Volumen XXIII Número 2

REVISTA

DE LA

UNION MATEMATICA ARGENTINA

(MIEMBRO DEL PATRONATO DE LA MATHEMATICAL REVIEWS)

Y DE LA

ASOCIACION FISICA ARGENTINA

□

SUMARIO

□

BUENOS AIRES
1967

UNION MATEMATICA ARGENTINA

La U. M. A. reconoce cuatro categorías de miembros: honorarios, protectores, titulares y adherentes. El miembro protector paga una cuota anual de $ 2000, por lo menos; el titular una cuota anual de $ 1000 y el adherente (estudiante solamente) una cuota anual de $ 500. Los pagos deberán efectuarse por cheque, giro u otro medio de gastos, a la orden de UNION MATEMATICA ARGENTINA, Casilla de Correo 3588, Buenos Aires.

Por ser la U. M. A. miembro del patronato de la Mathematical Reviews (sponsoring member), los socios de la U. M. A. tienen derecho a suscribirse a esa importante revista de bibliografía y crítica con 50 % de rebaja sobre el precio de suscripción que es de 50 dólares por año. Los socios de la U. M. A. pagarán por tanto sólo 25 dólares por año.

Los autores de trabajos reciben gratuitamente una tirada aparte de 50 ejemplares. Las correcciones extraordinarias de pruebas, son por cuenta de los autores.

ASOCIACION FISICA ARGENTINA

La A. F. A., asociación privada de investigadores, profesores y estudiantes de física y de astronomía, tiene por objeto fomentar el progreso de la investigación y de la enseñanza de dichas materias por medio de reuniones científicas periódicas y de la publicación de trabajos originales.

Podrán ingresar como socios activos quienes hayan efectuado investigaciones originales; pueden ser socios adherentes los profesores que no cumplan este requisito; y socios-estudiantes los que hayan aprobado el primer año de estudios de física o de astronomía.

Las solicitudes de ingreso, que deberán llevar la firma de dos socios activos o adherentes, habrán de dirigirse al secretario local que corresponda. Los socios activos abonarán una cuota anual de $ 1.200, los adherentes de $ 800, y los estudiantes de $ 500, pudiendo hacerlo en dos cuotas semestrales. En estas cuotas están incluidas las suscripciones a la "Revista de la U.M.A. y de la A.F.A." y a "Ciencia e Investigación".

La correspondencia relacionada con las colaboraciones (artículos originales, informes y reseñas bibliográficas) deben dirigirse al Dr. Daniel R. Bes, Facultad de Ciencias Exactas y Naturales, Perú 222, Buenos Aires.

Se solicita a las instituciones a que pertenecen los autores contribuyan con una cuota de $ 1.400 por página impresa, la que les dará derecho a recibir 100 apartados libres de cargo.

Abonnement à l'etranger (comprenant un volume complet): 12 dollars (Etats-Unis).

Prière d'adresser toute la correspondance scientifique, administrative et les échanges à l'adresse ci-dessous:

REVISTA DE LA UNION MATEMATICA ARGENTINA
Casilla de Correo 3588
Buenos Aires (Argentina)

ON THE THEORY OF INTERPOLATION SPACES

JAAK PEETRE

INTRODUCTION. In recent years various interpolation methods (i. e. constructions of interpolation spaces) have been given by many authors (see the bibliography, in particular [11], [12], [13], [14], [2], [5], [6], [9], [10]). In this article, which is based on three lectures given at the Universidad de Buenos Aires in May 1963, we consider two quite general interpolation methods called K-and J-methods, the introduction of which was suggested by the "equivalence theorem" of Lions-Peetre [16], [17], combined with some considerations in Peetre [21]. The K-and J-methods thus generalize the methods studied there. (It turns also out that K-methods are equivalent with the method of Gagliardo [6].) A preliminary account of the theory of K-and J-methods was given in [22]. In order to avoid unnecessary repetition we shall below concentrate on further developments not explicitely included in [22]..

The are two parts. In Part I we establish several interpolation theorems for K- and J- spaces give also an extention of the above mentioned "equivalence theorem" to these spaces. Theorems 1-5 are essentially contained in [22] while theorems 6-8 are new. As an application we obtain the interpolation theorems of M. Riesz [26] and Marcinkiewicz [18] as well as an extention of these theorems to Orlicz space. In Part II we consider more general spaces called n - and M - spaces. Some of the results of Part I can be easily carried over to the more general situation. The motivation for the introduction of N- and M-paces is that in this way we obtain a unified approach to K- and J-spaces on one hand and the "approximation spaces" of [21], [22] on the other hand. In particular we obtain general results (theorems 6-9) which cointain as a special case the "reiteration theorem" of [22] (which again generalizes the "reiteration theorem" of Lions-Peetre [16], [17]), as well as its analogue for the "approximation spaces" in [21], [22]. The enumeration of formulas etc. in the two Parts is independent.

We warn the reader that we are very negligant what concerns all questions of convergence, concentrating instead mainly on establishing the inequalities involved. It is of course clear that this is no serious limitation of the value of the theory established; in most cases the reader should have no difficulties in supplying missing details.

PART I

Some interpolation theorems for K- and J- spaces

Let A_0 and A_1 be two normed spaces both contained in one and the same complete normed space $\mathcal{A}$, the injection of A_i into A being continuos, $A_i \subset \mathcal{A}\,(i=0,\ 1)$. We can then form the *sum* $A_0 + A_1$ of A_0 and A_1 and the *intersection* $A_0 \cap A_1$ of A_0 and A_1. Each of these spaces is linear. In $A_0 + A_1$ we consider the family of (equivalent) norms

$$K(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}) \qquad (0 < t < \infty) \tag{1}$$

and in $A_0 \cap A_1$ the family of (equivalent) norms

$$J(t, a) = \max\,(\,\|a\|_{A_0}\ ,\ t\|a\|_{A_1}) \qquad (0 < t < \infty) \tag{2}$$

Fixing t (e.g. $t = 1$) they become normed spaces.

Let moreover $\Phi = \Phi[\phi]$ be a *function norm*, i.e. a positive (finite or infinite) functional defined in the set m_+ of all positive (finite or infinite) functions on $(0, \infty)$ measurable with respect to $\dfrac{dt}{t}$ such that the following axioms hold:

α) $\Phi[\phi] = 0 \leftrightarrow \phi(t) = 0$ a.e.; $\Phi[\phi] < \infty \rightarrow \phi(t) < \infty$ a.e.

β) $\Phi[a\phi] = \Phi[\phi] \quad (a > 0)$

γ) $\phi(t) \leqq \sum_{\nu=1}^{\infty} \phi_\nu(t)$ a.e. $\rightarrow \Phi[\phi] \leqq \sum_{\nu=1}^{\infty} \Phi[\phi_\nu]$

We say that Φ is of genus $\leqq f$ where $f = f(t)$ is a positive function if and only if the following inequality holds:

$$\Phi[\phi(\lambda t)] \leqq f(\lambda) \quad \Phi[\phi(t)]. \tag{3}$$

We denote by $(A_0, A_1)^{K}_{\Phi}$ the set of elements $a \in A_0 + A_1$ such that

$$\Phi[K(t, a)] < \infty \tag{4}$$

and by $(A_0, A_1)_\Phi^J$ the set of elements $a \in A_0 + A_1$ such that there exists a measurable with respect to $\frac{dt}{t}$ function $u = u(t)$ with values in $A_0 \cap A_1$ such that

(5) $$a = \int_0^\infty u(t) \frac{dt}{t} \quad (\text{in } A_0 + A_1), \quad \Phi[J(t, u(t))] < \infty.$$

Each of these spaces is linear. They become normed spaces if we introduce the norms

(6) $$\|a\|_{(A_0, A_1)_\Phi^K} = \Phi[K(t, a)]$$

and

(7) $$\|a\|_{(A_0, A_1)_\Phi^J} = \inf \Phi[J(t, u(t))].$$

We may call these spaces K- and J-spaces.

Let us set

(8) $$c_K = (\Phi[\min(1, t)])^{-1}$$

and

(9) $$c_J = \sup_{\Phi[\phi]=1} \int_0^\infty \min\left(1, \frac{1}{t}\right) \phi(t) \frac{dt}{t}$$

Then we have the following theorem.

Theorem 1. If $c_K < \infty$, *then* $(A_0, A_1)_\Phi^K \subset A_0 + A_1$ *and, if* $c_K > 0$, *then* $A_0 \cap A_1 \subset (A_0, A_1)_\Phi^K$. *If* $c_J < \infty$, *then* $(A_0, A_1)_\Phi^J \subset A_0 + A_1$ *and , if* $c_J > 0$, *then* $A_0 \cap A_1 \subset (A_0, A_1)_\Phi^J$.

All injections are continuous.

The proof may be found in [22].

We now turn to the following important interpolation theorem. Let Φ be of genus $\leqq f$.

Let, besides A_0 and A_1, B_0 and B_1 be another two normed spaces contained in one and the same normed space $\mathcal{B}$, the injection of B_i into $\mathcal{B}$ being continuous: $B_i \subset \mathcal{B}$ $(i=0, 1)$.

Theorem 2. Let Π *be a linear continuous mapping from* $A_0 + A_1$ *into* $B_0 + B_1$ *such that*

$$\|\Pi\, a\|_{B_i} \leqq M_i \, \|a\|_{A_i}, \; a \in A_i \qquad (i=0,1) \tag{10}$$

where M_0 *and* M_1 *are constants. Then*

$$\|\Pi a\|_B \leqq \gamma\, M_0 f\left(\frac{M_1}{M_0}\right) \|a\|_A, \; a \in A, \tag{11}$$

where 1º $A=(A_0, A_1)^K_\Phi$, $B=(B_0, B_1)^K_\Phi$, $\gamma=1$

or 2º $A = (A_0, A_1)^J_\Phi$, $B = (B_0, B_1)^J_\Phi$, $\gamma=1$

or 3º $A=(A_0, A_1)^J_\Phi$, $B=(B_0, B_1)^K_\Phi$,

$$\gamma = \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda}.$$

Proof: We note the following inequalities, which follow at once from (10):

$$K(t, \Pi\, a) \leqq M_0 K\left(\frac{M_1 t}{M_0}, \; a\right), \tag{12}$$

$$J(t, \Pi\, a) \leqq M_0 J\left(\frac{M_1 t}{M_0}, \; a\right), \tag{13}$$

$$K(t, \Pi\, a) \leqq \min\left(1, \frac{t}{s}\right) M_0 J\left(\frac{M_1 s}{M_0}, \; a\right). \tag{14}$$

Case 1º: Using (12) we get, in view of (3):

$$\|\Pi\, a\|_{(B_0,\, B_1)^K_\Phi} = \Phi[K(t, \Pi\, a)] \leqq M_0\, \Phi\left[K\left(\frac{M_1 t}{M_0}, \; a\right)\right] \leqq$$

$$\leqq M_0 f\left(\frac{M_1}{M_0}\right) \Phi\, [K(t, a)] = M_0\, f\left(\frac{M_1}{M_0}\right) \|\, a\, \|_{(A_0,\, A_1)^K_\Phi}.$$

Case 2°: We note that $\Pi a = \int_0^\infty \Pi u\left(\frac{M_1 t}{M_0}\right)\frac{dt}{t}$. Using (13) we get, in view of (3):

$$\|\Pi a\|_{(B_0,\ B_1)^{J}_{K}} \leqq \Phi[J(t, \Pi u\left(\frac{M_1 t}{M_0}\right))] \leqq$$

$$\leqq M_0\ \Phi[J\left(\frac{M_1 t}{M_0},\ \Pi u\left(\frac{M_1 t}{M_0}\right)\right)] \leqq M_0 f\left(\frac{M_1}{M_0}\right)\Phi[J(t, u(t))]$$

and the last term tends to

$$M_0 f\left(\frac{M_1}{M_0}\right)\|a\|_{(A_0,\ A_1)^{J}_{\Phi}}$$

if u is chosen conveniently.

Case 3°: We note again that $\Pi a = \int_0^\infty \Pi u\left(\frac{M_1 t}{M_0}\right)\frac{dt}{t}$.

Using (14) we get:

$$K(t, \Pi a) \leqq \int_0^\infty K(t, \Pi u\left(\frac{M_1 s}{M_0}\right))\frac{ds}{s} \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{t}{s}\right) M_0\ J\left(\frac{M_1 s}{M_0},\ u\left(\frac{M_1 s}{M_0}\right)\right)\frac{ds}{s} =$$

$$= \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) M_0\ J\left(\frac{M_1 t\lambda}{M_0},\ u\left(\frac{M_1 t\lambda}{M_0}\right)\right)\frac{d\lambda}{\lambda}$$

so that, in view of (3):

$$\|\Pi a\|_{(B_0,\ B_1)^{K}_{\Phi}} = \Phi[K(t, \Pi a)] \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) M_0\ \Phi[J\left(\frac{M_1 t\lambda}{M_0},\ u\left(\frac{M_1 t\lambda}{M_0}\right)\right)]\frac{d\lambda}{\lambda} \leqq$$

$$\leqq \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda)\frac{d\lambda}{\lambda} M_0 f\left(\frac{M_1}{M_0}\right)\Phi[J(t, u(t))]$$

and the last term tends to

$$\int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda)\frac{d\lambda}{\lambda} M_0 f\left(\frac{M_0}{M_1}\right)\|a\|_{(A_0,\ A_1)^{K}_{\Phi}}$$

if u is chosen conveniently.

