfico de las funciones $A_2(\delta)$ y $A_4(\delta)$ para el caso en que la cascada es $2 \rightarrow 2 \rightarrow 0$.

Los valores medidos de A_2 y A_4 determinan la relación de mezcla

$$\delta = -1.10 \pm 0.35$$

que debe ser comparada con (**)

$\delta = -1.22 \pm 0.22$ referencia (1)

$\delta = -0.96 \pm 0.10$ referencia (2)

demostrando que la experiencia es independiente del estado (sólido o líquido) de la fuente dentro de los errores experimentales.

(**) Estos valores son los adoptados por Nuclear Data Sheets, National Academy of Sciences, National Research Council, Washington, D. C.

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ANGULAR CORRELATION IN Hg^{108}

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Abstract: The angular correlation of the $2' \rightarrow 2 \rightarrow 0$ cascade in Hg^{198} has been measured in order to determine the mixing ratio in the $2' \rightarrow 2$ transition. The obtained result is $\delta = -1.10 \pm 0.35$.

Unlike in two previous experiments, a liquid source of Au^{108} was used. The agreement between the present work and the other two, has shown that the state of the source had not influenced the result of the experiment.

MEASURABLE TRANSFORMATIONS ON COMPACT SPACES AND O. N. SYSTEMS ON COMPACT GROUPS

by R. PANZONE and C. SEGOVIA

1. SUMMARY

This paper is devoted to the problem of finding a pointwise transformation which induces a given measure preserving Boolean isomorphism between the measure algebras of two measure spaces. We only consider spaces with a connection between measure and topology. Precisely, Hausdorff compact spaces with a regular measure on its σ-field of Borel sets and such that every open set is of positive measure. The applications are concerned with necessary and sufficient conditions on an orthonormal system of measurable functions on a compact abelian group G to be the image of the character group $G^\wedge$ under measurable transformations on the original group G. Therefore, in this point, the paper is a continuation of [1].

I. INTRODUCTION

2. *Nomenclature.* X and Y will always design compact Hausdorff spaces; μ and ν, regular probalitiy measures, positive on non void open sets. $B_a(X)$, $B_0(X)$, will design the σ-fields of Baire and Borel sets of X, respectively, and G_X, the compact space of all the multiplicative linear functionals on $L^\infty(X, B_0, \mu)$. In other words, the family of continuous functions on G_X, $C(G_X)$, is the Gelfand representation of $L^\infty(X, B_0, \mu)$. $A_X = A(X) = B_0(X) / N(X)$ will denote the σ-algebra of Borel sets mod. the null Borel sets, and $B_0(X)^*$, the completion of B_0. The extension of μ to $B_0{}^*$ will be denoted again by μ, occasionally by μ^*. $\Sigma(X)$, or Σ_X, will always represent a σ-field containing $B_a(X)$ and contained in $B_0(X)^*$.

N, Q, R, I and K will be the spaces of positive integers, rational and real numbers, the interval $[0,1]$, and the unit circle of the complex plane, respectively, all of them with their usual topologies. For Z completely regular, βZ will denote its Stone-Č ech compactification. $A < B$ means A included in B, and $|A|$, the power of A.

ω and Ω are, respectively, the first infinite ordinal number and the first non conuntable ordinal number.

3. *The problem.* Given a measure space (X, B_a, μ) and a measure preserving Boole σ-isomorphism from $B_a/N_a(X)$ onto itself, find a one-to-one pointwise transformation, measurable in both senses, which induces the prescribed Boole isomorphism, (cf. [5], p. 1022). For all the well known measure spaces the answer is affirmative, (cf. [7] and [5]). In general, the answer is no. We settle again the problem and try to determine what must be meant by a solution.

Let (X, B_a, μ) and (Y, B_a, ν) be measure spaces, and τ a measure preserving Boole σ-isomorphism from $B_a(X) / N_a(X)$ onto $B_a(Y) / N_a(Y)$. If we want to find a pointwise transformation which induces τ, *possible solutions* are:

I. There exist two transformations, $T_X : X \rightarrow Y$, $T_Y : Y \rightarrow X$, both measurables, such that $T_Y{}^{-1}$ induces τ and $T_X{}^{-1}$ induces τ^{-1}.

II. There exists an invertible, measurable transformation from a set of measure one, $S_X < X$, onto a set S_Y of measure one contained in Y.

III. There exist two sets, $O_X, O_Y, O_X < X, O_Y < Y$, of exterior measures one (i.e., thick sets) and a one-to-one mapping, T, invertible and measure preserving as a transformation from $(O_X, B_a(X) \frown O_X, \mu \mid O_X)$ onto $(O_Y, B_a(Y) \frown O_Y, \mu \mid O_Y)$.

The solution I admits two subcases: the T's are onto, the T's are into. The first one is immediately excluded since it may happen that $B_a(X) / N(X) \sim B_a(Y) / N(Y)$ and $|X| > |Y|$, cfr. next example a). The second one will be discussed in a moment. The solution II is again in generale false, even when $L^1(X)$ is separable, cf. ex. c). However, it holds for a significant class of spaces. Among them are the separable metric spaces. We do not know whether III is true. Nevertheless, for a wide class of spaces, which includes compact abelian groups and metric spaces, a slight variant of III holds.

Let us observe that there is no loss of generality in considering $X = Y$. Put $Z = X \smile Y$ with the sum topology, and with the measure $\mu/2$ on X and $\nu/2$ on Y. Define now a σ-isomorphiam from $A(Z)$ onto itself as follows: if $A \in A(X)$ and $B \in A(Y)$, then $\sigma(A + B) = \tau^{-1}(B) + \tau(A)$. Therefore $\sigma = \sigma^{-1}$ and, for example, a pointwise transformation of type II induces τ, if and only if a pointwise transformation of the same type induces σ.

4. In the proposed problem what is given beforehand is A_X ($= A_Y$), two measures, μ, ν on it, and a measure preserving σ-isomorphism τ. A restriction on the type of measure spaces to be considered is imposed. This restriction is reasonable and practical, but otherwise arbitrary. The condition on the positiveness of μ on open sets confines the null sets to proper limits, and in particular, avoids the possibility of adjoining to the space null sets topologically important. In other words, it excludes certain patologies. Finally we arrive to the σ-fields B_a's. It is not clear if it must be chosen the Baire σ-field, the Borel σ-field, or any other with the same σ-algebra, A_X Our position is to suppose that the measures spaces are (X, Σ_X, μ), (Y, Σ_Y, ν), with $B_a(X) < \Sigma(X) < B_0^*(X)$, $B_a(Y) < < \Sigma(Y) < B_0^*(Y)$ and μ, ν restrictions to the Σ's of regular Borel measures. Of course, Σ_X mod. null sets is A_X. The discussion on the type of solutions can be repeated. If the Σ's coincide with the $B'_a{}^s$ we do not know if there is a counterexample to the first solution. However, if $\Sigma = B_a$ for one space and $\Sigma = B_0^\#$ for the other, a solution of type I may be impossible as is shown in example b). A certain variant to the solution of type I is true.

5. *Examples.* a) Let $X = N^*$ be the one-point compactification of N, and $Y = \beta N$, with $\mu(n) = \nu(n) = 2^{-n-1}$, $n \in N$. It is well-known that $| \beta N | = 2^c$, and obviously, their measure algebras are equivalent.

b) We want to give an example of a measure-preserving isomorphism from $A(X)$ onto $A(Y)$ which is not induced by a measurable transformation. Let the measure spaces be (I, B_0, m) and $(I, B_0^\#, m)$ where m is the Lebesgue measure. Assume τ is the identity application from $A(X) = A(Y)$ onto itself. If τ is induced by a measurable transformation T from (X, B_0) into $(Y, B_0^\#)$, then, as it is easy to see, $Tx = x$ a.e.. Therefore, $T(X)$ is a set of measure one, and, consequently, it contains a null set of the continuum power. Then, $T(X)$ contains 2^c null sets, and B_0 must contain 2^c Borel sets, contradiction.