The proof is complete.

Taking $A_0 = B_0$, $A_1 = B_1$, $\Pi =$ identity mapping we get as a concequence.

Theorem 3. We have $(A_0, A_1)^J_\Phi \subset (A_0, A_1)^K_\Phi$ *, with continuous injection, provided*

$$\int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty. \tag{15}$$

Indeed we have then the inequality

$$\| a \|_{(A_0, A_1)^K_\Phi} < \int_0^\infty \min\left(1, \frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \| a \|_{(A_0, A_1)^J_\Phi}, \quad a \in (A_0, A_1)^J_\Phi. \tag{16}$$

The following theorem is a sort of converse.

Theorem 4. We have $(A_0, A_1)^K_\Phi \subset (A_0, A_1)^J_\Phi$ *, with continuous injection, provided* $c_K < \infty$ *and*

$$\min\left(1, \frac{1}{\lambda}\right) f(\lambda) \to 0 \quad as \quad \lambda \to 0 \quad or \quad \infty. \tag{17}$$

Indeed we have the inequality

$$\| a \|_{(A_0, A_1)^J_\Phi} \leqq 4 \| a \|_{(A_0, A_1)^K_\Phi}, \quad a \in (A_0, A_1)^K_\Phi. \tag{18}$$

This follows easily from the proof of theorem 1 (cf. [22]) and the following lemma.

Lemma 1. Let $a \in A_0 + A_1$ *be such that*

$$\min\left(1, \frac{1}{t}\right) K(t, a) \to 0 \quad as \quad \lambda \to 0 \quad or \quad \infty. \tag{19}$$

Then there exists a measurable with respect to $\frac{dt}{t}$ *function* $u = u(t)$ *with values in* $A_0 \cap A_1$ *such that*

$$a = \int_0^\infty u(t) \frac{dt}{t} \ (\text{in } A_0 + A_1), \quad J(t, u(t)) < 4 K(t, a). \tag{20}$$

For details we refer to [22].

Remark 1. Note that (15) and (17) are fulfilled in the important special case $f(\lambda) = \lambda^\theta$, $0 < \theta < 1$. This leads in view of [21], [23], to the "equivalence theorem" of Lions-Peetre [16], [17] mentioned in the Introduction.

With the aid of theorem 4 we can give the following complement to theorem 2.

Theorem 5. Assume that (17) *holds true. Then the conclusion of theorem 2 holds also in the following case:* 4º $A = (A_0, A_1)^K_\Phi$, $B = (B_0, B_1)^J_\Phi$, $\gamma = 4$.

Let us now observe that, in view of the definition (1), $K(t,a)$ is *concave* considered as a function of t. Therefore $K(t, a)$ can be represented in the form

$$K(t, a) = \int_0^t k(s, a)\, ds \tag{21}$$

where $k(t, a)$ is *non-increasing* considered as a function of t, provided we impose also some auxiliary condition which assures that $K(t,a) \to 0$ as $t \to 0$. (This is always the case in example 1 below).

Theorem 6. We have $a \in (A_0, A_1)^K_\Phi$ *if and only if* $\Phi[t\, k(t, a)] < \infty$, *provided*

$$\int_0^1 f(\lambda) \frac{d\lambda}{\lambda} < \infty. \tag{22}$$

Proof: i) Since $K(t, a) \geqq t\, k(t, a)$ we get

$$\| a \|_{(A_0,\, A_1)^K_\Phi} = \Phi[K(t, a)] \geqq \Phi[t\, k(t, a)]$$

and the "only if" part follows.

ii) Let us make a change of variable in the integral (21):

$$K(t, a) = \int_0^1 t\, \lambda\, k(t\lambda, a) \frac{d\lambda}{\lambda}. \tag{23}$$

Therefore

$$\| a \|_{(A_0,\, A_1)^K_\Phi} = \Phi[K(t, a)] \leqq \int_0^1 \Phi[t\lambda k(t\lambda, a)] \frac{d\lambda}{\lambda} \leqq$$

$$\leqq \int_0^1 f(\lambda) \frac{d\lambda}{\lambda} \Phi[t\, k(t, a)]$$

and the "if" part follows.

We illustrate the above results in a concrete case.

Example 1. Let $A_0 = L_1$, $A_1 = L_\infty$ (with respect to some positive measure on some locally compact space). Then one can prove (cf. [22]) that $k(t,a) = a^*(t)$ where $a^*(t)$, as customary, denotes the non-increasing rearrangement of a on $(0, \infty)$ with the measure dt, i.e. a^* and a are equimeasurable (cf. e.g. [7]).

a) (Lebesgue spaces) Let us take

$$\Phi[\phi] = \left(\int_0^\infty \left(\frac{\phi(t)}{t}\right)^p dt\right)^{\frac{1}{p}} = \left\| \frac{\phi}{t} \right\|_{L_p}$$

One sees easily that Φ is of genus $\leqq \lambda^{1-\frac{1}{p}}$. Then

$$\Phi[t\, k(t, a)] = \| a^* \|_{L_p} = \| a \|_{L_p}$$

so that by theorem 6 $(L_1, L_\infty)^K_\Phi = L_p$ provided $p > 1$. Applying theorem 2 one gets as a special case the interpolation theorem of M. Riesz [26].

b) (Orlicz spaces). Let $M(\lambda)$ be a positive, non-decreasing convex function and let $\xi(\lambda)$ be a positive increasing function such that $M(\lambda\mu) \leqq \xi(\lambda)\, M(\mu)$. Let us take

$$\Phi[\phi] = \inf_{r>0} r \max\left(\int_0^\infty M\left(\frac{\phi(t)}{r\,t}\right) d\,t, \frac{1}{r}\right) = \left\| \frac{\phi}{t} \right\|_{L_M}$$

(which is Luxemburg's definition of the norm in Orlicz space, cf. e.g. [8]). One sees easily that Φ is of genus $\leqq \frac{\lambda}{\xi^{-1}(\lambda)}$. Then

$$\Phi[t\, k(t, a)] = \| a^* \|_{L_M} = \| a \|_{L_M}$$

so that by theorem 6 $(L_1, L_\infty)^K_\Phi = L_M$ provided $\int_0^1 \frac{d\lambda}{\xi^{-1}(\lambda)} < \infty$.

Applying theorem 2 one gets as a special case a sort of generalization to Orlicz space of the interpolation theorem of M.. Riesz [26].

Remark 2. A quite different approach to such interpolation theorems can be based on an idea in Cotlar [3], p. 197.

We discuss next some extentions of theorem 2 in the case 3º.

Theorem 7. Let Π *be a continuous linear mapping from* $A_0 + A_1$ *into* $B_0 + B_1$ *such that*

$$K(t, \Pi a) \leqq Q\left(\frac{t}{s}\right) M_0 J\left(\frac{M_0 s}{M_1}, a\right) \tag{24}$$

where $Q(\lambda)$ *is a positive function and* M_0 *and* M_1 *are constants. Then* (11) *holds with*

$$A = (A_0, A_1)^J_\Phi \ , \ B = B_0, B_1)^K_\Phi \ , \gamma = \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda}.$$

Proof: Identical with the proof of theorem 2 (case 3º).

Theorem 8. Assume, instead of (24), *that* Π *satisfies*

$$t\,k(t, \Pi a) \leqq q\left(\frac{t}{s}\right) M_0 \ J\left(\frac{M_1 s}{M_0}, a\right). \tag{25}$$

where $q(\lambda)$ *is a positive function and* M_0 *and* M_1 *are constants. Then* (24) *holds with*

$$Q(\lambda) = \int_0^1 q(\lambda\mu) \frac{d\mu}{\mu}. \tag{26}$$

Therefore hold also the conclusions of theorem 7.

Proof: Using (23) we get at once

$$K(t, a) \leqq \int_0^1 q\left(\frac{t\lambda}{s}\right) \frac{d\lambda}{\lambda} M_0 \ J\left(\frac{M_1 s}{M_0}, a\right)$$

and (24), with Q defined by (26), follows.

Example 2. An important special case is $q(\lambda) = \min(1, \lambda)$. Then $Q(\lambda) = \lambda$ if $\lambda \leqq 1$, $= 1 + \log\lambda$ if $\lambda > 1$.

Example 3. Let A_0, A_1,ϕ be as in example 1 and $q(\lambda)$ as in example 2. Applying theorem 8 we can now get as a special case the interpolation theorem of Marcinkiewicz [18] as well as a generalization of it to Orlicz space.

Remark 3. We conclude Part I with a few observations of heuristic nature intended to facilate the proper understanding of the above results. First we wish to point out that theorem 2 in the case 3º and theorem 7 are related to each other roughly as the theorems of M. Riesz and Marcinkiewicz. We also wish to point out that the special case $f(\lambda) = \lambda^{\Theta}$ (thus essentially the case considered in Lions-Peetre [16], [17] is related to the general case roughly in a similar way as Lebesgue spaces L_p to Orlicz spaces L_M.

PART II

A general reiteration theorem.

Let $\mathscr{A}$ be a complete normed space. We consider two arbitrary families of norms [1] in $\mathscr{A}$, $N(t, a)$ and $M(t, a)$ $(0 < t < \infty)$. Let Φ be a function norm (see Part I). We denote then by F^N_Φ the set of elements $a \in A$ such that

$$\Phi[N(t, a)] < \infty \tag{1}$$

and by E^M_Φ the set of elements $a \in A$ such that there exists a measurable with respect to $\frac{dt}{t}$ function $u = u(t)$ with values in $\mathscr{A}$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t} \text{ (in } \mathscr{A}), \ \phi[M(t, u(t))] < \infty. \tag{2}$$

Each of these spaces is linear. They become normed spaces if we introduce the norms

$$\| a \|_{F^N_\Phi} = \Phi[N(t,a)] \tag{3}$$

and

$$\| a \|_{E^M_\Phi} = \inf \Phi[M(t, u(t))]. \tag{4}$$

We may call this spaces N- and M-spaces.

---

(1) We use the word norm in a very wide sense including in this concept also what is usually called semi-norm (the value 0 is permited) and pseudonorm (the value ∞ is permited).

Let us discuss the two principal examples of N - and M-spaces.

Example 1. Let A_0 and A_1 be two normed spaces both contained in $\mathcal{A}$, the injection of A_i into $\mathcal{A}$ being continuous $(i = 0,1)$. We may take $N(t, a) = K(t, a)$, $M(t, a) = J(t, a)$. Then wen have

$$F^N_\Phi = (A_0, A_1)^K_\Phi \ , \ E^M_\Phi = (A_0, A_1)^J_\Phi \ .$$

Example 2. Let W_n $(n = 0, 1, 2, \ldots)$ be a family of linear subspaces of $\mathcal{A}$ such that $0 = W_0 \subset W_1 \subset W_2 \subset \ldots$ We may take

$$N(t,a) = \inf_{w \in W_n} \| a - w \|_{\mathcal{A}} \tag{5}$$

and

$$\begin{aligned} M(t, a) &= \| a \|_{\mathcal{A}} \text{ if } a \in W_n, e^{-n} \leqq t < e^{-n-1} \text{ or } t > 1 \\ &= \infty \text{ if } a \in W_n, e^{-n} \leqq t < e^{-n-1} \end{aligned} \tag{6}$$

We will start with some straight forward generalizations of certains results of Part I.

Theorem 1. Let Π *be a continuous linear mapping from* $\mathcal{A}$ *into* $\mathcal{A}$ *such that*

$$M(t, \Pi\, a) \leqq Q\left(\frac{t}{s}\right) M_0\, N\left(\frac{M_1 s}{M_0}, a\right) \tag{7}$$

where $Q(\lambda)$ *is positive function and* M_0 *and* M_1 *are constants. Suppose* Φ *is of genus* $\leqq f$. *Then*

(8)

$$\| \Pi\, a \|_{F^N_\Phi} \leqq \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} M_0\, f\left(\frac{M_1}{M_0}\right) \| a \|_{E^M_\Phi} \ , \ a \in E^M_\Phi \ .$$

Proof: Identical with the proof of theorem I. 2 (Case 3º).

If $\Pi =$ identity mapping, we get as a consequence.

Theorem 2. Assume that

$$M(t, a) \leqq Q\left(\frac{t}{s}\right) N(s,a) \tag{9}$$

where $Q(\lambda)$ is a positive function. Then we have $E^M_\Phi \subset F^N_\Phi$, with continuous injection, provided

$$(10) \qquad \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty .$$

Indeed we have the inequality

$$(11) \qquad \| a \|_{F^N_\Phi} \leqq \int_0^\infty Q\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \| a \|_{E^M_\Phi} , \; a \in E^M_\Phi .$$

Example 3. In the case of example 1 we may take $Q(\lambda) = \min (1, \lambda)$ and in the case of example 2 $Q(\lambda) = 0$ if $\lambda \leqq 1$, 1 if $\lambda < 1$.