c) $X = I$, $\mu =$ the Lebesgue measure on the Baire sets. $Y = \beta Q$ and ν the Baire measure induced on $B_a(Y)$ by the continuous transformation T which is the extension to βQ of the natural mapping from Q onto the rational numbers of I. (Assume F is a closed set of positive measure contained in X, then ν is defined by $\nu(T^{-1} F) = = \mu(F)$.) ν is the restriction to $T^{-1}(B_a(I))$ of a regular Borel measure on βQ and in such a way that every set of $B_0(\beta Q)$ is equivalent to some element of $T^{-1}(B_a(I))$. Almost all the arguments to prove this appear later in the lext. Assume now that F is a closed set, $F < \beta Q$, $\nu(F) > 0$. Since every point of $Q < \beta Q$ is of (extended) measure zero, there exists a closed set $F_1 < F$, of positive measure and such that $F_1 \frown Q = \emptyset$. Therefore $|F_1| = 2^c$ (cf. [3], ch. 9). Hence, $|F| = 2^c$. The regularity of ν implies that every measurable set of positive measure is of power 2^c. Any image if I into βY is of power $\leqq c$; hence, is not measurable, or of measure zero. Observe that in this example (and also in example a)) $B_0(\beta Q) \neq B_a(\beta Q)$. In fact, no point of $\beta Q - Q$ is a $G\delta$, (cf. 3, ch. 9).

d) Let G be the functions from R into K with the pointwise topology i.e., $G = K^R$. G is a compact abelian group, which has not a countable basis, and $L^2(G)$, and therefore $L^1(G)$, is not separable. However, G is seperable, since the simple functions with rational values taken on rational intervals are a countable, dense set.

6. *Solutions.* A space $(X, \Sigma(X), \mu)$ will be called *rich* if for any Borel measurable set M of positive measure: $|M| \geqq |A(X)|$. It is not true that any compact space is rich, take for example the Alexandroff compactification N^* of N, and an arbitrary probability measure on N. Examples of rich spaces are those separable, metric spaces, i.e., closed subsets of I^ω, that admit probability measures vanishing on every point. In fact, every $G\delta$ is denumerable or has power of the continuum (cf. [14], 320, 465), therefore, any set of positive measure has power c. Since such spaces have a countable basis, $|B_0(X)| = c$, and therefore they are rich spaces.

Rich spaces are also the compact abelian groups with the only exception of the discret ones. This will be proved later.

Other concept we need is that of null extensión. Let (Z, Φ_1, α_1) and (Z, Φ_2, α_2) be arbitrary probability spaces and suppose that $\Phi_1 < \Phi_2$. If $\alpha_1 = \alpha_2$ on Φ_1 and if every set $B \in \Phi_2$ is α_2-equivalent to a set $A \in \Phi_1$, then we say that (Z, Φ, α_1) is a *null contraction* of, or admits as a *null extensión* (Z, Φ_2, α_2). We say that the spaces (Z, Φ_i, α_i), $i = 1,2$, or the σ-fields Φ_i are *comparable* if there exists a common null extension (Z, Φ_3, α_3). Null extensions define a partial order. We shall see later that there does not exist necessarily the supremum null extension of two given spaces, neither that comparability is an equivalence relation. An example of null extensión is the usual completion of a measure space. Other: let Z be a compact, Hausdorff space such that $B_a(Z) \neq B_0(Z)$, μ a measure on B_a and ν its regular extension to B_0. (Z, B_0, ν) is a null extension of

(Z, B_a, ν) as is easily seen taking into account that if U is open, K compact and $U \supset K$, there exists K_0, compact and G_δ such that $U \supset K_0 \supset K$. Assume that there exists a null Baire set such that one of its subsets is not Borel measurable. Then, the completion of B_a, B_a^*, is different of B_0. However, they are comparable, for they are null contractions of $B_0(Z)^*$. A trivial example of this is $Z = K^\Omega$ with μ concentrated at the point 0. The set $[1/2, 3/4] \times \times \Pi [K \alpha ; 1 < \alpha]$ is a Baire set of measure zero and contains a non measurable subset. (For this particular example $B_a^* \neq B_0^*$; consider the point 0).

For the spaces which interest us, two Σ's between $B_a(X)$ and $B_0(X)^*$ are, obviously, comparable.

A space (X, Σ, μ) we shall say to be *a.e. separated* if it admits a finite-valued family F of functions, dense in L^1 and such that for any pair of points a, b, of a set of measure one, there exists an $f \in F$ with $f(a) \neq f(b)$. If F is countable, we shall say that X is *a.e. ω-separated.* Every compact metric space X admits a countable family of continuous functions that separates X. However not every separable space is a.e. ω-separated. In fact, example d) shows a separable space whose L^1 is not separable.

Now we are ready to state some of the main theorems of the paper. Their proofs will be given in part III.

Theorem A. Let (Y, B_0, ν) and (X, B_0, μ) be a.e. ω-separated. Assume τ is a measure preserving σ-isomorphism from $A(Y)$ onto $A(X)$. There are two measurable sets $X_0 < X$, $Y_0 < Y$, both of measure one and an invertible measurable transformation from X_0 onto Y_0 such that for any $C \in B_0(Y)$, $T^{-1} (C \frown Y_0) \in \tau(C)$, and for any $D \in B_0(X)$, $T(D \frown X_0) \in \tau^{-1}(D)$.

This result is due to P. Halmos and J. von Neumann.

Observations. 1) The preservation of measure is irrelevant. What is really important is that $A(X)$ and $A(Y)$ be isomorphic. If τ does not preserve measure, $\mu\, 0\, \tau$ defines a measure on $B_0(Y)$ such that $d\, (\mu\, 0\, \tau) = f\, d\, \nu$, where f is a Radon-Nikodym derivative with $1/\epsilon \geqq f \geqq \epsilon > 0$, a.e.. 2) In relation with this result, cf. the paper by Halmos and von Neumann, [7], where the authors give neccessary and sufficient conditions on an arbitrary probability space to be in one-to-one measurable correspondence with a subset of the unit interval I. Of course, a.e. ω-separability becomes, in this situation, a necessary assumption.

Theorem B. Suppose τ *is a measure preserving Boole* σ*-isomorphism from* (Y, Σ_Y, ν) *onto the rich space* (X, Σ_X, μ)*. There exists a* σ*-field* Γ *comparable with* Σ_X*, and a measurable pointwise transformation* T *from* (X, Γ, μ) *into* (Y, Σ_Y, ν)*, such that* T *induces* τ*.*
(X, Σ_X, μ) will be called **-rich* if $|N| < |A_X|$, $N < X$, implies $\mu^{\#}(N) = 0$. Every **-rich space is rich.*

Theorem C. Suppose τ *is a measure preserving Boole* σ*-isomorphism from* (Y, Σ_Y, ν) *onto* (X, Σ_X, μ)*,* X *and* Y *being *-rich spaces.*

a) There exists a σ*-field* Γ *comparable with* Σ_X*, and a measurable pointwise transformation* T *from* (X, Γ, μ) into (Y, Σ_Y, ν)*, such that* T *induces* τ*. There are two thick sets* $\mathcal{X} < X$*,* $\mathcal{Y} < Y$*, such that* T *is one-to-one from* $\mathcal{X}$ *onto* $\mathcal{Y}$*, and it is an invertible measure preserving transformation from* $(\mathcal{X}, \Gamma \frown \mathcal{X}, \mu)$ *onto* $(\mathcal{Y}, \Sigma_Y \frown \mathcal{Y}, \nu)$*, (also,* $(T \mid \mathcal{Y})^{-1}$ *induces* τ^{-1}*).*

b) Moreover, there exists a σ*-field* Φ *comparable with* Σ_Y*, such that* T *restricted to* $\mathcal{X}$ *is an invertible measure preserving transformation from* $(\mathcal{X}, \Sigma_X \frown \mathcal{X}, \mu)$ *onto* $(\mathcal{Y}, \Phi \frown \mathcal{Y}, \nu)$*,and* $\mathcal{X}$ $(\mathcal{Y})$ is a *thick set with respect to sup* (Σ_X, Γ) *sup* (Σ_Y, Φ)*).*

II. RICH SPACES AND NULL EXTENSIONS

This part is devoted to examples and to exhibit large classes of spaces which satisfy some of the properties of the title. It is independent of next part III, where the theorems mentioned above are proved.

7. *Null extensions.* More artificial but also more illuminating is the following example. Let $X = I$ and $\mu =$ Lebesgue measure. The proof given in [4], § 16, shows the existence of a sequence of pairwise disjoint sets, $\{E_j\}$, $j = 1,2, \ldots$, such that $\sum_{i=1}^{n} E_{j_i} \overline{\epsilon} B_0(I)^*$, whatever be n and $j_1, \ldots, j_n$, with $\sum_{1}^{\infty} E_i = I$. Moreover, for any n, $\mu_{\#}(\sum_{i=1}^{n} E_{j_i}) = 0$. The family of sets $M_1 = \{A \Delta B;\ A \in B_0^*, B \subset E_1\}$ is a σ-field, and $\mu_1(A \Delta B) = \mu(A)$ defines a measure on M_1. It is a null extension of (I, B_0^*, μ). $E_2 \overline{\epsilon} M_1$. In fact, if $E_2 \in M_1$, $E_2 = (M - E_1') + E_1''$, where $M \in B_0$, $E_1' < M \frown E_1$, $E_1'' < E_1$, $E_1'' \frown M = \emptyset$. Then, $E_2 + E_1' = M + E_1''$, and therefore, $E_1'' = \emptyset$. Since $E_2 + E_1' = M < E_2 + E_1$, we have $\mu_{\#}(M) = \mu(M) = 0$. This implies the measurability of E_2, contradiction. Repeating the argu-

ments, they exhibit a strictly increasing sequence: $(I, B_0^*, \mu) < < (I, M_1, \mu_1) < \ldots < (I, M_n, \mu_n) < \ldots$, each space a null contraction of the following and $E_i \in M_i$. Since $\Sigma E_i = I$ this sequence does not admit a bound, i.e., there is no common null extension. Moreover, if we adjoin to B_0^*, instead of E_1, $\Sigma_3^\infty E_j$, we obtain (I, M, μ'). It follows immediately that this is a null extension of (I, M_n, μ_n), $n \geqq 3$, but it is not comparable with (I, M_1, μ_1). In particular, the null extensions define a partial order which is not a lattice.

8. *Rich spaces.* We show here that every non discret compact group is rich.

Theorem 1. *a) Le G be a non finite compact abelian group and $G^\wedge$ its character group. $G^\wedge$ contains a subgroup $H^\wedge$, direct product of cyclic groups, with $| H^\wedge | = | G^\wedge |$.*

b) $| G | = 2^{|G^\wedge|}$.

Proof. a) We call M the family of all sets $A < G^\wedge$ such that the group generated by A, $[A]$, is direct product of cyclic subgroups of $G^\wedge$, $Z(\alpha)$, with $\alpha \in A$. If B is a chain of M, ordered by inclusion, and $D = \cup \, [A \in B]$, then D is a bound of B. In fact, if

$$Z(\alpha_0) \cap [Z(\alpha_1) \cup Z(\alpha_2) \cup \ldots \cup Z(\alpha_n)] = S$$

there exists $A \in B$ such that $A \ni \alpha_j$, $j = 0, 1, \ldots, n$, and therefore $S = \{1\}$. Call H a maximal element of M and $H^\wedge = [H]$. Since H is maximal, for any $\alpha \in G^\wedge - H^\wedge$, there exists $n \geqq 2$ such that $\alpha^n \in H^\wedge$. Put

$$E^n = \{ \alpha;\ \alpha^n \in H^\Delta \text{ and } \alpha^m \overline{\in} H^\Delta \text{ for } 1 \leqq m < n \}.$$

Then, $G^\wedge - H^\wedge = \Sigma_{n=2}^\infty E_n$. If $| H^\wedge | < | G^\wedge |$, then $| G^\wedge - H^\wedge | = | G^\wedge |$, and for some n, $| E_n | = | G^\wedge |$. Suppose p is the least index for which $| E_p | = | G^\wedge |$. The application $\alpha \to \alpha^p$ maps E_p into $H^\wedge$. Pick out a $\beta \in H^\wedge$ such that the set

$$\Gamma = \{ \alpha \in E_p;\ \alpha^p = \beta \}$$

is of cardinality $| G^\wedge |$. If $\alpha_0 \in \Gamma$, the set $\alpha_0^{-1}. \Gamma$ is also of cardinality $| \Gamma | = | G^\wedge |$. Since $| H^\wedge \smile E_2 \smile \ldots \smile E_{p-1} | < | \alpha_0^{-1} \Gamma |$, there exists $\delta \in \Gamma$ sucha that $\alpha_0^{-1} \delta \overline{\in} H^\Delta \smile E_2 \smile \ldots \smile E_{p-1}$. Hence, $\alpha_0^{-1} \delta \in E_p$, and $H + \{ \alpha_0^{-1} \delta \} \in M$, contradiction.

b) Let be $D = \{ x;\ \alpha(x) = 1 \text{ for any } \alpha \in H^\Delta \}$. D is a closed subgroup of G and $(G/D)^\wedge = H^\wedge$. Then, G/D is isomorphic to

$\Pi\ [Z\ (a)^{\wedge}; a \in H]$ where the Z's are (non trivial) compact groups. Then, $|G| \geqq |G/D| \geqq 2^{|H|} = 2^{|G^{\wedge}|}$. The application $f: G \to K^{G^{\wedge}}$ defined by $f(x) = (a(x))_{a \in G^{\wedge}}$ is one-to-one and therefore, $|G| \leqq \leqq |K^{G^{\wedge}}| = 2^{|G^{\wedge}|}$, *Q.E.D.*

Calling $A(G)$ the Boolean σ-algebra of Baire sets mod. the null sets with respect to the Haar measure, we have:

Theorem 2. *a*) $|A(G)| = |G^{\wedge}|^{\aleph_0}$

b) *If* M *is a Baire measurable set and* $\mu(M) > 0$, then $|M| = |G|$.

c) *The same statement holds for Borel sets.*

(The appearing exponent represents the first infinite cardinal number).

Proof. a) Since $A(G)$ is isomorphic to $A(K^{G\Delta})$, (cf. [1], intr., Lemma 2), it is enough to prove a) for the group $K^{G\Delta}$. *Let* $\{ a_n \}$ be a sequence of positive numbers, $a_n < 1$, such that $\Pi_1^{\infty}\, a_n = \frac{1}{2}$. Consider the family of infinite, countable subsets of $G^{\wedge}$. Suppose $S = \{ \sigma_n \}$ isone of these sets. Consider the Baire set in $K^{G^{\wedge}}$ defined by: $B = \bigcap_1^{\infty} B_n$, $B_n = \{ x;\ x \in K^{G\Delta},\ x_{\sigma_n} \in (0, a_n) \}$.

Then, the Haar measure of B equals $\Pi_1^{\infty}\, a_n = \frac{1}{2}$. Suppose $S' = \{ \sigma'_n \}$ is another countable subset, $S' \neq S$. Then, the set B' defined in analogous fashion is different from B. In fact, assume $\sigma_m \in S - S'$, then, $B \frown B' < B' \frown B_m$, and measure of $B' \frown B_m = = (\frac{1}{2})\ a_m < \frac{1}{2}$. Since the family of infinite, countable subsets of $G^{\wedge}$ is of cardinality $|G^{\wedge}|^{\aleph_0}$, we have $|A(K^{G^{\wedge}})| \geqq |G^{\wedge}|^{\aleph_0}$. On the other hand we have, $|A(K^{G\Delta})| \leqq |L^2(K^{G^{\wedge}})| \leqq \leqq c.\ |G^{\wedge}|^{\aleph_0} = |G^{\wedge}|^{\aleph_0}$. The last inequality follows from the fact that any element of L^2 is developable in a Fourier series of functions of $G^{\wedge}$.

b) If G has countable basis, then it is metrizable and as we have already seen (§ 6) any Baire set of positive measure has power *c*. If G has not a countable basis, and M is a Baire set of positive measure, G contains a compact subgroup Y such that G/Y has a countable basis and $M.Y = M$ (cf. [4], p. 287). Using the properties of the quotient measure and the metrizability of G/Y, we have, $|M| = c.\,|Y|$, and $|G| = |G/Y|\,.\,|Y| = c.\,|Y| = |M|$.

c) It follows from the regular completion of Haar measure, ([4], p. 287).

Corollary. Any non-finite compact abelian group is rich.

Proof. $|M| = |G| = |2^{G^{\wedge}}| \geqq |G^{\wedge}|^{\aleph_0} = |A(G)|$.

9. For any space (X, B_a, μ), there exists (Y, B_a, ν) such that $(X \times Y,\ B_a(X) \times B_a(Y),\ \mu \times \nu)$ is rich. In fact, take $Y = 2_I$ with $| I | = A(X)\ |^{x_0}$, and ν the usual product measure. The result follows immediately from $B_a(X) \times B_a(Y) = B_a(X \times Y)$.