Let us denote by $\overset{0}{\mathcal{A}}$ the space of elements $a \in \mathcal{A}$ such that there exists a constant R and a measurable with respect to $\frac{dt}{t}$ function function $u = u(t)$ with values in $\mathcal{A}$ such that

$$(12) \qquad a = \int_0^\infty u(t) \frac{dt}{t} , \; M(t, u(t)) \leqq R\, N(t, a).$$

Example 4. In the case of example 7 $a \in \overset{0}{\mathcal{A}}$ with $R = 4$ provided (see Lemma I.1)

$$(13) \qquad \min\left(1, \frac{1}{t}\right) K(t, a) \to \text{ as } t \to 0 \text{ or } \infty$$

and in the case of example 2 $a \in \overset{0}{\mathcal{A}}$ with $R = 2$ provided (cf. [22])

$$(14) \qquad N(t, a) \to 0 \text{ as } t \to 0.$$

We can now give a converse of theorem 2.

Theorem 3. If $a \in F^N_\Phi$ implies $a \in \overset{0}{\mathcal{A}}$ with R independent of a, then $F^N_\Phi \subset E^M_\Phi$, with continuous injection. Indeed we have the inequality

(15) $$\| a \| E_{\Phi}^{M} \leqq R \| a \| F_{\Phi}^{N}, \quad a \in F_{\Phi}^{N}.$$

Example 5. In the case of example 1 it suffices that f satisfies (see theorem I.4)

(16) $$\min\left(1, \frac{1}{\lambda}\right) f(\lambda) \to 0 \text{ as } \lambda \to 0 \text{ or } \infty.$$

In the case of example 2 it suffices that

(17) $$f(\lambda) \to 0 \text{ as } \lambda \to 0.$$

Let $f = f(\lambda)$ be any positive function. Let A be a normed space contained in $\mathcal{A}$.

Definition 1. We say that A is of class $\mathcal{D}_f^N$ *if and ouly if*

(18) $$N(t, a) \leqq Df(t) \; \| a \|_A$$

where D is a constant, and that A is of class $\mathcal{C}_f^M$ *if and only if*

(19) $$\| a \|_A \leqq C f\left(\frac{1}{t}\right) M(t, a)$$

where C is a constant.

Example 6. Assume that Φ is of genus $\leqq f$. In the case of example 1, $(A_0, A_1)_{\Phi}^{K}$ is of class $\mathcal{D}_f^K$ provided $c_K < \infty$ and of class $\mathcal{C}_f^J$ provided $c_K > 0$; $(A_0, A_1)_{\Phi}^{J}$ is of class $\mathcal{D}_f^K$ provided $c_J < \infty$ and of class $\mathcal{D}_f^J$ provided $c_J > 0$. (Here c_K and c_J are as in (I.8) and (I.9)!) This follows easily from the proof of theorem I.1 (cf. [22]).

Spaces of classes $\mathcal{D}_f^N$ and $\mathcal{C}_f^N$ are characterized by the following theorems.

Theorem 4. A *is of class* $\mathcal{D}_f^N$ *if and only if*

$$(20) \qquad A \subset F^N_\Phi, \quad \Phi[\phi] = \sup \frac{\phi(t)}{f(t)}.$$

Theorem 5. A is of class $\mathcal{C}^M_f$ *if and only if*

$$(21) \qquad A \supset E^M_\Phi, \quad \Phi[\phi] = \int_0^\infty f\left(\frac{1}{t}\right) \phi(t) \frac{dt}{t}.$$

The proof of these theorems is obvious.

Let us from now on assume that $f(\lambda)$ is of the from λ^α. We shall write $\mathcal{D}^N_\alpha$ and $\mathcal{C}^M_\alpha$ instead of $\mathcal{D}^N_{\lambda^\alpha}$ and $\mathcal{C}^M_{\lambda^\alpha}$. Let $\alpha_0 < \alpha_1$ be given. If Φ is any function norm we define Ω by

$$(22) \qquad \Omega[\phi] = \Phi\left[t^{\alpha_0} \varphi\left(t^{\alpha_1 - \alpha_0}\right)\right].$$

If Φ is of genus $\leqq f$ then Ω is of genus $\leqq r$ where r is given by

$$(23) \qquad r(\lambda) = \lambda^{-\frac{\alpha_0}{\alpha_1 - \alpha_0}} f\left(\lambda^{\frac{1}{\alpha_1 - \alpha_0}}\right).$$

We can now announce our main results.

Theorem 6. Let A_i *be of class* $\mathcal{D}^N_{\alpha_i}$ $(i = 0,1)$. *Then*

$$(24) \qquad \| a \|_{F^N_\Phi} \leqq D_0\, r\left(\frac{D_1}{D_0}\right) \| a \|_{(A_0, A_1)^K_\Phi}, \quad a \in (A_0, A_1)^K_\Phi$$

so that $(A_0, A_1)^K_\Phi \subset F^N_\Phi$ *with continuous injection.*

Theorem 7. Let A_i *be of class* $\mathcal{C}^M_{\alpha_i}$ $(i = 0,1)$. *Then*

$$(25) \qquad \| a \|_{(A_0, A_1)^J_\Phi} \leqq \frac{1}{\alpha_1 - \alpha_0} C_0\, r\left(\frac{C_1}{C_0}\right) \| a \|_{F^M_\Phi}, \quad a \in E^M_\Phi$$

so that $E^M_\Phi \subset (A_0, A_1)^J_\Phi$.

Since the proof of theorem 7 is similar though slightly longer (cf. [22] for details) we shall only indicate the proof of theorem 6.

Proof of theorem 6: Let $a = a_0 + a_1$. Then we have

$$N(t,a) \leqq N(t, a_0) + N(t, a_1) \leqq D_0\ t^{\alpha_0}\ \| a_0 \|_{A_0} + D_1\ t^{\alpha_1}\ \| a_1 \|_{A_1} =$$
$$= D_0\ t^{\alpha_0} \left(\| a_0 \|_{A_0} + \frac{D_1}{D_0}\ t^{\alpha_1 - \alpha_0}\ \| a_1 \|_{A_1} \right).$$

Making vary a_0 and a_1 we get

$$N(t, a) \leqq D_0\ t^{\alpha_0}\ K \left(\frac{D_1}{D_0}\ t^{\alpha_1 - \alpha_0}\ ,\ a \right)$$

from which the result easily follows by (22) and (23).

Theorem 8. Let A_i be of class $\mathcal{D}^{N}_{\alpha i}$ and of class $\mathcal{C}^{M}_{\alpha i}$ $(i = 0,1)$.

Then $E^{M}_{\Phi} \subset (A_0, A_1)^{J}_{\Phi} \subset (A_0, A_1)^{K}_{\Phi} \subset F^{N}_{\Phi}$, with continuous injections, provided

$$\int_0^\infty \min\left(\frac{1}{\lambda^{\alpha_0}}, \frac{1}{\lambda^{\alpha_1}}\right) f(\lambda)\, \frac{d\lambda}{\lambda} < \infty. \tag{26}$$

Proof: Apply theorem 2 (or theorem I.3).

Theorem 9. Let again A_i be of class $\mathcal{D}^{N}_{\alpha i}$ and of class $\mathcal{C}^{M}_{\alpha i}$ $(i = 0,1)$. Suppose that the assumptions of theorem 3 are fulfilled. Then $F^{N}_{\Phi} = E^{M}_{\Phi} = (A_0, A_1)^{K}_{\Phi} = (A_0, A_1)^{J}_{\Phi}$, with continuous injections, provided (26) *holds*

Proof: Apply theorem 3.

Example 7. Consider the case of example 1. Let $\widetilde{A}_i$ be of class $\mathcal{D}^{K}_{\alpha i}$ and of class $\mathcal{C}^{J}_{\alpha i}$ $(i = 0,1)$. Then $(A_0, A_1)^{K}_{\Phi} = (A_0, A_1)^{J}_{\Phi} =$

$= (\widetilde{A}_0, \widetilde{A}_1)^{K}_{\Omega} = (\widetilde{A}_0, \widetilde{A}_1)^{J}_{\Phi}$ provided (26) and (16) hold. This is the "reiteration theorem" of [22]. (A "reiteration theorem" of somewhat different nature connected with the "complex variable" methods of [2], [9],]14[was recently found by Lions [15].)

Remark 1. With the aid of the reiteration theorem we can also extend the results of example I.1 to the case $A_0 = L_{p_0}$, $A_1 = L_{p_1}$.

Example 8. Consider the case of example 2. Let A_i be of class $\mathcal{D}^K_{\alpha i}$ and of class $\mathcal{C}^J_{\alpha i}$ $(i = (0,1)$. Then $F^N_\Phi = E^M_\Phi = (A_0, A_1)^K_\Omega =$ $= (A_0, A_1)^J_\Omega$ provided (26) and (17) hold. This is analogue of the "reiteration theorem" for the "approximation spaces" (cf. [21], [22]).

We conclude by applying example 8 in a concrete case.

Example 9. Let $\mathcal{A}$ be L_p with respect to the Haar measure dx on the additive group of real numbers, i.e. the intervall $(-\infty, \infty)$. Denote W^m_p the space of functions a whose generalized derivatives up to order m are in L_p: $\left(\frac{d}{dx}\right)^j a \in L_p$ if $0 \leqq j \leqq m$. Let W_n be the space of functions a in L_p such that the generalized Fourier transform vanishes outside $(-e^n, e^n)$;i.e. a is entire of exponential type e^n. Then (trivial) L_p is of class $\mathcal{D}^M_0$ and of class $\mathcal{C}^N_0$ and (using Fourier transforms) W^m_p is of class $\mathcal{D}^M_m$ and of class $\mathcal{C}^N_m$. Therefore $F^N_\Phi = E^M_\Phi = (L_p, W^m_p)^K_\Omega = (L_p, W^m_p)^J_\Omega$ where Ω is given by (22) with $\alpha_0 = 0$, $\alpha_1 = m$ provided $\int\limits_0^\infty \min\left(1, \frac{1}{\lambda^m}\right) f(\lambda) \frac{d\lambda}{\lambda} < \infty$ and $f(\lambda) \to 0$ as $\lambda \to 0$. Let us specialize to $\Phi[\phi] =$ $= \sup \frac{\phi(t)}{t^\alpha}$, $0 < \alpha < m$. Then we may take $f(\lambda) = \lambda^\alpha$ so the above assumptions of f are fulfilled. On the order hand it is known (cf. [12], [17], [25], [22]) that in this case $(L_p, W^m_p)^K_\Omega =$ $= (L_p, W^m_p)^J_\Omega$ is the space of functions $a \in L_p$ satisfying the following Hölder type condition: $\sup\limits_h h^{-\alpha} \| (\Delta(h))^m a \|_{L_p} < \infty$ where $\Delta(h)$ is the operation of taking differences of increment h: $\Delta(h)a(x) =$ $= a(x+h) - a(x)$. In this way we are lead to the classical theorems of Jackson and Bernstein in the constructive theory of functions (cf. e g. [1]). One can also consider the case of ν variables $(\nu \geq 1)$, in which way we obtain various results found in recent years by Nikolskij and his school (cf. e. g. [19]), as well as other extentions. The full details will be published in a forthcoming paper.

Stockholm, July 1963.

BIBLIOGRAPHY

[1] N. AKHIESER, *Vorlesungen über Approximationstheorie.* Deutsche Übersetzung, Berlin, 1953.

[2] A. P. CALDERÓN, *Intermediate spaces and interpolation.* Studia Math. Special Senies 1 (1963), 31-34.

[3] M. COTLAR, *Condiciones de continuidad de operadores potenciales y de Hilbert.* Notes, Buenos Aires, 1959.

[4] C. FOI,S and J. L. LIONS, *Sur certains théorèmes d'interpolation.* Acta Szeged 22 (1961), 269-282.

[5] E. GAGLIARDO, *Interpolazione di spazi di Banach e applicazione.* Ricerche Mat. 9 (1960), 58-81.

[6] E. GAGLIARDO, *Una structura unitaria in diverse famiglie di spazi funzionali.* I: Ricerche Mat. 10 (1961), 245-281.

[7] G. H. HARDY, J. E. LITTLEWOOD and G. POLYA, *Inequalities.* 2 ed., Cambridge, 1952.

[8] M. A. KRASNOSELSKIJ and J. B. RUTITSKIJ, *Convex functions and Orlicz spaces.* Moscov, 1958. (Russian).

[9] S. G. KREIN, *On an interpolation theorem in operator theory.* Dokl. Akad. Nauk SSSR 130 (1960), 491-494. (Russian.)

[10] S. G. KREIN, *On the concept of a normal scale of spaces.* Dokl. Akad. Nauk SSSR 132 (1960), 510-513. (Russian.)

[11] J. L. LIONS, *Espaces intermédiaires entre espaces hilbertiens et applications.* Bull Math. Soc. Sc. Math. Phys. R. P. Roumanie 50 (1959), 419-432.

[12] J. L. LIONS, *Théorèmes de traces et d'interpolation.* I: Ann. Scuola Normale Sup. Misa 13 (1959), 389-403; II: ibid. 15 (1960), 317-331; III: J. Math. Pures Appl. 42 (1963), 195-203; IV: Math. Ann. 151 (1963), 42-56; V: to appear in Summa Brasil, Math.

[13] J. L. LIONS, *Sur les spaces d'interpolation; dualité.* Math. Scand. 9 (1961), 147-177.

[14] J. L. LIONS, *Une construction d'espaces d'interpolation.* C. R. Acad. Sci. Paris 251 (1960), 1853-1855.

[15] J. L. LIONS, *Une propriété de stabilité pour des espaces d'interpolation; applications.* C. R. Acad. Sci. Paris 256 (1963), 855-857.

[16] J. L. LIONS and J. PEETRE, *Propriétés d'espaces d'interpolation.* C. R. Acad. Sci. Paris 253 (1961), 1747-1749.

[17] J. L. LIONS and J. PEETRE, *Sur une classe d'espaces d'interpolation.* Inst. Hautes Etudes Sci. Publ. Math. 19 (1964), 5-68.

[18] J. MARCIKIEWICZ, *Sur l'interpolation d'opérateurs.* C. R. Acad. Sci. Paris 208 (1939), 1272-1273.

[19] S. M. NIKOLSKIJ, *On imbedding, continuation and approximation theorems for differentiable functions of several variables.* Uspechi Mat. Nauk 16:5 (1961), 55-104. (Russian).

[20] J. PEETRE, *Espaces intermédiaries et la théorié constructive des fonctions.* C. R. Acad. Sci. Paris 256 (1963), 54-55.

[21] J. PEETRE, *Nouvelles propriétés d'espaces d'interpolation.* C. R. Acad. Sci. Paris 256 (1963), 1424-1426.

[22] J. PEETRE, *A theory of interpolation of normed spaces. Notes,* Brasilia, 1963.

[23] J. PEETRE, *Sur le nombre de paramétres dans la définition de certains espaces d'interpolation.* Ricerche Mat. 12 (1963), 248-261).

[24] J. PEETRE, *On an interpolation theorem of Foias and Lions.* To appear. in Acta Szeged.

[25] J. PEETRE and A. PERSSON, *Interpolation av Banach-rum.* Notes, Lund, 1963.

[26] M. RIESZ, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires.* Acta Math. 49 (1926), 465-497.

ON JORDAN OPERATORS AND RIGIDITY OF LINEAR CONTROL SYSTEMS

by H. O. FATTORINI *
University of Buenos Aires

INTRODUCTION

Let E be a vector space over a field K. A linear operator A in E will be called a *Jordan operator* if there exists a non-null polynomial $P(\lambda) = a_0 + a_1 \lambda + \ldots$ with coefficients in K such that

$$P(A) = a_0 + a_1 A + \ldots = 0 \tag{1}$$

We present in this paper a result on these operators (Theorem 1.2). It is established for the case $K =$ real or complex numbers, E a Banach space, A a bounded operator, although it is easily seen to be valid, with an additional assumption, for general K, E and A. (See the observations after the proof). Theorem 1.2 is proved with the help of a result in [5] (Theorem 15) which, for the sake of completeness, is included here together with Lemma 14 as Theorem 1.1 and Remark 1 respectively. We establish next a version of Theorem 1.2 for certain unbounded operators A (Theorem 2.2) and point out its connections with control theory. Theorem 2.2 is a generalization of Theorem 2.2 of [4] from the case in which the "control" space F has dimension 1 to the case of arbitrary finite dimension.

Paragraph § 1 is fairly self contained and makes use only of elementary notions of linear topological algebra; paragraph § 2 is

* Most of the results of this paper were obtained at the Courant Institute of Mathematical Sciences, New York University, with the support of a Ford Foundation Pre-Doctoral Fellowship.

closely related to [4], Section § 2 and uses notations, definitions and results in that paper.

§ 1. *The case of bounded A*

We shall suppose throughout this paragraph (unless otherwise stated) that K is the field of real (complex) numbers, E is a real (complex) Banach space and A is a bounded operator.

Theorem 1.1. Assume that for every $u \in E$ there exists a polynomial $p(\lambda) = p(u;\lambda) \neq 0$ such that $p(A)u = 0$. Then A is a Jordan operator.

Proof: Let $p(u;\lambda)$ be the *minimal* polynomial of A at u, i.e. the generator of the ideal I_u (of the ring of polynomials in one indeterminate with coefficients in K) consisting of all polynomials $p(\lambda)$ with $p(A)u = 0$. Recall that $p(u;\lambda)$ is uniquely defined, save by multiplication by a nonzero element of K. We have

$$p(cu;\lambda) = p(u;\lambda) \;,\; c \in K \;,\; c \neq 0 \tag{1.1}$$

$$p(u;\lambda)\, p(v;\lambda) \text{ is divisible by } p(u+v;\lambda) \tag{1.2}$$

(1.1) is clear; (1.2) follows from the relation

$$p(u;A)\, p(v;A)\,(u+v) = p(v;A)\, p(u;A)u + p(u;A)p(v;A)v = 0$$

Let us observe next that the degree of $p(u;\lambda)$ *is* bounded independently of u. In fact, let
$E_N = \{\, u \in E \mid \deg\, p(u;\lambda) \leqq N \,\}$, and let $\{\, u_n \,\}$ be a sequence in some E_N convergent to some element $u \in E$.
Normalize $p(u_n;\lambda) = a_{0,n} + a_{1,n}\lambda + \ldots$ by, say, the condition $|a_{0,n}| + |a_{1,n}| + \ldots\ldots = 1$. By passing, if necessary to a subsequence we can suppose that $a_{k,n} \to a_k$ as $n \to \infty$; by the normalization condition $|a_0| + |a_1| + \ldots = 1$ and therefore $p(\lambda) = a_0 + a_1\lambda + \ldots \neq 0$. But

$$p(A)u = \lim p(u_n;A)u_n = 0$$

hence $\deg\, p(u;\lambda) \leqq \deg\, p(\lambda)$ and $u \in E_N$. This shows that each E_N is closed. Since $\bigcup_N E_N = E$ the category theorem of Baire implies that some E_N contains a sphere, say $\{ u \in E \mid |u - u_0| \leqq \rho \}$ But if v is any element of E $p(v;\lambda) = p(\rho v/|v|;\lambda)$ divides $p(u_0 - \rho v/|v|;\lambda)\, p(u_0;\lambda)$ which shows that

$$\deg \mathrm{p}(v;\lambda) \leqq 2N.$$

Let us pass now to the construction of the polynomial P in (1). Choose $u \in E$ such that

$$\deg p(u;\lambda) = \sup \{ \deg p(v;\lambda) \ ; \ v \in E \} \tag{1.3}$$

We shall show that $p(u;\lambda) = P(\lambda)$. In fact, let w be any element of E such that $p(w;\lambda) = p_0(\lambda)^m$, $m \geqq 1$ where $p_0(\lambda)$ is an *irreducible* polynomial. In view of (1.2) we have

$$p(u+w;\lambda) p(w;\lambda) = p(u;\lambda)\, q(\lambda) \tag{1.4}$$

$$p(u;\lambda)\, p(w;\lambda) = p(u+w;\lambda)\, r(\lambda) \tag{1.5}$$

where q, r, are polynomials. We get from (1.4) and (1.5) that

$$q(\lambda)\, r(\lambda) = p(w;\lambda)^2 = p_0(\lambda)^{2m}$$

$$so \ \ q(\lambda) = p_0(\lambda)^k, \ r(\lambda) = p_0(\lambda)^j, \ k, j \geqq 0, k + j = 2m$$

Then

$$p(u;\lambda) = p(u+w;\lambda)\, p_0(\lambda)^h,$$

$-m \leqq h \leqq m$. By virtue of (1.3) $h \geqq 0$. But then $p(u;A)\, w = = p_0(A)^h p(u+w;A)\ (u+w) - p(u;A)\, u = 0$, so $p(u;\lambda)$ is divisible by $p(w;\lambda)$.

Let now v be any element of E, $p(v;\lambda) = \Pi^n_{k=1} p_k(\lambda)^{m_k}$ where $p_1, \ldots p_n$ are different irreducible polynomials. It is plain that if $w = \Pi_{k \neq j}\, p_k(\lambda)^{m_k}$, $p(w;\lambda) = p_j(\lambda)^{m_k}$ By virtue of the precedings considerations $p(u;\lambda)$is divisible by all the polynomials $p_j(\lambda)^{m_j}$, and hence by $p(v;\lambda)$ itself. This ends the proof of Theorem 1.

Remark 1 Clearly, Theorem 1.1 remains valid for general K, E and A if we *assume*

$$\sup \{ \deg p(u;\lambda) \ ; \ u \in E \} < \infty \tag{1.6}$$

On the other hand, if (1.6) is false the conclusion of Theorem 1.1 might not hold. In fact, let E consist of all sequences $\{a_0, a_1, \ldots\}$ of elements of K such that $a_k = 0$ except for a finite number of indices, $A\{a_0, a_1, \ldots\} = \{a_1, a_2, \ldots\}$. Then for each $u \in E$ there exists $n = n(u)$ such that $A^n u = 0$; however, it is easy to see that A is not a Jordan operator.

Remark 2 We only need to assume in Theorem 1.1 the existence of a function $f(u; \lambda)$ for each $u \in E$, analytic in $\sigma(A)$ such *that* $f(A)u = 0$(2). In fact, any such f can be written $f = gp$, where g has no zeros in $\sigma(A)$ and p is a polynomial. Then $f(A) = g(A)p(A)$ and, since $g(A)$ is one-to-one $f(A)u = 0$ implies $p(A)u = 0$.

Remark 3 It is clear from the proof of Theorem 1.1 that we need to assume the existence of $p(u; \lambda)$ (or $f(u; \lambda)$, see Remark 2) only for u in a subspace of the second category of E.

Theorem 1.2 Let $m \geqq 1$. *Assume that for every m-ple* $(u_1, u_2, \ldots u_m)$ *there exists a m-ple of polynomials* $(p_1, \ldots p_m)$ *not all zero such that* $\Sigma^m_{k=1} p_k(A)u_k = 0$. *Then A is a Jordan operator.*

Proof: Let E^m be the Banach space of all m-ples $(u_1, u_2, \ldots u_m)$ of elements of E (pointwise operations) normed with, say, $|(u_1, u_2, \ldots u_m)| = \max(|u_1|, |u_2|, \ldots |u_m|)$. *Let* $E_N{}^m = \{(u_1, u_2, \ldots u_m) \in E^m$ such that there exists polynomials $p_1, p_2, \ldots p_m$ not all zero with $\Sigma^m_{k=1} p_k(A)u_k = 0$ and $\max_k \deg p_k \leqq N\}$. It is easy to show like in the proof of Theorem 1.1 that each $E_N{}^m$ is closed; thus by Baire's cathegory theorem some $E_N{}^m$ contains a sphere. This implies again that the degree of the polynomials $p_1, p_2, \ldots p_m$ in the statement of Theorem 1.2 can be supposed bounded by a constant N independent of $(u_1, u_2, \ldots u_m)$.

We end now the proof by induction. If $m = 1$ we are in the case considered in Theorem 1.1. Let $m > 1$ and let $(u_1, u_2, \ldots u_{m-1})$ be any $(m-1)$ — ple of elements of E.

Consider the m-ple

$$(u_1, u_2, \ldots, A^{N+1}u_{m-1}, u_{m-1})$$

By the preceding considerations, there exists a m-ple $(p_1, p_2, \ldots p_m)$ of polynomials, not all zero and such that $\Sigma^{m-2}_{k=1} p_k(A)u_k + (p_{m-1}(A)A^{N+1} + p_m(A))u_{m-1} = 0$, $\max_k \deg p_k \leqq N$. Since

(2) See [2], VII for the necessary notions of operational calculus.

deg $p_m \leqq N$ the polynomials above cannot be all zero, and thus our inductive step is achieved. Theorem 1.2 is proved.

Remarks 1 and 3 after Theorem 1.1 have evident generalizations to this case. As regards to Remark 2 we only need to assume in Theorem 1.2 for each $(u_1, \dots u_m) \in E^m$ the existence of m functions $f_1, \dots f_m$, analytic in a domain $D \supset \sigma(A)$ (independent of $(u_1, \dots u_m)$), not all zero, such that $\Sigma f_k(A) u_k = 0$. The proof is substantially similar to that of Theorem 2.2 below.

§ 2. *Rigidity of linear control systems*

We consider in this paragraph linear control systems

$$u'(t) = Au(t) + Bf(t), \ t \geqq 0 \tag{2.1}$$

Here A is the infinitesimal generator of a strongly continuous *semigroup* $T(t)$ of bounded operators in the complex Banach space E, $u(t)$ is a point in the space E describing the state of the system at the time t, $f(t)$ is a function (the *input* or *control*) with values in some other Banach space F and the linear bounded operator $B : F \rightarrow E$ is a "transmission mechanism" through which f acts on (2.1).