III. MAIN THEOREMS

This part is devoted principally to prove the theorems stated in § 6. We shall say that *a compact set $K \in B_0(X)$ is faithful* if the restriction of μ to K is positive on (relative) non void open sets.

It is easy to see that the intersection of all the compact sets K_α contained in a given compact set K and such that $\mu(K_\alpha) = \mu(K)$, is a faithfull compact set with the same measure than K, (it follows from the regularity of μ applied to $\frown K_\alpha$).

10. *Proof of theorem A.* Let $\{ f_n$ $(\{ g_n \})$ be a sequence of finitevalued functions of $L^1(X)$ $(L^1(Y))$ which separates the points of the set R_X (S_Y) of measure one. The Boole isomorphism τ from $A(Y)$ onto $A(X)$ induces a norm-preserving operator from $L^1(Y)$ onto $L^1(X)$ which we call 0_τ. Suppose $\{ (\tau g_n)\ (x) \}$ is a family of concrete, finite-valued functions of $L^1(X)$ with $(\tau g_n)\ (x) \in 0_\tau\ (g_n)$. Since this family is dense in $L^1(X)$, for every f_k there exists a subsequence τg_{n_i} such that $\tau g_{n_i} \to f_k$ a.e.. Then, it follows immediately that R_X contains a set of measure one, S_X, which is separated by $\{ \tau g_n \}$. We can always suppose that not all of the g_n's $(\tau g_n$'s$)$ vanish at a point of S_Y (S_X), (it suffices to eliminate certain null set).

Using Lusin's theorem it is possible to find a compact set $Y_1 < Y$ such that for every n, $g_n(x)$ is continuous on Y_1, and verifying $\nu(Y_1) > 1/2$. Call X_1 a measurable set of τY_1. Again by, Lusin's theorem we find a compact set $X_2 < X_1$, $\mu(X_2) > 1/2$, and such that every $\tau g_n(x)$ is continuous on X_2 Call Y_2 a set contained in Y_1 and belonging to $\tau^{-1} X_2$. There exists $Y_3 < Y_2$, compact and of measure greater than $1/2$ (μ is regular). Let be $X_3 \in \tau Y_3$ and $X_3 < X_2$, etc.. The sets $X'' = \frown X_n$ and $Y'' = \frown Y_n$ are equimeasurable and are elements of classes in correspondence by τ. Besides, they are compact. Call X' (Y') the faithful compact set contained in X'' (Y'') of the same measure as X'' (Y'').

Since the restrictions to Y' of the g_n's vanish simultaneously at no point, the uniform closure of the algebra generated by them coincides with $C(Y')$, (Stone-Weierstrass theorem).

Idem for the $\tau\, g_n$'s and $C(X')$. From the construction and the faithfulness of X' and Y', it follows that $C(X')$ is isomorphic to $C(Y')$, and also that the associated homeomorphism $T' : X' \to Y'$, between X' and Y' is measure preserving. (Precisely, between $(X', B_0(X'), \mu \mid X')$, $(Y', B_0(Y'), \nu \mid Y')$). Considering now $S_X - X'$ and $S_Y - Y'$, we repeat the process and find X'' and Y'' with the same properties as X' and Y', contained respectively in $S_X - X'$ and $S_Y - Y'$, and with measures greater than $\mu\,(S_X - X')/2$, and so on. Then, $X_0 = \cup X^{(n)}$, $Y_0 = \cup Y^{(n)}$, $T = \cup T^{(n)}$, *Q.E.D.*.

To be used in the proof of theorems *B* and *C*, we give next a brief account of well-known results and prove certain auxiliary facts.

11. *Boolean measure spaces.* Given (X, B_0, μ), there exists an extremally (') disconnected compact Hausdorff space $G(X)$, such that the fa mily of its clopen sets determines a Boolean algebra isomorphic to $A(X)$, cf. [6]). Besides, $c(G)$ is isomorphic to the algebra $L^\infty\,(X, B_0, \mu)$. Moreover, the operation $\bigvee_\alpha C_\alpha = cl\,(\bigcup_\alpha C_\alpha)$ joined to the usual complementation make a complete algebra of the clopen algebra just considered. (Following the nomenclature of [2], $G(X)$ is a hyperstonian space of countable genus.). The family of Baire sets in $G(X)$ is generated by the clopen sets and the Borel sets are generated by the open sets. We denote by $\bar{\mu}$ the measure induced by μ on $B_a(G(X))$, *or* its regular extensión to $B_0(G)$. It holds:

a) every Borel set is clopen regular, i.e., the sup and inf of the measures of the clopen sets contained and containing, respectively, a given Borel set $A < G$, coincides with $\bar{\mu}\,(A)$, ($\bar{\mu}$ is completion regular in the sense of [4]);

b) a Borel set is of measure zero if and only if it is nowhere dense;

c) the boundary of any Borel set is of measure zero;

d) every regular open set (i.e., a set which coincides with the interior of its closure) is clopen;

e) every meager set is nowhere dense. These properties imply: 1) every bounded Borel measurable function on $G(X)$ is $\bar{\mu}$-equivalent to a continuous function; 2) every measurable function is $\bar{\mu}$-equi-

(') By definition, the closure of an open set is open, or equivalently, $C(G)$ is a conditionally complete lattice.

valent to an upper semi-continuous function. It also holds: every isomorphism of the Boolean algebra of the clopen sets onto itself is given by a homeomorphism α of $G(X)$.

Of course, α defines a complete isomorphism on the algebra of the clopen sets.

12. *Theorem* 1. *If* (X, B_0, μ) *is a space as in* § *2, then* X *is the quotient of* $G(X)$ *by a certain decomposition* D: $G(X) / D = X$.

Proof. Since μ is faithful, $\| f - f' \|_1 \neq 0$ whenever $f, f' \in C(X)$ and $f \neq f'$. Every measurable bounded function F on X has as image under the Boole isoporphism established between X and $G(X)$, a uniquely determined continuous function on $G(X)$, TF. This defines an isomorphism from $C(X)$ into $C(G(X))$. Therefore, there exists a continuous mapping δ from G onto X, such that: $TF(g) = F(\delta g)$, (Banach-Stone theorem). Each preimage $\delta^{-1}(x)$, $x \in X$, is an element of D, (and is a maximal set where every TF is constant). The compactness of G implies that the topology of X is the quotient topology.

Every point of $G(X)$ corresponde to an ultrafilter in the measure algebra $A(X)$, or equivalently, to a maximal ideal. A point can be also interpreted as a maximal ideal in $L^\infty (X, B_0, \mu)$, or, what is the same, as a multiplicative linear functional. Theorem 1 shows that $\delta^{-1}(x)$, $x \in X$, is exactly the family of ultrafilters of $A(X)$ which contain the filter of neighbourhoods of x. The mapping from G to X is, in general, many-to-one. However, since an atom in B_0 must correspond to an atomic clopen set which necessarily is a one-point set, it follows:

Corollary Any atom of (X, B_0, μ) *is concentrated at a point.* Then, if (X, B_0, μ) has atoms, it is rich only in the trivial case.

13. *Theorem 2.* With the hypotesis of theorem 1, it holds:

a) $\delta^{-1} (B_a(X)) < B_a (G(X))$.

b) $\delta^{-1} (B_0(X)) < B_0(G) < B_0(G)^* = B_a(G)^*$, where the completions are taken with respect to the induced measure $\bar{\mu}$.

c) $Q \in B_0(X)^*$ implies $\mu(Q) = \bar{\mu} (\delta^{-1}(Q))$.