We shall understand by a solution of (2.1) with initial data $u(0) = u \epsilon E$ and input f in some space $L^p(0, \infty; F)$, $1 \leqq p \leqq \infty$, the expression

$$u(t) = T(t)u + \int_0^t T(t-s)\, Bf(s)\, ds \tag{2.2}$$

where $T(t)$ is the semigroup generated by A (see [4])

A point $v \in E$ will be called *reachable from* u if there exists f such that the solution $u(t)$ of (2.1) starting at u (say, for $t = 0$) satisfies $u(t) = v$ for some $t \geqq 0$.

Definition The linear control system (2.1) will be called *rigid* if any point v, reachable from another point u in the time t by means of some control f is not reachable from u in the same time by any control different from f.

It follows easily from the representation (2.2) for the solution of (2.1) (and the replacement of $t - s$ by s in the integral) that the system (2.1) will be rigid if and only if the map

$$f \rightarrow \int_0^t T(s)\, Bf(s) \,.\, ds \tag{2.3}$$

from $L^p(0, t; F)$ to E is one-to-one for all $t > 0$

Let us pass now to establish the relation between these notions and the results in § 1. In view of the last observation in the proof of Theorem 2.2 in [4] we need only to consider the case $p = 2$. Observe next that if F is m-dimensional unitary space, the space $\mathcal{L}\,(F; E)$ (3) of all linear bounded operators from F to E can be algebraically and topologically identified with the space E^m defined in the proof of Theorem 1.2 by means of the correspondence that assigns to the element $(u_1, \ldots, u_m) \in E^m$ the operator in $\mathcal{L}\,(F; E)$

$$B(x_1, \ldots, x_m) = \Sigma_{k=1}^{m} x_k\, u_k\,, \quad (x_1, \ldots, x_m) \in F \tag{2.4}$$

It is a consequence of the functional calculus for infinitesimal generators (see [4], § 2) that if

$$f(s) = (f_1(s), \ \ldots, f_m(s)) \in L^2\,(0, \infty\,; F)$$

and B is the operator (2.4)

$$\int_0^t T(s)\, Bf(s)\, ds = \Sigma_{k=1}^{m} \hat{f}_k(A) u_k$$

where the functions $\hat{f}_k$ (the Fourier transforms $\hat{f}_k(\lambda) = \int f_k(s)\, \exp(\lambda s)\, ds$ of f_k) belong to the space H^2 of the left half-plane (see [4], § 2 and [3])

Finally, let us recall the notion of *operator of admissible meromorphic type,* generalization of that of Jordan operator for the unbounded case. An infinitesimal generator A is said to be of admissible meromorphic type if the resolvent $R(\lambda; A)$ is a meromorphic function with poles of order m_k at points λ_k and

$$-\Sigma\, m_k\, Re\, \lambda_k \,/\, (1 + |\, \lambda_k \,|^2) < \infty$$

(see again [4], § 2). The preceding considerations make clear the equivalence of

(3) We endow $\mathcal{L}\,(F; E)$ with the uniform topology of operators.

Theorem 2.1. Let A be an infinitesimal generator satisfying conditions (2.1.*a*), (2.1.*b*) *of* [4], § 2. *Assume A is not of admissible meromorphic type. Then the linear control system* (2.1) *is rigid for all operators* $B \in \mathcal{L}(F; E)$ *except for those in a subset of the first category of* $\mathcal{L}(F; E)$

and

Auxiliary Theorem 2.2 Let A satisfy the same conditions of Theorem 2.1. Assume there exists a subset L of the second category of E^m *such that for every* $(u_1, \ldots, u_m) \in L$ *there exist m functions* $f_1, \ldots, f_m$ *in* H^2, *not all zero and such that* $\Sigma_{k=1}^{m} f_k(A) u_k = 0$. *Then A is of admissible meromorphic type.*

For the proof, we shall make use of

Lemma 2.3. Let $\{ f_n \}$ *be a sequence in* H^2 *of the half-plane* $Re\, \lambda \leqq 0$ *such that* $| f_n |_{H^2} \leqq 1$. *Then there exists a subsequence* $\{ f_m \}$ *such that:*

(a) $\{ f_m \}$ *converges weakly to a function* $f \in H^2$, $| f |_{H^2} \leqq .v$

(b) $f_m(A)$ *converges to* $f(A)$ *in the uniform topology of operators.*

Proof: The fact that there exists a subsequence $\{ f_m \}$ satisfying (*a*) is an elementary fact of the theory of H^2 (in fact, Hilbert) spaces. To show (*b*), let us consider the representation (2.11) of [4]

$$f_m(A) = \frac{1}{2\pi i} \int_{P(c,\theta)} f_m(\lambda)\, R(\lambda; A)\, d\lambda \qquad (2.5)$$

where $P(c, \theta)$ is the contour $c + | y | \operatorname{cotg} \theta + iy$, $-\infty < y < \infty$ for suitable $c < 0$, $\theta > \pi/2$ (see [4], § 2). Cauchy's formula

$$f_m(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f_m(it)}{it - \lambda}\, dt \qquad (2.6)$$

and the weak convergence of $\{ f_m \}$ imply that $\{ f_m \}$ converges uniformly on compacts of $Re\, \lambda < 0$ to f. It is then clear that (*b*) will hold for $\{ f_m \}$ if we can show

$$\lim_{n \to \infty} \left| \int_{P(c,n,\theta)} f_m(\lambda)\, R(\lambda; A)\, d\lambda \right| = 0 \qquad (2.7)$$

uniformly with respect to *m*, where $P(c, n, \theta)$ is the intersection of $P(c, \theta)$ with the region $|\lambda| \geqq n$. But in view of (2.1.b) of [4]

$$|R(\lambda; A)| \leqq C/|\lambda|$$

for $\lambda \in P(c, \theta)$ and some constant C. Furthermore, (2.6) and the Cauchy-Schwarz inequality imply

$$||f_m(\lambda)| \leqq C/|\lambda|^{1/2} \tag{2.9}$$

for $\lambda \in P(c, \theta)$ and some constant C, uniformly with respect to m. (2.8) together with (2.9) imply 2.7) and, a fortiori, Lemma 2.3

Proof of Theorem 2.2. Define subsets $L_{M,N}$ ($M = 1, 2, 3, \ldots$ $N = 1, 2, \ldots m$) of L as follows: $L_{M,N} =$ $(u_1, \ldots, u_m) \in L$ such that there exist functions $f_1, f_2, \ldots, f_m$ in H^2, not all zero and such that (a) max $_k |f_k|_{H^2} \leqq 1$ (b) $|f_N(c)| \geqq 1/M$, c a fixed point outside $\sigma(A)$, (c) $\Sigma_{k=1}^{m} f_k(A) u_k = 0$. It is easy to see that every $(u_1, \ldots, u_m) \in L$ belongs to some $L_{M,N}$ (if the corresponding functions $f_1, \ldots, f_m$ all vanish at c multiply them by a convenient power of $(\lambda - c)^{-1}$) and that each $L_{M,N}$ is closed (to do this we proceed in a way similar to that of Theorem 1.2 and make use of Lemma 2.3) Again by an application of Baire's theorem we deduce that some $L_{M,N}$ has an interior point, and this can be easily seen to imply (possibly after a rearrangement of indices) that the functions $f_1, \ldots, f_m$ in the statement of Theorem 2.2 can be chosen in such a way that $f_m(c) \neq 0$.

The proof ends now like that of Theorem 1.2. Let $(u_1, \ldots u_{m-1})$ be any $(m-1)$ —ple of elements of E, and let $g(\lambda) = (\lambda - c)$ $(\lambda + c)^{-2} \in H^2$. Then, if $f_1, \ldots, f_m$ are the functions corresponding to the m-ple

$$(u_1, \ldots, u_{m-2}, g(A) u_{m-1}, u_{m-1})$$

we have $\Sigma_{k=1}^{m-2} f_k(A) u_k + (f_{m-1}(A) g(A) + f_m(A) u_{m-1} = 0$, the functions $f_1, \ldots, f_{m-2}, f_{m-1} g + f_m$ not all zero. This allows us to reduce the case of m-ples to the case of $(m-1)$ ples, and when $m = 1$ Theorem 2.2 reduces to Theorem 2.2 of [4].

Remark Theorem 1.2 states that when A is not of admissible meromorphic type and F is finite-dimensional then (2.1) is rigid for "most" operators B in $\mathcal{L}$ $(F; E)$. The situation changes when

F is of infinite dimension; for instance, if $E = F$ it is not difficult to see that (2.1) is not rigid when B has a bounded inverse or is not one-to-one.

BIBLIOGRAPHY

[1] A. E. TAYLOR, *Introduction to Functional Analysis*, Wiley, New York, 1958.

[2] N. DUNFORD and J. T. SCHWARTZ, *Linear Operators*, vol. 1, Interscience, New York, 1957.

[3] K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall Inc., Englewood Cliffs, 1962.

[4] H. O. FATTORINI, *Control in finite time of differential equations in Banach space*, Comm. Pure Appl. Math. XIX (1966) 17 - 34.

[5] I. KAPLANSKY, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor, 1954.

NOTES ON THE MEASURE EXTENSION PROBLEM,

by J. C. MERLO and R. PANZONE

1. INTRODUCCIÓN. The so called measure problem has been proposed by Lebesgue in 1904 ([L]) and solved by Vitali ([V]) the next year, and asks for a traslation invariant finite measure in $\mathcal{P}$ ([0,1)), (see next paragraph for the nomenclature).

Vitali's theorem asserts that the only solution is $m = 0$, (a proof of it can be seen in [H], p. 70). Observing that the conditions on which is enunciated the problem imply that every point must have measure zero, Banach and Kuratowski proposed *the generalized measure problem*: define in $\mathcal{P}$ ([0,1)) a real valued, signed measure, null on every point. The same authors proved ([BK]) the following: under the continuum hypothesis the only solution is $m = 0$.

The problem was generalized even more by Ulam who proved: *if X is a set whose power is weakly accesible, then, on $\mathcal{P}$ (X), may be defined only one real valued, signed measure, vanishing at each point*: $m = 0$, (cf. [U] and [B]).

Something more can be said if we impose more restrictive conditions on m, and precisely: if X is a set whose power is strongly accesible, it cannot be defined on $\mathcal{P}$ (X) a$0 - 1$ valued measure, vanishing at every point and non trivial, (Ulam-Tarski, [U]).

The problem can be more generally posed as follows, (cf. [$B1$] and [LM]): *let (X, $\mathcal{B}$, P) be a probability space and $\mathcal{A}$ a σ-algebra of $\mathcal{P}$ (X) containing $\mathcal{B}$. Does there exist a measure (hence, a probability) $\bar{P}$ on $\mathcal{A}$ such that $\bar{P} = P$ on $\mathcal{B}$?*

(No generatlity is lost considering probabilities instead of finite signed measures as it follows immediately from Jordan-Hahn decomposition theorem for signed measures.). When $\mathcal{B} = \{\phi, X\}$ and $\mathcal{A} = \mathcal{P}$ (X) we are in the Ulam's case and when besides $X = [0,1)$, in the case considered by Banach and Kuratowski. Therefore, the problem has in general no non-trivial solution. However, if $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$, where $\mathcal{C}$ is a finite partition of X, there are infinite solutions (Los-Marcewski), and the same holds when $\mathcal{C}$ is denumerable partition (Bierlein).

This paper is a set of notes on this problem and gives also a general view of if.

2. NOMENCLATURE. By a measure we mean a σ-additive, non negative set function P defined on a σ-algebra of subsets of a set X and σ-finite. Whenever $P(X) = 1$, it will be called a probability.

A measure algebra is said to be purely atomic if the Boolean algebra of its sets mod. null sets (its Boolean algebra associated) is generated by a denumerable family of atoms. A measure will be called purely atomic (atomic) if its measure algebra associated is purely atomic (has atoms), and it will be called discret if all its mass is concentrated on a finite or countably infinite set of points. Finally, a measurable set S will be said indecomposable if and only if for every measurable $T \subseteq S$, $T = \phi$ or $T = S$.

If $\mathcal{B}$ and $\mathcal{C}$ are algebras of subsets of X, $\mathcal{B} \vee \mathcal{C}$ indicates the σ-algebra generated by them.

I will designate the half-closed unit interval $[0,1)$ and $\mathcal{P}(X)$ the family of subsets of the set X. βX will mean the Stone-Čech compactification of X, if X has a completely regular topology.

Aph$_\alpha$ will indicate the α-the infinite cardinal. We shall not enter into the definitions of weakly and strongly accesible cardinal numbers; it will suffice to us to observe shat Aph$_1$ is weakly accesible and every Aph$_\alpha \leqq c =$ the continuum power, is strongly accesible.

3. AUXILIARY RESULTS. a) Assume $(\Omega_i, \mathcal{A}_i)$, $i = 1, 2$, are measurable spaces and $T: \Omega_1 \rightarrow \Omega_2$, is a measurable application such **that** $T^{-1}(\mathcal{A}_2) = \mathcal{A}_1$. Assume P_2 is a probability measure on Ω_2. Then, $P_1(T^{-1}B) = P_2(B)$, $B \in \mathcal{A}_2$, defines a probability P_1 on $\mathcal{A}_1$ if and only if $P_2^*(T(\Omega_1)) = 1$, (Doob).

The proof is straightforward.

b) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{B}$ a σ-algebra, $\mathcal{B} \subseteq \mathcal{A}$ Suppose $(\Omega, \mathcal{B}, P)$ is not purely atomic. Let B be a subspace of $L^\infty(\Omega, \mathcal{A}, P)$ such that every function of $L_1(\Omega, \mathcal{B}, P)$ verifies:

$$\|f\|_1 = \underset{\|b\|_\infty = 1,\, b \in B}{\text{supremum}} \int f b \, dP \quad .$$

Then, there exists $b^* \in B^*$ such that for any $f \in L^1 (\Omega, \mathcal{A}, P)$, it is possible to find $b \in B$ with :

$$b^* (b) \neq \int f \, b \, dP .$$

In other words, there exists a bounded linear functional in B^* not representable as a function of $L^1 (\Omega, \mathcal{A}, P)$,

The proof of this results is given in § 6, [*NP*], although the statement is slightly different.

c) If f_n is a non-negative submartingale sequence and $f \in L^1$ closes it on the right, then f_n converges to f *a.e.* and in L^1. Besides, f_n is uniformly integrable if and only if f_n converges in probability to f which closes the submartingale on the right and $\int f_n \, dP \rightarrow \int f \, dP$. (Uniformly integrable means $\int f_n \, dP \rightarrow 0 \; \{ f_n \geqq a \}$ uniformly in n, when $a \rightarrow \infty$). (cf [*Le*] p. 394 and p. 528).

4. The measure problem for finitely additive measures. We want to exhibit now the several possibilities that appear combining algebras, σ-algebras, finitely additive an σ-additive measures.

Problem. *Let X be a set and Σ an algebra or σ-algebra of sets of X. Let Φ be another subalgebra of $\mathcal{P}$ (X) and $\overline{\Sigma}$ the algebra generated by Σ and Φ. Let m be a finitely additive or σ-additive signed measure of bounded variation defined on Σ. Define a signed measure of bounded variation $\overline{m}$ on $\overline{\Sigma}$ such that on Σ, $m = \overline{m}$.*

The problem has many subcases which we designate with we designate with $(a, b/c, d)$ where $a, b, c, d,$ stand for the numbers 0,1, and with the following convention: $a(b)$ represents Σ $(\overline{\Sigma})$ and will be equal to 0 if Σ $(\overline{\Sigma})$ is an algebra and 1 if it is a σ-algebra; $c(d)$ represents $m(\overline{m})$ with value 1 or 0 depending whether or not m is σ-additive. The decomposition theorems associated to the names of Hahn and Jordan (cf. [*D S*] pp. 98 and 129; [*Le*], pp. 86-87) assert that for any values of a and b, it is sufficient to solve the problem for non-negative measures. Hence, we suppose that $\overline{m}$ and m are $\geqslant 0$, and besides that $m(X) = 1$.

Oustanding results in this situation are the following theorems.

Banach-Hausdorff theorem. Let R^n be the euclidean space of dimension n. For $n = 1(2)$, they can be defined on $\mathcal{P}$ (R^n) two finitely additive positive measures μ, $\overline{\nu}$, traslation (and rotation) invariant, taking the same values on intervals and such that μ is an extension of Lebesgue measure and $\overline{\nu}$ is not, ([*Bch*]); for

$n > 2$ the only finitely additive measure on $\mathcal{P}$ (R^n), vanishing at each point and rotation invariant, is the trivial one, ([*Hf*], p. 469).

ALEXANDROFF THEOREM. Let X be a compact space and m a finitely additive and regular measure defined on an algebra Σ, then m is σ-additive, and therefore admits a unique extension to the σ-algebra generated by Σ, (cf [*DS*], p. 138).

For a generalization of this result cf. *T.* 3 *B,* [*B* 1]. With the problem of determining in what case a finitely additive measure in a Boolean algebra is σ-additive deal Kelley's results (cf. [*K*] and [*Lu*], § 6). The same type of result is of great importance in the theory of measures in topological vector spaces. Cf. for example [*G V*], chapter IV.

CARATHEDORY THEOREM. Every σ-additive measure on an algebra admits a unique extension to the σ-algebra generated by Σ.

These theorems are examples of the cases (00/00), (00/01), and (01/11), respectively. Example for the case (11/11) is the result of Los and Marcewski already mentioned in the introduction.

Suppose that $d = 1$, i.e. $\overline{m} \in \sigma$-additive. Then, a case $(a\,0/c1)$ admits a solution whenever it is already a case type $(a1/c1)$, and in this last case it admits a solution if there is also a solution after replacing $\overline{\Sigma}$ by its generated σ-algebra (as one can see applying Caratheodory's extension theorem). That is, a case $(a1/c1)$ is always reduced to a case $(a\,1/11)$. Another application af Caratheodory's theorem shows that a case $(a\,1/11)$ can be reduced to a case (11/11).

Therefore, any case $(a\,0/c1)$ can be reduced to a case (11/11), and every case of this last type is also type $(a0/c1)$. Concluding, any measure problem with $d = 1$ is finally reduced to two problems. First, to determine that P on Σ is σ-additive as in Alexandroff, theorem, second, solve a problem of type (11/11). In this paper we shall be essentially concerned with the case (11/11).

The case $d = 0$ cannot be discused as before and it is vinculated with a set of astonishing results, cf. for example [*B T*], [*v N*], [*B c h*], [*H f*], and Hadwiger's book, [*H r*]. To show the difference of both cases it is enough to compare the theorem of Ulam and Tarski with the following result due also to Tarski: for any inifite set X there exists a non-discret, finitely additive, 0-1 valued measure, defined on $\mathcal{P}$ (X). Let us prove it.

Theorem 1. *Le* $\mathcal{B}$ *be an algebra of subsets of* X *and* P *a finitely additive measure on* $\mathcal{B}$. *There exists a finitely additive measure* $\overline{P}$ *on* $\mathcal{P}(X)$ *such that* $\overline{P} = P$ *on* $\mathcal{B}$. *If* P *is* $0-1$ *valued,* $\overline{P}$ *can be chosen* $0-1$ *valued.*

Proof. There is a lattice isomorphism between the family of (real valued) bounded functions on X and the space of continuous functions in $C(\beta X)$. Under this isomorphism the space of bounded $\mathcal{B}$ measurable functions is in correspondence with a subspace S of $C(\beta X)$. P induces a bounded linear functional on $L^\infty(X, \mathcal{B}, P)$ and therefore on S, which can be extended to $C(\beta X)$. By the Riesz representation theorem this extension can be represented by a regular Borel measure μ. The restriction of μ to the clopen sets of βX is a finitely additive measure. By a result of Čech two sets of X are disjoint if and only if the clopens which are their closures are disjoint. Therefore, the restriction μ_0 of μ to the clopen sets can be also understood as defined on $\mathcal{P}(X)$, and defining there a finitely additive measure $\overline{\mathcal{P}}$. From the construction it follows that $\overline{P} = P$ on $\mathcal{B}$. Let K be the support of μ on βX. By definition of support, every clopen set intersecting K has μ-positive measure. If P is 0-1 valued, every clopen which is the closure of a set of $\mathcal{B}$, must contain K or be disjoint to it. Let ν be the δ-measure corresponding to a point of K. ν is also an extension of the linear functional associated to P and its restriction to $\mathcal{P}(X)$ is 0-1 valued.

Suppose B is an indecomposable infinite positive set of $\mathcal{B}$. Its closure on βX contains a point y in $\beta X - X$. Take a δ-measure concentrated on y and of magnitude $P(B)$ and proceed as above for P restricted to $X - B$. The union of these two partial extensions is an extension of $P, \overline{P}$, whose restriction is null on every point of B. (Naturally, it is not σ-additive). *Q.E.D.*

5. Extreme case. We consider in this section the case with $\mathcal{B} = \{X, \phi\}$ *and* $\mathcal{A} = \mathcal{P}(X)$. We note with Ω the first infinite non-countable ordinal.

Thorem 2. *Assume* $|X| = Aph_1$.

I) *There exists a family of subsets of* $X, \{A^i_k\}, i, k = 1, 2, \ldots,$ *such that* $A^i_k \wedge A^i_j = \phi$ if $k \neq j$, whatever be i; $\sum_{k=1}^{\infty} A^i_k = X$ and $\left| \bigcap_{i=1}^{\infty} (A^i_1 + \ldots + A^i_{k_i}) \right| \leq Aph_0$ *whatever be the sequence* $k_1, k_2, \ldots,$ (Banach-Kuratowski).

II) *There exists a family* $\mathcal{F}$ *of sequences of positive integers with* $|\mathcal{F}| \geqq Aph_1$ *and such that for any sequence of positive integers* $S = (S_1, s_2, \ldots)$, *it holds*:

$|\{(t_1, t_2, \ldots) \in \mathcal{F};\ t_i \leqq s_i,\ \textit{for every}\ i\}| \leqq Aph_0$, (Banach-Kuratowski).

III) *There exists a sequence of functions* $\{f_n\}$, *defined on* X, *such that* $0 \leqq f_n \leqq 1$, $f_n(x)$ *converges to* 0 *for each* $x \in X$, *and if* $Y \subseteq X$ *is a set where* f_n *converges uniformly then* $|Y| \leqq Aph_0$, (Sierpinski).

1) *It does not exist a non-discret probability measure defined on* $\mathcal{P}(X)$, (Ulam).

2) *There exists a denumerable family of subsets of* X, $\{A_n\}$ *such that the generated* σ*-algebra* $\mathcal{B}(\{A_n\})$ *contains the one-point sets and cannot be the domain of definition of a non-discret probability measure*, (Bierlein).

3) *It is possible to define on* X *a real valued function* f *such that* $\mathcal{B}(f) =$ *the least* σ*-algebra on which* f *is measurable, contains the one-point sets and is not domain of definition of a non-discret probability measure.*

4) *Let* P *be a probability on* $\mathcal{B} \subset \mathcal{P}(X)$. *If* P *is extendable to* $\mathcal{P}(X)$ *there exists a subspace* B *of* $L^\infty(X, \mathcal{P}(X), P)$ *determiming for* $L^1(X, \mathcal{B}, P)$ (*) *such that dual* B^* *of* B *is equal to* $L^1(X, \mathcal{B}, P)$.

5) *For any* σ*-algebra* $\mathcal{B} \subset \mathcal{P}(X)$, *any measure on* $\mathcal{B}$ *extendable to* $\mathcal{P}(X)$ *is purely atomic.*

Then, the propositions I), II), and III), are equavalent and they imply 1), 2), 3) and 4), which are equivalent and true.

Asuming the continuum hypothesis, I) holds. The equivalence of I), II) and III) also holds for $|X| = c$.

PROOF. I) $\leftrightarrow$ II) : cf. [BK], pp. 130-131. I) $\leftrightarrow$ III) : [S], p. 279. In the proofs no use is made of the magnitude of the power of X. I) holds assuming the continuum hypothesis: cf. [BK], p. 130. III) $\rightarrow$ 1) : suppose P is a non-discret probability on $\mathcal{P}(X)$. Without loss of generality we can assume that $P(\{x\}) = 0$ for every $x \in X$, and therefore, the countable sets will have measure zero. If $f_n \rightarrow 0$ pointwise, by Egoroff theorem, f_n converges uni-

(*) If means that $\|f\|_1 = \sup \left\{ \int f g\, dP\ ;\ \|g\|_\infty \leqq 1,\ g \in B \right\}$.

formly on a set of measure $1-\epsilon$, and hence on a set of power Aph_1, contradicting III).

2) $\rightarrow$ 1) : trivial. 1) $\rightarrow$ 2) : cf. [B 1], p. 33.2) $\leftrightarrow$ 3) : cf. [B1], *Th.* 1*B*, p. 32, where the equivalence is proved using a useful result of Banach (see [*Mi*]). 1) is proved in [*U*]. Let us see that 4) is equivalent to 1) and 5).

1) $\rightarrow$ 4) : from the hypothesis, it follows that $(X, \mathcal{B}, P)$ is purely atomic, and therefore, there is a determining subspace B (in $L^\infty(X, \mathcal{B}, P)$) for $L^1(X, \mathcal{B}, P)$ such that $B^* = L^1$ ("the space of bounded sequences tending to zero when $n \rightarrow \infty$"). 4) $\rightarrow$ 5) $\rightarrow$ 1). Assume that P is a probabiliy measure on $\mathcal{B}$ and extendible to $\mathcal{P}(X)$ and suppose that it is not purely atomic. From § 3, *b*), it follows that every determining subspace for $L^1(X, \mathcal{B}, P)$ admits a linear functional which is not representable as a function of $L^1(\mathcal{B})$, contradicting 4). Hence $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ is purely atomic.

Extending it to a probability on $\mathcal{P}(X)$ and using the theorem of Ulam-Tarski mentioned at the introduction, it follows that P is a discret measure. (We can assume Ulam-Tarski theorem since its proof is independent of the proof of Ulam theorem).