Proof. Let us consider the family U of sets of the form $\{ g \in G ; b > (TF)(g) > a, F \in C(X) \}$. Since every open Baire set is σ-compact, it follows that every open Baire set of X is of the form $\{ x;\ 1 > f(x) > 0 \}$. Therefore, the σ-field generated by U coincides with $\delta^{-1}B_a(X)$, and a) follows. The first inclusión of b) follows immediately. The equality is nothing but the completion regularity of $\overline{\mu}$. Let us prove c). We suppose $\overline{\mu}$ is the (regular) measure on $B_0(G)$ that coincides with μ on the clopen sets, i.e., on a copy of $A(X)$, and we want to see that on $B_0(X)^*$, $\mu(Q) = \overline{\mu}(\delta^{-1} Q)$. In other words, that μ is a quotient measure. Define $\nu(Q) = \overline{\mu}(\delta^{-1}(Q))$ for $Q \epsilon B_a(X)$. We assert that $\delta^{-1}(Q)$ belongs to the same class of equivalence (mod. $\overline{\mu}$) to which belongs the clopen set, C_Q, image of the class of $A(X)$ containing Q. Suppose Q is compact. Then it is a G_δ, and therefore its characteristic function is (everywhere) limit of a sequence of continuous functions. For any $F(x)$ bounded and measurable, $TF(x)$ is the uniform limit of a sequence of simple functions whose values are taken on clopen sets. Now, it follows easily that not only C_Q is equivalent to $\delta^{-1}(Q)$ (mod. $\overline{\mu}$), but also that $\delta^{-1}(Q) > C_Q$.

Then $\nu(Q) = \overline{\mu}(C_Q) = \mu(Q)$. The regularity of Baire measures implies $\nu(Q) = \mu(Q)$ for every $Q \epsilon B_a(X)$. Defining now $\nu(Q) = \overline{\mu}(\delta^{-1}(Q))$ for $Q \epsilon B_0(X)$, we obtain a regular Borel measure on X. Since $\nu = \mu$ on the Baire sets, they coincide on B_0, and also $\nu^* = \mu^*$, *Q.E.D.*

Theorem 2 is nothing but a case of the theorems sought, for the pair (X, B_0, μ), $(G, B_0, \overline{\mu})$. The measure preserving pointwise transformation is here the continuous and closed mapping δ. Observe that for S clopen, $\overline{\mu}(S) \leqq \mu(\delta(S))$ since $\delta^{-1} \delta S > S$, and in general it is false that the equality holds. In fact, this would imply that any clopen set is equivalent to a saturated compact, set, and therefore, that any Baire set of X is equivalent to a compact set. This and the next considerations suggest the necessity of a special designation for the clopen sets which have the same measure as its projections, we call them *faithful clopens,* shortly, *f-clopens.* The image K of a clopen C_K is a compact set such that, for any open set V, $V \frown K \neq \emptyset$ implies $\mu(V \frown K) > 0$, i.e., K is a faithful compact set. Reciprocally, if $K < X$ is a faithful compact set, the clopen $\overline{\mu}$-equivalent to $\delta^{-1}(K)$, C_K, is an f-clopen contained in $\delta^{-1}(K)$ and $\delta(C_K) = K$. Except when K is a clopen set, $\delta^{-1}(K)$ contains C_K strictly. Obviously, the σ-field generated by the f-clopens, and the σ-field generated by the faithful compact sets of X are null contractions of $B_0(G)$ and $B_0(X)$, respectively.

14. This paragraph is dedicated to the proof of theorem B. the arguments and results will be improved and repeated in next

sections. However, the ideas can be more easily seen in the present simpler case.

With the notations of theorem 1, we say that a set $\triangle < G$ is a selector if it contains one, and only one point in each class of D.

Theorem 3. If $(X; B_0, \mu)$ *is a rich space, then* G_X *contains a dense selector set* $\triangle_X$.

Proof. Let ω_α be the first ordinal number such that $|\omega_\alpha| = |A(X)|$ and suppose that the clopen sets of G are numbered from 1 to ω_α. Let C_η be a clopen, $\eta < \omega_\alpha$. Assume that for $\beta < \eta$ we have selected a set $\triangle_\beta$ of points of G with: 1) any class of D, $\delta^{-1}(x)$ for $x \in X$, contains at most one point of $\triangle_\beta$; 2) each C_ν, $\nu < \beta$, contains a point of $\triangle_\beta$; 3) $\triangle_\beta < \triangle_\varepsilon$ for $\beta < \epsilon < \eta$.

If η is a limit ordinal number, then define $\triangle_\eta = \smile [\triangle_\beta; \beta < \eta]$. Suppose η is not a limit number, then $|\eta - 1| < |\omega_\alpha|$. Since X is rich, C_η intersects, at least, a set of power $|\omega_\alpha|$ of classes belonging to D (C_η is equivalent to a set M of positive measure of $\delta^{-1}(B_0(X))$ and $\delta(M)$ contains a positive faithful compact set K. Its associated clopen C_K is contained in C_η, and since $\delta(C_K) = K$, C_K interesects a family of power $\geqq |\omega_\alpha|$ of classes of D.).

Then, there exists $x \in X$, such that $\delta^{-1}(x) \frown \triangle_{\eta-1} = \emptyset$ and $\delta^{-1}(x) \frown C_\eta \neq \emptyset$. Taking a point in $\delta^{-1}(x) \frown C_\eta$ and adding it to $\triangle_{\eta-1}$ we obtain $\triangle_\eta$. Obviously, $\smile [\triangle_\eta; \eta < \omega_\alpha]$ is dense in G, and is not a selector set in that eventually there are classes of D with void intersection with $\smile \triangle_\eta$. Now, the theorem follows immediately chosing a point in those remaining classes, *Q.E.D.*

A particular case of the problem we are considering is when Y is replaced by $(G_X, B_0, \bar{\mu})$. The solution of this case is crucial, also for the general case, and shows the reasonableness of the solutions offered by theorems B and C. $\triangle_X$ inherits from G_X a σ-field, $B_0(G) \frown \triangle_X$. Since $\triangle_X$ is in one-to-one correspondence with X, $B_0(X)$ is comparable to the σ-field $\Gamma = (\delta \mid \triangle_X)(B_0(G) \frown \triangle_X)$. Now observe that *a set in G_X is dense if and only if it is a thick set*. Therefore, $(\triangle_X, B_0 \frown \triangle_X, \bar{\mu})$ and $(X, \Gamma\ \mu)$ are isomorphic measure spaces under the mapping $T = (\delta \mid \triangle_X)^{-1}$. This illustrates theorem B. If we replace $B_0(G)$ by $\Phi = \delta^{-1}(B_0(X))$ (which is a null contraction of $B_0(G)$), then $(G, \Phi, \bar{\mu})$ and $(X, B_0\, \mu)$ are related as mentioned in the second part of theorem C with $T = (\delta \mid \triangle_X)^{-1}$, and $\mathcal{Y} = \triangle_X$.

With respectto the topological relation between $\triangle_X$ and G_X we have, since G_X is extremally disconnected, that G_X is the Stone-Čech compactification. of $\triangle_X$, assumed this set has the topology induced by that of C_X, (cf. [3], 6 M).

Another justification of the necessity of something like a selector set is the following. As we already noted (§ 13), $\delta : G \rightarrow X$ is strictly measure increasing for some sets of G. From the regularity of μ and $\bar{\mu}$, and the fact that f-clopens go onto faithful compact sets of the same measure, we see that the strict increase observed is equivalent to the existence of sets of measure zero of G whose images are sets of positive measure. However, when δ is restricted to $(\triangle_X ; \Phi = B_0^*(G) \frown \triangle_X ; \bar{\mu})$ there is an improvement. Precisely, *a set of measure zero of* Φ *has a* δ*-image which is a null set, or a non-measurable set,* and therefore, it is natural to replace $B_0(X)$ by a comparable σ-field, (cf. § 7). In fact, if P is a null set of $B_0^*(G)$ call $N = P \frown \triangle_X$, and assume that $\delta(N) \in B_0^*(X)$ and has positive measure. Therefore, there exists a positive faithful compact K, contained in $\delta(N)$. Then, the f-clopen C_K such that $\delta(C_K) = K$ verifies $C_K \frown \triangle_X < N$. This contradicts the thickness of $\triangle_X$.

Proof of theorem B. Firstly observe that $G_X = G_Y$. Call δ_X (δ_Y) the function defined as in theorem 1 into the space $X(Y)$, and take $\triangle_X$ as in theorem 2. Denote with Γ the σ-field on X defined by $(\delta_X \mid \triangle_X)\, (B_0^*(G) \frown \triangle_X)$. From theorem 1, since $\triangle_X$ is a thick set, Γ is a null extension of Σ_X ($< B_0^*(X)$). The transformation T of theorem B is defined by

$$T = \delta_Y \circ (\delta_X \mid \triangle_X)^{-1} \qquad \textit{Q.E.D.}$$

15. *Proof of theorem C.* From the hypothesis and § 11 we know that there exists a measure preserving homeomorphism, H, from G_X onto G_Y. Assume, as in theorem 2, $|\omega_\alpha| = |A(X)|$ ($= |A(Y)|$). We want to show the existence of a set $\triangle'_X$ dense (and therefore, thick) in G_X with the property that, it and $H(\triangle'_X) = \triangle'_Y$ have, at most, one point in each class of D_X and D_Y, respectively. Suppose we have found a family $\triangle_\epsilon$, $\epsilon < \eta < \omega_\alpha$, $\triangle_\epsilon < \triangle_Y$, with: 1) $H^{-1}(\triangle_\epsilon)$, $\triangle_\epsilon$, contain at most one point in each class of D_X, D_Y; 2) if we have numbered the clopen sets of G_Y from 1 to ω_α, then every clopen

$C\gamma < G_Y$ with $\gamma < \eta$ contains a point of $\smile [\triangle_\varepsilon \; ; \; \epsilon < \eta]$; 3) $\epsilon_1 < \epsilon_2$ implies $\triangle_{\varepsilon_1} < \triangle_{\varepsilon_2}$. Then, if $cl\,(\smile \triangle_\varepsilon) = G_Y$ nothing is to prove. If not, there are two possibilities, first, η is a limit number, and then, $\triangle_\eta = \smile [\triangle_\varepsilon \; ; \; \epsilon < \eta]$. Secondly, η is not a limit number. In this case take the clopen $C < Y$, with minimal index and disjoint to $cl\,(\smile \triangle_\varepsilon)$. Since X is a *—rich space, the saturation of $H^{-1}(\triangle_{\eta-1})$, S, is of exterior measure zero, and therefore, $H(S)$ is also of exterior measure zero. Hence, the set $V = H(S) \cup \delta_Y{}^{-1}(\delta_Y(\triangle_{\eta-1}))$ is of exterior measure zero. Consequently, C contains a point $y' \in C - V$ and the sets $H^{-1}(\triangle_{\eta-1}) \cup H^{-1}(y')$, $\triangle_{\eta-1} \cup \{y'\}$ contain at most a point in each class of D_X, D_Y, respectively. Define $\triangle_\eta = \triangle_{\eta-1} \cup \{y'\}$.
This transfinite process permits to construct $\triangle'_Y$, and $\triangle'_X = H^{-1}(\triangle'_Y)$ as asserted.
We call $\mathcal{X} = \delta_X(\triangle'_X)$ and $\mathcal{Y} = \delta_Y(\triangle'_Y)$. Φ is defined as $(\delta_Y \mid \triangle_Y)\ [H(\delta_X{}^{-1}(\Sigma_X))]$. Using theorem 1, it is easily seen that $H(\delta_X{}^{-1}(\Sigma_X))$ is comparable to, and contained in $B_0(G_Y)^*$. Hence, Φ is comparable with Σ_Y.

Complete now $\triangle'_X$ to a selector set $\triangle_X$. Γ is defined as in theorem B: $\Gamma = \mathrm{T}^{-1}(\Sigma_Y)$, where, $\mathrm{T} = \delta_Y o H o\, (\delta_X \mid \triangle_X)^{-1})$.

From the very definitions of $\mathcal{X}$, Φ, Γ and T, we obtain theorem C. We leave the remaining details to the reader.

Proposition. There exists a measure preserving homeomorphism between (X, B_0, μ) *and* (Y, B_0, ν) *if, and only if, there exists a homeomorphism* Λ *between* G_X *and* G_Y*, which preserves equivalence classes, i.e.,* $\Lambda(\delta_X{}^{-1}(x)) = \delta_Y{}^{-1}(y)$.

16. Every compact group 2^α, $|\alpha| \geqslant \aleph_0$ (1), is a *—rich space if $|\alpha|^{\aleph_0} = |\alpha|$ and is not, if $|\alpha|^{\aleph_0} > |\alpha|$. Suppose the equality holds and assume the generalized continuum hypothesis. Then $|2^\alpha| = 2^{|\alpha|} > |\alpha|^{\aleph_0} = |\alpha|$. Since $|A(2^\alpha)| = |\alpha|^{\aleph_0}$, if N is a subset of 2^α of power less than $|A(2^\alpha)|$ then $|N| < |\alpha|$. The closed subgroup N' generated by N in 2^α is of power $< |2^\alpha|$, and therefore, $|2^\alpha / N'| = |2^\alpha|$ and $\mu(N') = 0$. Let us suppose now that $|2^\alpha| = |\alpha|^{\aleph_0} > |\alpha|$, and $|N| \leq |\alpha|$. If $|N| < |\alpha|$, as before, $\mu^*(N) = 0$. Assume $|N| = |\alpha|$, and that α is the first ordinal number of its cardinal class. The subsets,

$N_\beta = \{ x \in 2^\alpha \; ; \; i > \beta$ implies $x_i = 0$ except for a finite number of values $\}$ for $\beta < \alpha$, permit to define N as the union of the N_β,

(1) The first infinite cardinal number.

$\beta < \alpha$. Then $|N| = |\alpha|$, and if P is an open set, $P > N$, then $\mu(P) = 1$. We have, each subset N of the compact group 2^α, $|\alpha| \geqq \aleph_0$, such that $|N| < |\alpha|$ is of exterior measure zero, and when $|N| = |\alpha|$, $\mu^*(N) = 0$ if $|\alpha|^{\aleph_0} = |\alpha|$. If $|\alpha|^{\aleph_0} > |\alpha|$, there exists $N < 2^\alpha$, $|N| = |\alpha|$, such that $\mu^*(N) = 1$.

17. *Compact groups.* In [1], we were involved with the problem of introducing a convolution-type operation on a general measure space. This led us to the following situation. Assume F and G are compact abelian groups with normalized Haar measures ν and μ, respectively. Let $F^\sim = \{\eta_i\}$ be an a.e. multiplicative group of measurable functions on F isomorphic (as a group) to $G^\wedge$ = the dual group of G. By an *a.e. multiplicative group* we mean a (complex-valued) family of functions such that: if $\eta_i, \eta_j \in F^\sim$ then $\eta_i . \eta_j$ equals a.e. to a function of $F^\sim$; for any $\eta_i \in F^\sim$, there exists $\eta_k \in F^\sim$ such that $\eta_i \eta_k$ equals 1 a.e.

Problem: Give necessary and/or sufficient conditions for $F^\sim$ *to* be the image of G^Δ under a measurable transformation from F into G.

Some answers are given by theorems 4 to 7 of part II, [1]). Theorem 2 of the same part asserts that $F^\sim$ is a complete orthonormal system of functions of L^2 if and only if there is a Boole isomorphism between $A(G)$ and $A(F)$ which sends G^Δ onto $F^\sim$. Then, with the additional hypothesis that $F^\sim$ is a complete orthonormal system, we are able to apply, for example, theorem D of this paper.

However, the deep relations between algebraic, topological and measure-theoretic properties of a group permit to generalize theorem 7 of [1] in a way that we cannot obtain by a direct application of the lettered theorems of this paper. And, precisely in that no passage to comparable σ - fields is needed. The key of the alternative method is Pontrjagin's duality theorem.

Theorem E. Let $F^\sim = \{\eta_i(y)\}$ *be an orthonormal system of functions of* $L^2(F)$, *which is an a.e. multiplicative group isomorphic to* $G^\wedge = \{e_i(x)\}$. *a) Then, there exists a measure preserving transformation* T: $(F, B_a, \nu) \to (G, B_a, \mu)$, *such that for every* i, $\eta_i(y) = \epsilon_i(Ty)$ *a.e.* $y \in F$. *b) If* $|G^\Delta| = \aleph_0$, *and* $F^\sim$ *is complete then* T *can be chosen tobe a measure preserving transformation* *from* (F, B_0, ν) *into* (G, B_a, μ), *and in such a way that there exist* $F_0 < F$, $\nu(F_0) = 1$, $T(F_0) \in B_a$, and,

$$T/F_0\colon\ (F_0, B_a \frown F_0, \nu) \to (T(F_0),\ B_a \frown T(F_0), \mu)$$

is an invertible measure preserving transformation.