6. General extreme case. It has been proved by Luxemburg (cf. [*Lu*], T. 4.5), that a complete Boolean algebra with the Egoroff property has at most a denumerable set of atoms, if it is assumed the continuum hypothesis. We shall not enter into the definition of Egoroff property.

It will suffice for us to say that every Boolean measure algebra has Egoroff property. A corollary of Luxemburg's result is that any complete Boolean measure algebra is isomorphic to the family of subsets of a denumerable set. However this result is trivial. The following theorem is in close connection with Luxemburg's result.

Theorem 3. *a) If* $(X, \mathcal{P}(X), P)$ *is a probability space and the continuum hypothesis holds(or better, c is weakly accesible), then* P *is purely atomic. b) If besides* X *is strongly accesible, then* P *is a discret measure.*

Proof. b) follows from a) using Ulam-Tarski theorem. Let us prove a). If P is not purely atomic, then we can assume, without loss of generality, that it is not atomic, *i.e.*, it has no atom. Therefore, it can be constructed a family of sets A_{rs}, r, s, rational numbers, $r < s$, which can be put in a one-to-one, measure and inclusion preserving, correspondence with the rational intervals of I, $\{[r, s]\}$. Let $\mathcal{D}$ be the family of sets D which are maximal with respect to the property of beeing contained or disjoint to every A_{rs}. Then, $|\mathcal{D}| \leq c$. Since P is defined on $\mathcal{P}(X)$, it is a fortiori defined on the family of subsets S having the property $D \in \mathcal{D}$ and $S \wedge D \neq \phi \rightarrow S \supseteq D$.

Hence, P induces a measure Q on $(Y = X/\mathcal{D}, \mathcal{P}(Y))$. From the construction it follows that there is a one-to-one correspondence τ between Y and a subset Z of real numbers, of Lebesgue exterior measure equal to one. Moreover, since Q is defined on $\mathcal{P}(Y)$, it means that the measure induced by the Lebesgue measure m on Z can be extended to $\mathcal{P}(Z)$. Now, defining $\overline{m}(A) = 0$ if A is a subset of $I - Z$ and $\overline{m}(A) = Q(\tau^{-1}(A))$ if $A \subseteq Z$, we get an extension to $\mathcal{P}(I)$ of the Lebesgue measure, (recall that $m^*(Z) = 1$). Then, a) follows from the assumption of weakly accessibility of c and the Ulam's theorem, Q.E.P.

From the proof it follows that the continuum hypothesis in theorem 3 could be omited, if the following problem had a negative solution.

PROBLEM 1. *Is it possible to find a probability measure which extends the Borel measure to* $\mathcal{P}(I)$?

And so, the problem for X, at least for non atomic measures, is reduced to the same problem for the unit interval. Observe that any sucho extension must be non-atomic and therefore taking into account Maharam theorem ([*M*]), it turns out that problem 1 is equivalent to the following;:

PROBLEM 2. *Is it possible to find a homogeneous probability which extends Lebesgue measure to* $\mathcal{P}(I)$?

We leave the details to the reader, (cf. Banach-Hausdorff theorem in § 4).

7. CRITERIONS FOR MEASURE EXTENSION. Let $(X, \mathcal{B}, P)$ be a probability space and $\mathcal{C}$ an algebra of subsets of X. Our purpose is now to discuss some methods to extend P to $\mathcal{C}$, in other words, to extend P to $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$.

In relation with the first and second method cf. [*B* 1], specially Ths. 3*A* and 2*B*.

We shall agree in this section that any σ-algebra $\mathcal{B}$ on which a measure is defined separates points, i.e., for any two points x, y, there exists $B \epsilon \mathcal{B}$ such that $x \epsilon \overline{B} \ni y$. This is not an essential restriction since except for atoms, the condition is satisfied by the completion of the measure and in a non--one-point atom several solutions are at hand. For example, if the atom has only a denumerable set of points we can add a non denumerable set of them problem and is of immediate application in the extreme cases.

7.1. CONSISTENCY CRITERION. $F(\mathcal{B})$ will design a family of $\mathcal{B}$ -measurable functions such that the least σ-algebra on which the functions of $F(\mathcal{B})$ are measurable ir $\mathcal{B}$ itself. $\Pi(\mathcal{B})$ will design a product of real lines: $\Pi \{R_f;\ f \epsilon F(\mathcal{B})\}$. Let $\Phi(\mathcal{B})$ be the application related to $F(\mathcal{B})$ and $\Pi(\mathcal{B})$ defined by $\Phi(x) = (\ldots, f(x), \ldots) \varepsilon \Pi(\mathcal{B})$, $f \epsilon F(\mathcal{B})$. Then $(\Pi(\mathcal{B}), \mathcal{S}, P\Phi^{-1}(\mathcal{B})$ is a probability space, ($\mathcal{S}$ is the family of Borel stts).

THEOREM 4. *A necessary and sufficient condition for the existence of a probability* $\overline{P}$ *on* $\mathcal{A} = \mathcal{B} \vee \mathcal{C}$ *with* $\overline{P}/\mathcal{B} = P$ *is the existence of a probability* μ *on* $(\Pi(\mathcal{B}) \times \Pi(\mathcal{C}), \mathcal{S})$ *such that*: 1) $\Phi(\mathcal{A})(X) = (\Phi(\mathcal{B}), \Phi(\mathcal{C}))(X)$ *is of exterior* μ*-measure one;* 2) *its projection on* $\Pi(\mathcal{B})$ *coincides with* $P\Phi^{-1}(\mathcal{B})$.

PROOF. The sufficiency follows from a), § 3. The necessity is trivial.

A necessary and sufficient condition for the existence of a probability μ on $\Pi(\mathcal{A})$ is the existence of consistent finite distributions on $\Pi_{i=1}^{n} R_{f_i} \times \Pi_{i=1}^{m} R_{g_i}$, $g_i \epsilon \mathcal{F}(\mathcal{C})$, $f_p \epsilon \mathcal{F}(\mathcal{B})$, (Kolmogoroff theorem). The projection of μ on $\Pi(\mathcal{B})$ will coincide with $P\Phi^{-1}(\mathcal{B})$ whenever the μ-probability of any set A defined on $\Pi_{i=1}^{n} R_{f_i}$, whatever be f_i and n, is equal to $P((f_1,\ldots,f_n)^{-1}(A))$.

COROLLARY. a) *Let* $(X, \mathcal{B}, P)$ *be a probability space and* A *a non-measurable subes tof* X. *There exists (infinitely many indeed) a probability* $\overline{P}$ *which extends* P *to* $\mathcal{B} \vee \{A\}$, (Los-Marcewski).

b) *Let* $\{A_\alpha\}$ *be a family of disjoint subsets of* X *such that the complement of a denumerable family* $\{A_n\}_{n=1}^{\infty}$, *is contained in a set* M *of interior measure zero. Then, there exist a measure* $\overline{P}$ *on* $\mathcal{B} \vee \{A_\alpha\}$ *such thet* $\overline{P}$. *extendeds* P, *(Bierlein).*

PROOF. Set V the sample space $(\Pi(\mathcal{B}), \mathcal{S}, P\Phi^{-1}(\mathcal{B})$ and $W = V \times R$, where R denotes the real line. Let us define μ and apply next the preceding theorem. Call $K(K')$ the measurable hull of A $(X - A)$. If $B \epsilon \mathcal{B}$, and $B \subseteq X - K$, put $\mu(B \times X\ \{0\}) =$

$= P(B)$, if $B \subseteq X - K'$, $\mu(B \times \{1\}) = P(B)$ and if $B \subseteq K \wedge K'$ define μ $(B \times \{1\} = \alpha P(B)$, $\mu(\times \{0\}) = (1-\alpha)P(B)$, where α will be fixed at once. Condition 2) of theorem 4 is then fulfilled. Set now $\Phi(\mathcal{A}) = (\Phi(\mathcal{B}), \mathcal{X}_A)$, i.e., $\Phi(\mathcal{A})(x) = \Phi(\mathcal{B})(x) \times \{0\}$ for $x \in A$, and $\Phi(\mathcal{A})(\mathcal{X}) = \Phi(\mathcal{B})(x) \times \{1\}$ for $x \in A$.

Choosing now α verifying:
$P(K) = P^*(A) \geqq \mu(K \times \{1\}) = \alpha \geqq P_*(A) = P(X - K')$, (**) it follows condition 1) of Th. 4. Observe that (**) is a neceassary condition. The same procedure applies for corollary b). We shall restrict curselves to the definition of μ. One way of doing it is so. Set $\Phi(\mathcal{A})(x) = \Phi(\quad)(x) \times \{n\}$ if $x \in A_n$, $n = 0,1,\ldots$, with $A_0 = M$. Let K_n be the measurable hull of A_n. If $B \in \mathcal{B}$ and $B \subseteq K_n - \bigcup_{1}^{n-1} K_j$, $n = 2, 3, \ldots$, define $\mu(B \times \{n\}) = P(B)$, if $B \subseteq K_1$, $\mu(B \times \{1\}) = P(B)$.

7.2. CONDITIONAL EXPECTATION METHOD. Suppose that $\mathcal{B} \subseteq \mathcal{A}$ and $(X, \mathcal{A}, P)$ is a probability space. Then, the conditional expectation operator $E(./\mathcal{B})$ has the following properties: 1) $E(./\mathcal{B}) : L^\infty(\mathcal{A}) \to L^\infty(\mathcal{B})$ is a contraction operator and preserves L^1-norms; 2) $\sum_{j=1}^{\infty} A_j = A_0$ implies $E(\Sigma A_j/\mathcal{B}) = \Sigma E(A_j/\mathcal{B}) = E(A_0/\mathcal{B})$; 3) $E(O/\mathcal{B}) = = 0 . E(1/\mathcal{B}) = 1$, *a.e.*.

Trying to use the conditional expectation operator to extend a measure, one gets:

THEOREM 1. *Let $\mathcal{F}$ be a family of $\mathcal{B}$-measurable functions such that:*

a) $f \in \mathcal{F} \to 1 \geq f \geq 0$ *a.e.*, b) *there exits an application φ from $\mathcal{A} \supseteq \mathcal{B}$ into $\mathcal{F}$ such that $\varphi(\Omega) = 1$, $\varphi(\phi) = 0$ and $\varphi(\sum_1^\infty A_j) = \sum_1^\infty (A_j)$, c) $A \in \mathcal{B} \to \varphi(A) = \chi(A)$, a.e.. If there is a probability P on $\mathcal{B}$, then $\bar{P}(A) = \int \varphi(A)\, dP$ defines a probability P on $\mathcal{A}$ with $\bar{P}/\mathcal{B} = P$.*

The proof is trivial. From this theorem easily follows the corollary of section 7.1, (cf. [B1]).

7.3. MARTINGALE CRITERION. Assume that $\mathcal{B}_n \subseteq \mathcal{B}_{n+1}$, $n = 1, 2, \ldots$, is a denumerable set of non-decreasing σ-subalgebras of $P(X)$ P_n, Q_n, are probabilities measures defined on $\mathcal{B}_n$. Suppose the P's admit a common extension $\bar{P}$ to $\mathcal{A} = \bigvee_{n=1}^{\infty} \mathcal{B}_n$. The problem is to establish conditions on the Q's so that they also admit a common extension to $\mathcal{A}$. It holds:

THEOREM 6. *Under the hypothesis mantioned above, if* $Q_{n+1}/\mathcal{B}_n = Q_n$ *and* $Q_n << P_n$, $n = 1, 2, 3, \ldots$, *then, the following propositions are equivalent.*

1) *there exist* $\bar{Q} << \bar{P}$ *defined on* $\mathcal{A}$ *such that* $\bar{Q}/\mathcal{B}_n = Q_n$ *for every* n.

2) $\{ f_n = dQ_n/dP_n \}$ *is martingale closed on the right with o closure function in* $L^1(X, \mathcal{A}, P)$.

3) $\{ f_n \}$ *is fundamental in* L^1.

4) $\{ f_n \}$ *is uniformly integrale.*

In these cases, $f_n = dQ_n/dP_n$ *converges a.s. and in* $L^1(X, \mathcal{A}, \bar{P})$ *to* $d\bar{Q}/d\bar{P}$.

PROOF. If $\{ f_n \}$ is L^1-fundamental then it is uniformly integrable (cf. [*Le*], p. 163). Observe now that the conditional expectation $E(f_{n+1}/\mathcal{B}_n)$ is a.e. equal to f_n, and therefore that $\{ f_n \}$ is a martingale sequence. Using c), § 3, $f_n \xrightarrow[\bar{P}]{} f$ and f closes the martingale on the right and belongs to $L_1(\bar{P})$. If $\{ f_n \}$ is a martingale closed on the right by an $L_1(P)$-function, then again by c), § 3, $f_n \xrightarrow{1} f$. Hence, 2), 3) and 4) are equivalent. Defining $\bar{Q}(A) = \int_A f\, d\bar{P}$, they, trivially, imply 1). Assume that 1) holds. Then, from f_n is closed on the right by $d\bar{Q}/d\bar{P} \in L^1(\bar{P})$, *Q.E.D.*

Similar results can be obtained if one asks instead of $Q_{n+1}/\mathcal{B}_n = Q_n$ that $Q_{n+1}(B) \geq Q_n(B)$ for every $B \in \mathcal{B}$. In this case, $dQ_n = f_n$ is a submartingale sequence.