Proof. We have already observed that under the hypothesis, F and G are Boole σ - isomorphic. With this in mind, b) follows from theorem A.. In fact, $|\, G^{\wedge} \,| = x_0$ mplies that G has a countable basis and therefore, it is metrizable and separable. Then, theorem A applies. It only remains to observe that in this case $B_0 = B_a$.

Let us prove a). First of all we replace $F^{\sim}$ by a system (which we call again $F^{\sim} = \{\, \eta_i(y) \,\}$), of Baire measurable functions equivalent to the given system and everywhere multiplicative. This is possible and a proof is given in theorem 4, part II, [1]. Then

$$\epsilon_i \to \eta_i,\ \epsilon_j \to \eta_j,\ \epsilon_k \to \eta_k \text{ and } \epsilon_i \,.\, \epsilon_j = \epsilon_k \text{ implies}$$
$$\eta_i(y) \,.\, \eta_j(y) \,\acute{=}\, \eta_k(y) \text{ for every } y \in F.$$

The function $a(\epsilon_i) = \eta_i(y)$ for a fixed y, is a homomorphism from $G^{\wedge} = \{\ \epsilon_i\ \}$ into the unit circle, and therefore, there exists a point of G, Ty, such that

$$\epsilon_i(Ty) = a(\epsilon_i) = \eta_i(y), \text{ for every } i..$$

Hence, $T^{-1}(\epsilon_i{}^{-1}(M)) = \eta_i{}^{-1}\,(M)$ is a Baire set, whenever M is a Baire set of the unit circle. Since, the family

$$\{\, \epsilon_i{}^{-1}(M)\,;\ \epsilon_i \in G^{\wedge},\ M \text{ a Borel set} \,\}$$

includes a subbasis for the topology of G, it generates the family of Baire sets, and therefore, T is Baire measurable.

From the orthonormality and isomorphism of $F^{\sim}$ and $G^{\wedge}$, we get, $\int \epsilon_i(Ty)\,d\nu = \int \eta_i(y)\,d\nu = \int \epsilon_i(x)\,d\mu$, and $\int h(Ty)\,d\nu = = \int h(x)\,d\mu$, for every function $h(x) \in C(G)$. Then, $\nu(T^{-1}A) = \mu(A)$ for any compact $G\delta$, and consequently, for any Baire set. Q.E.D.

If T is an isomorphism from F onto G, then $F^{\sim}$ coincides with $F^{\wedge}$ and the linear operator $U\colon L^1(G) \to L^1(F)$, $U(g)(y) = g(Ty)$, is an isomorphism between the group algebras and has norm equal to one. Besides, $U1 = 1$. Conversely, if U is an isomorphism of norm not greater than one between the group algebras $L^1(G)$, $L^1(F)$, and $U1 = 1$, then U is defined as above, for T an isomorphism between F and G. In fact, from [13] it follows that U is of the form: $U(g)(y) = \eta(y) . g(Ty)$, where $\eta \in F^\wedge$. Replacing g by 1 we obtain $\eta \equiv 1$.

We finish this paragraph with theorem 4 wich is related with theorem A. .It deals with a type of result suggested by what has been told in the preceding proof. We do not know if it holds in general, (cf. next note 1).

Theorem 4. Let X and Y be compact spaces with a countable basis, and T a measure preserving transformation from (X, B_0^*, μ) into (Y, B_0^*, ν) which induces a (measure preserving) Boole σ-isomorphism from $A(Y)$ onto $A(X)$. Then, there exists a transformation with the same properties than T, S, such that $\mu\{ x; S(x) \neq T(x) \} = 0$, and for any $E \in B_0^*(X)$, $S(E) \in B_0^*(Y)$ and $\nu(S(E)) = \mu(E)$. S can be chosen to be one-to-one on a set of measure one.

Proof. We shall sketch the proof. Since X has a countable basis, B, the field R generated by B is countable. Then. for every measurable set $E < X$, $\mu(E) = \inf \{ \Sigma_1^\infty \mu(A_n);\ E < \bigcup A_n, A_n \in \mathrm{R} \}$. If $A \in \mathrm{R}$, there exists by hipothesis $B \in B_0^*(Y)$, such that $\mu(A) = \nu(B)$, and $\mu(A \bigtriangleup T^{-1}(B)) = 0$. Call $H = \mathrm{U} \{ A \bigtriangleup T^{-1}(B);\ A \in \mathrm{R} \}$. Then, H is a measurable set and $\mu(H) = 0$. Define, $S'(x) = T(x)$ if $x \overline{\in} H$, $S'(x) = u$ for any $x \in H$, *and* $\nu(\{u\}) = 0$. (If no point of Y is of measure zero, then Y is purely atomic, and the theorem is trivial, cf. corollary to theorem 1). Now, it follows that S' is a measure preserving transformation, that $\mu\{ x; S'x \neq Tx \} = 0$, and that for every null set N, $S'(N)$ is a null set. Next, that $S'(M)$ is a measurable set of the same measure as M, for any $M \in B_0^*(X)$. Define now $U(S'x)$, choosing a point in $(S')^{-1}(S'x)$. After defining U on $Y - S'(X)$, we obtain a measure preserving transformation from Y into X. Applying the result already obtained, we get U': $Y \rightarrow X$, U' different of U at most in a set of measure zero, and with the same properties as S'. Then, U' is one-to-one on a set of Y of measure one. The theorem follows at once.

18. *Note 1.* Assume $L^1\ (X, B_a, \mu)$ is separable. To fix ideas, assume also that X has a countable basis. Therefore, $\prod_{i=1}^{N} X(i)$,

$1 \leqq N \leqq \omega$, has a countable basis. Then, any Boole automorphism of $\Pi^N{}_{i=1} X(_i)$ onto itself is induced by a pointwise transformation. The same is true for any product $\Pi\, [X(_i)\,;\; i \in J]$ and with respect to the σ - field of Baire sets. If the Boole automorphism preserved subfields depending of only one coordinate, then the pointwise transformation in $\Pi\, X(_i)$ would be easily constructed by means of the pointwise transformations in each $X(_i)$. Therefore, it suffices to see that J can be divided into disjoint countable sets, J_γ , such that the family of measurable sets depending only on the coordinates of J_γ is preserved under the Boole automorphism. This last fact can be obtained taking into account that T has a countable basis, and that any Baire set depends only on a countable number of indices. We leave the verification to the reader.

Note 2 (*). We required the compact space X to admit a faithful measure. Trivially, not every compact space is of this kind; however, two very important classes of spaces have this property: compact groups and compact, separable metric spaces (put a positive mass on any point of a countable dense set). A space which admits a proper measure must verify *the Souslin condition*:

(S) There exists, at most, a countable family of disjoint open sets.

This condition coincides for ordered spaces without jumps, with the property that has been used, with the same name, in [11], and permits to enunciate *Souslin problem*:

(SP) Is a totally ordered set without jumps that satisfy (S) isomorphic to a subset of the real numbers?

The problem remains open and a thorough study has been made in [11].

Coming now to the converse of a proposition that we stated above, we can ask if any compact, Hausdorff space with property (S) admits a proper measure. We do not know whether it is true or not. However, if it is true to prove it is, at least, as difficult as to solve Souslin problem. In fact, firstly, if a totally ordered set not isomorphic to a subset of the real line, without jumps, and with property (S) exists, then it also exists another such com-

(*) The results of this note were kindly communicated to us by Prof. R. A. Ricabarra.

plete ordered set without intervals isomorphic to the real numbers, say a set C .Secondly, the existence in this C, (which is compact in its order topology), of a faithful measure, implies, as shown for instance in [12, th. c, p. 60], the existence of a non countable disjoint family of non trivial intervals. Contradiction. We remark that "Theorem C" just quoted gives the partially ordered version of our statement, and goes much beyond what we actually need here.

Note 3. Some of the main theorems can be generalized to measurable homomorphisms, and insomorphisms into.

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SOBRE UN PROBLEMA DE B. GRÜNBAUM

por FAUSTO ALFREDO TORANZOS (h.)

ABSTRACT: If we call "rhomboid" a quadrilateral that has a couple of equal adjacent sides and perpendicular diagonals, we show (prop. 3): "A closed convex curve L such that whenever three edges of a rhomboid R support L, the fourth edge of R also supports L, is a circle". This characterization was suggested by a problem stated by B. Grünbaum on [4].

Besicovitch [1] y Danzer [2] han dado dos demostraciones distintas de que la circunferencia es la única curva convexa cerrada C tal que no existe rectángulo con exactamente tres vértices sobre C. Branko Grünbaum [4] planteó el siguiente problema, en cierto sentido dual del anterior:

PROBLEMA I: *¿Es la circunferencia la única curva convexa cerrada C tal que: (*) cada vez que tres lados de un rombo R se apoyan en C, el cuarto lado de R también se apoya en C?*

Si definimos rombo como un paralelogramo equilátero, la siguiente proposición responde negativamente al problema I.

PROPOSICION 1: *La propiedad (*) caracteriza a las curvas de ancho constante entre todas las curvas convexas cerradas del plano.*

Demostración: Sea M una curva de ancho constante y R un rombo de vértices $a, b, c, d,$ tal que los lados (a, b), (b, c), y (c, d) se apoyan en M. Trazando la recta r de apoyo de M paralela a (b, c) determinamos un paralelogramo circunscripto en M. Pero la distancia de (b, c) a r es la misma que la de (a, b) a (c, d), luego el paralelogramo coincide con R y r $\supset$ (a, d). El recíproco es trivial.

Una forma de generalizar el problema I consiste en reemplazar la expresión "curva convexa cerrada" por "curva simple cerrada". Obviamente, la proposición 1 también responde en forma negativa a este enunciado, pero surge naturalmente una nueva pregunta:

PROBLEMA II: *¿Existe alguna curva plana, simple, cerrada y no convexa que verifique* (*)?

El resultado siguiente, más fuerte que la proposición 1, contesta negativamente al problema II.

PROPOSICION 2: *La propiedad* (*) *caracteriza a las curvas de ancho constante entre todas las curvas simples cerradas del plano.*

Demostración: Sea L una curva simple cerrada que verifique (*). Si consideramos a $K = \text{conv } L$ (cápsula convexa de L) como intersección de los semiplanos cerrados determinados por rectas de apoyo de L, resulta que K tiene ancho constante. Pero L debe contener todos los puntos extremales de K, y como es bien sabido, todo punto de front K (frontera de K) es extremal (para este y otros resultados elementales referentes a conjuntos de ancho constante remitimos al lector a [5] Nº 7, o bien a [3] chap. 7), luego $L \supset \text{front } K$. Pero como L es simple $L = \text{front } K$.

Otra generalización del problema I consiste en sustituir el rombo por una figura más general.

DEFINICION 1: *Llamaremos romboide a todo cuadrilátero que tiene un par de lados adyacentes iguales y las diagonales perpendiculares.*

Es claro que el par de lados adyacentes, opuesto al dado también está constituido por lados iguales. Todo rombo es un romboide.

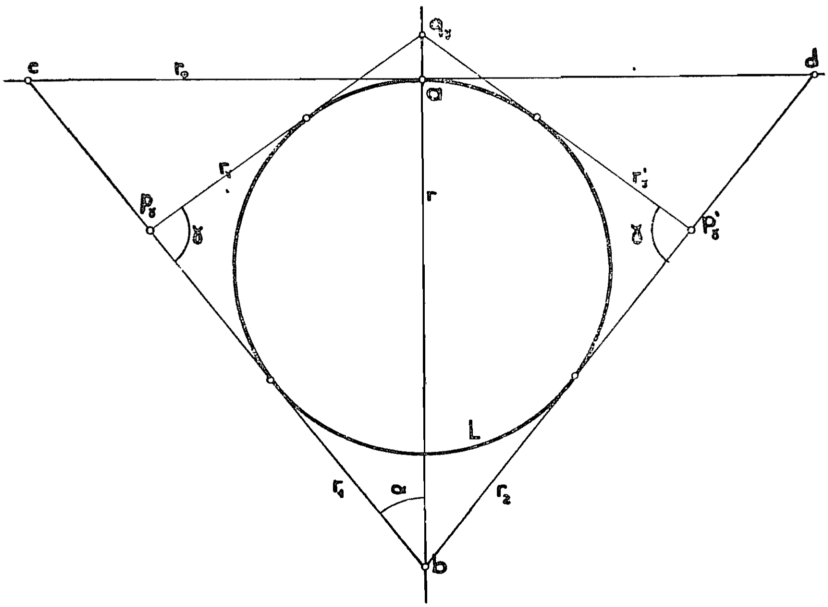
Llamaremos (**) a la propiedad (*) donde se sustituye la palabra "rombo" por "romboide".

PROBLEMA III: *¿Es la circunferencia la única curva convexa cerrada del plano que verifica* (**)?

Después de un lema preparatorio demostraremos la respuesta afirmativa al problema III.

LEMA 1: *Sea L una curva convexa cerrada que verifica* (**), *r_0 una recta de apoyo de L y a el (único) punto de contacto de r_0 con L. Sea r la perpendicular a r_0 por a, b un punto de r exterior a L, y r_1 y r_2 las rectas de apoyo de L que pasan por b. Entonces el triángulo formado por r_0, r_1 y r_2 es isósceles con base en r_0.*

Demostración: Sean $c = r_0 \cap r_1$ y $d = r_0 \cap r_2$. Bastará entonces con que demostremos que $|(a, c)| = |(a, d)|$. Si α es el ángulo formado por r y r_1, para cada γ tal que $\pi > \gamma > \frac{\pi}{2} - \alpha$ existe un punto p_γ de r_1 comprendido entre el punto de contacto y c, tal que la recta r_γ que pasa por p_γ y forma con r_1 un ángulo γ es de apoyo de L.



Si $p_\gamma' \in (b, d)$ es tal que $|(p_\gamma', b)| = |(p_\gamma, b)|$, entonces por (**) la recta r_γ' de apoyo de L que pasa por p_γ' forma con r_2 un ángulo γ. Sea $q_\gamma = r_\gamma \cap r_\gamma'$ el cuarto vértice del romboide circunscripto a L así definido. Es claro que, independientemente del ángulo γ elegido, los puntos de contacto de r_γ y r_γ' con L están en distinto semiplano respecto de r. Consideramos ahora una sucesión $\{\gamma_i\}$ tal que $\gamma_i \to \frac{\pi}{2} - \alpha$. Entonces r_{γ_i} y r_{γ_i}' convergen hacia r_0, mientras que p_{γ_i} y p_{γ_i}' tienden a c y d respectivamente. Por la rotundidad (es decir, no contiene segmentos) de L los puntos de contacto de r_{γ_i} y r_{γ_i}' se acercan tanto como se quiera, y puesto que q_{γ_i} está entre ambos, resulta que en el límite los tres puntos se confunden en uno solo: a. Entonces resulta:

$$|(a, c)| = \lim |(p_{\gamma_i}, q_{\gamma_i})| = \lim |(p_{\gamma_i}', q_{\gamma_i})| = |(a, d)| \quad \text{q.e.d.}$$

PROPOSICION 3: *Una curva convexa plana que verifica la propiedad (**) es una circunferencia.*

Demostración: Es claro que, en virtud de la proposición 1, tal curva L tiene ancho constante. Por simplicidad de notación nos referiremos en adelante a $K = \text{conv}\ L$, es decir al conjunto convexo K tal que front $K = L$. Dada una dirección arbitraria del plano sean r y r' las dos ractas de apoyo de K paralelas a esa dirección, a y a' sus puntos de contacto con L y l la recta que pasa por a y a' y (por ser L de ancho constante) es perpendicular a r y r'. Si llamamos S (respectivamente S') la semirrecta de l, exterior a K y con origen en a (resp. a'), para cada $p \in S$ (resp. $p \in S'$) sea $K(p)$ la región del plano (ángulo) que contiene a K comprendida entre las dos semirrectas de apoyo de K con origen en p. Definimos $K(\infty)$ como la banda comprendida entre las dos rectas de apoyo de K paralelas a l, y llamamos $l' = S \cup S' \cup \{\infty\}$. Es claro, a partir del lema 1, que para todo $x \in l'$ $K(x)$ es simétrico respecto de l, y como además vale que $K = \bigcap_{x \in l'} K(x)$ resulta que l es eje de simetría de K. Pero como la dirección de l es arbitraria, podemos encontrar análogamente un segundo eje de simetría de K, perpendicular a l. El punto de intersección de estas dos rectas será centro de simetría de K, y como K es de ancho constante, es un círculo q.e.d.

Es claro que, razonando como en la proposición 2, podemos debilitar la hipótesis de la proposición 3, sustituyendo "convexa" por "simple".

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