REFERENCES

[B] BANACH S., *Uber additive Massfunktionen in abstrakten Mengen,* Fun. Math., XV, (1930), 97-101.

[Bch] BANACH, S., *Sur le problème de la mesure,* Fund. Math., IV, (1923), 7-33.

[Bl] BIERLEIN, D., *Über die Forsetzung von Wahscheinlichkeitsfeldern,* Zeitsch. f. Wahrsch. u.v.G., 1, (1962), 28-46.

[BK] BANACH, S. et KURATWSKI, C., *Sur un généralisation du problème de la mesure*. Fund. Math., XIV, (1929), 127-131.

[BT] BANACH, S. et TARSKI, A., *Sur la décomposition des ensembles de points en parties respectiment congruentes*, F. M., VI, (1924), 244-277.

[D] DIXMIER, J., *Sur certains espaces considérés par M. H. Stone*, Summa Bras. Math., (1951), vol. II, fasc. 11, 1-30.

[DS] DUNFORD, N. and SCHWARTZ J. T., *Linear operators, I.* New York, (1958).

[GJ] GILLMAN, L. and JERISON, M., *Rings of continuous functions*, New York, (1960).

[GV] GELFAND I. M. and VILENKIN N. Y., *Generalized functions*, IV, (1961), (Russian).

[H] HALMOS, P., *Measure theory*, New York, (1956).

[Hf] HAUSDORFF, F., *Gundzüge der Mengenlehre*, Leipzig, (1914).

[Hl] HALMOS, P., *Boolean algebras*, Univ. of Chicago, (1959).

[Hr] HADWIGER, H., *Vorlesungen über inhalt, oberfläche und isoperimetrie*, Berlin, (1957).

[HvN] HALMOS, P. and v. NEUMANN, J., *Operators methods in clasical mechanics, II*, Ann. of Math., vol. 43, (1942), 332-350.

[K] KELLEY, J. L., *Measures on Boolean algebras*, Pacific J. of Math., 9 (1959), 1165-1177.

[L] LEBESGUE, H., *Leçons sur l'intégration et la recherche des fonctions primitives*, Paris, (1904-1928).

[Le] LOÈVE, M., *Probability theory*, New York, (1963).

[LM] LOS, J. and MARCEWSKI, E., *Extensions of measure*, F. M. XXXVI, (1949), 267-276.

[Lu] LUXEMBURG, W. A. J., *On finitely additive measures in Boolean Algebras*, Journal f. d. reine und a. Math. 213, (1964), 165-173.

[M] MAHARAM, D., *On homogeneuos measure algebras*, Proc. N.A.S. U.S.A., vol. 28, (1942), 108-111.

[Mi] MARCEWSKI, E., *Sur les suites d'ensembles excluant l'existence d'une mesure par S. Banach*, Colloquium Math., (1948), 102-108.

[NP] NIRENBERG, R. and PANZONE, R., *On the spaces L^1 which are isomorphic to a B^**, *Revista* U.M.A., (1963), 119-130.

[S] SIERPINSKI, W., *Sur un théorème de MM. Banach et Kuratowski*, F. M., XIV, (1929), 277-280.

[Sz] SZPILRAJN, E., *Sur l'extension de la mesure lebesguienne*, F. M., (1935) 550-558.

[U] ULAM, S., *Zur Masstheorie in der allgemeinen Mengenlehre*, F. M., XVI, (1930), 140-150.

[Ul] ULAM, S. M., *A collection of mathematical problems, New York*, (1960).

[V] VITALI, G., *Sul problema della mesura dei gruppi di punti di una retta*, Bologna, (1905).

[vN] von NEUMANN, J., *Zur allgemeinen Theorie des Masse*, F. M., XIII, (1929), 73-116.

UNIVERSITY OF BUENOS AIRES

CRONICA

SEGUNDA CONFERENCIA INTERAMERICANA DE EDUCACIÓN MATEMÁTICA

Durante los días 4 al 12 de Diciembre de 1966 tuvo lugar en Lima (Perú) la Segunda Conferencia Interamericana de Educación Matemática. Participaron en ella representantes de todos los países latinoamericanos. La Conferencia estuvo presidida por Marshall H. Stone, presidente del Comité Interamericano para la Educación Matemática (C I A E M) y se nombró presidente honorario al Dr. Carlos Cueto Fernandini, Ministro de Educación Pública del Perú, a cuyo cargo estuvo el discurso de apertura.

De la Argentina participaron L. A. Santaló, que tuvo a su cargo la ponencia sobre "Los problemas que encuentra la reforma de la matemática, referentes a profesores y a programas, en América Latina" y Renato Völker, miembro del Comité organizador, quien disertó sobre "El nuevo curriculum y la formación de profesores en Argentina". Como invitados especiales asistieron a la Conferencia los profesores Hans Georg Steiner (Alemania), George Papy (Bélgica), Erik Kristensen (Dinamarca), André Revuz (Francia) y Pedro Abellanas (España).

Las deliberaciones se centraron principalmente en los problemas de la enseñanza de la matemática al nivel secundario y primer año universitario, si bien se hicieron varias referencias acerca de la importancia del problema en la escuela primaria. Se analizó también la obra realizada en los distintos países en cuanto a educación matemática desde la Primera Conferencia Interamericana celebrada en Bogotá en 1961, resultando un balance muy alentador en cuanto a cursos de perfeccionamiento realizados, textos publicados y ensayos puestos en marcha.

En la última sesión se nombró el nuevo comité del C I A E M que resultó constituido de la manera siguiente: César Abuauad (Chile), Ricardo Losada (Colombia), Manuel Meda (México), Leopoldo Nachbin (Brasil), L. A. Santaló (Argentina, vicepresidente), J. J. Schäffer (Uruguay, secretario), Edgardo Sevilla (Honduras), M. H. Stone (U.S.A., presidente) y José Tola (Perú). Entre las recomendaciones aprobadas figura la que "el C I A E M auspicie, en cada país, la constitución de un comité que fomente en escala nacional o regional actividades conducentes al desarrollo del medio matemático y que, además, preste la cooperación mencionada en las recomendaciones de la Conferencia de Ministros de Educación y de Ministros encargados del planeamiento económico en los países de América Latina y del Caribe, celebrada en Buenos Aires del 20 al 30 de junio de 1966".

Además, una comisión especial integrada por los profesores E. Suger, G. Papy, G. Steiner, B. Jones, R. Read, E. Kristensen, R. D. James, A. Colamarco, M. Meda, J. Arias, M. de Souza y C. Imaz, redactó el siguiente programa ideal, cuyo ordenamiento y forma de exposición se dejan librados a cada profesor:

EDAD 12-15 AÑOS:

1. Noción de conjunto. Operaciones con conjuntos.
2. Relaciones. Función, Equivalencia, Orden, Composición.
3. El anillo de los números enteros. Potencias. Divisibilidad.
4. Operación binaria. Ilustración del concepto de grupo.
   Solución de la ecuación $a^*x = b$.
   Aplicaciones a la geometría y a los sistemas de números.
5. Introducción progresiva y descriptiva de los axiomas de la geometría. Incidencia, paralelismo, ordenación. Proyección paralela, traslación...
6. Introducción progresiva y descriptiva del campo de los números reales y de los racionales. La ecuación lineal y la cuadrática.
7. El espacio vectorial del plano.
8. Coordenadas. Ecuación de la recta. Desigualdades. Semiplano, algunas aplicaciones (programación lineal).
9. Algunas formas de representar una función (tabulación, gráfica, expresiones analíticas...) Operaciones con funciones numéricas.
10. Geometría métrica del plano. Producto escalar. Teorema de Pitágoras.
11. Geometría analítica en bases ortogonales (recta, circunferencia, ...).
12. Solución de sistemas de ecuaciones lineales.

EDAD 15-18 AÑOS:

1. Estudio de los números reales.
2. Espacio Euclideano. Bases ortogonales. Desigualdad de Cauchy-Schwarz.
3. Transformaciones lineales del plano. Matrices de orden 2. El grupo de transformaciones ortogonales. Semejanza.
4. Números complejos.
5. Trigonometría.
6. Análisis combinatorio. Nociones de probabilidad.
7. Algoritmo de Euclides. Teorema de la factorización única.
8. Polinomios. Teorema de residuo.
9. Introducción progresiva y descriptiva de algunos conceptos topológicos. Los espacios topológicos usados en análisis elemental.
10. Funciones continuas. Límites. Sucesiones.
11. Derivación de funciones de una variable real.
12. Integración (preferentemente como límite de sumas).
13. Funciones elementales especiales
    (exponenciales, logarítmicas, circulares...)
14. Determinantes.
15. Geometría del espacio usando el espacio vectorial euclideano de 3 dimensiones. Geometría analítica en R^3
16. Probabilidad y Estadística elemental.

Para desarrollar este programa se cuenta con unos estudios de la escuela primaria que den al estudiante una preparación sólida en el manejo de las operaciones aritméticas y un conocimiento intuitivo de las figuras geométricas. La calculatoria elemental aprendida en la escuela primaria se ejercitará continuamente, para que no sea olvidada por el alumno.

BIBLIOGRAFIA

RUEL V .CHURCHILL, *Series de Fourier y problemas de contorno.* (260 páginas), McGraw-Hill Book Company, 1966.

Los textos del profesor R. Churchill sobre temas de matemáticas aplicables a la física son bien conocidos y han alcanzado gran difusión durante más de un cuarto de siglo. El libro que vamos a analizar es una traducción española de la segunda edición inglesa aparecida en 1963 (la primera databa de 1941).

Su objeto es el estudio de los problemas de contorno de ecuaciones en derivadas parciales utilizando el método de separación de variables; no presupone otros conocimientos que los de un alumno que ha seguido los dos primeros cursos de análisis matemático en una universidad argentina.

El primer capítulo (Ecuaciones en derivadas parciales de la física) plantea en forma general y somera los problemas de contorno y muestra como los problemas físicos conducen a las ecuaciones de las cuerdas de las membranas, del calor y de Laplace, ésta última se plantea también en coordenadas cilíndricas y esféricas.

El segundo capítulo (Superposición de soluciones) da la idea formal del método de superposición de soluciones usando series e integrales y hace aplicaciones a la resolución de la ecuación de las cuerdas.

El tercer capítulo (Sucesiones ortogonales de funciones) trata el problema del desarrollo en serie de funciones ortonormales. Este capítulo se resiente de la fecha de nacimiento de la obra; en aquellos tiempos los conceptos básicos del álgebra (a la que entonces se le decía moderna) parecían fuera de lugar en un libro de enseñanza para físicos e ingenieros. La situación hoy día no es esa y por ello creemos hubiera sido preferible rehacer enteramente ste capítulo y encuadrarlo en la teoría de los espacios prehilbertianos. Lo mismo se puede decir del estudio hecho en este capítulo del problema de Sturm-Liouville desarrollado sin definir el concepto de aplicación lineal entre dos espacios vectoriales.

Los capítulos cuarto (Series de Fourir) y quinto (Otras propiedades de las series de Fourier) desarrollan la teoría de las series de Fourier trigonométricas; se establece el teorema sobre convergencia puntual de las series de Fourier de funciones con derivadas laterales, se estudia la derivación e integración de las series y se terminan enunciando, sin demostración, algunos resultados complementarios.

El capítulo sexto (Integrales de Fourier) expone muy sucintamente algunas propiedades de las integrales de Fourier.

El Capítulo séptimo (Problemas de contorno) trata de la resolución, por separación de variables, de problemas de contorno en el caso de las ecuaciones de las cuerdas, unidimensional del calor y de Laplace con dos variables. Termina con una aplicación de la integral de Fourier a la ecuación del calor.

El Capítulo octavo (Funciones de Bessel y sus aplicaciones) estudia las propiedades básicas de las funciones J de Bessel (recurrencia, representaciones integrales, raíces,...) y el desarrollo de funciones en series de Fourier-Bessel. Hace después aplicaciones a la ecuación del calor en coordenadas polares con simetría radial y a la vibración de una membrana circular.

De un tipo completamente análogo, el capítulo noveno (Polinomios de Legendre y aplicaciones) estudia las propiedades de estos polinomios, las series de Fourier-Legendre y sus aplicaciones a casos partciulares de la ecuación de Laplace en coordenadas esféricas. Las funciones asociadas son simplemente mencionadas y no se ocupa de los armónicos esféricos.

El último capítulo (Unicidad de las soluciones) demuestra la unicidad de las soluciones de algunos problemas de contorno resueltos previamente.

Hay un gran número de ejercicios sobre los temas matemáticos y sobre las aplicaciones físicas. En general están bien elegidos y en varias ocasiones complementan el texto; en algunos casos creemos que los ejercicios hubieran debido ser incorporados al texto; por ejemplo la desigualdad de Schwarz está relegada a un ejercicio. La bibliografía es bastante completa y hay un índice alfabético.

El libro es fundamentalmente claro y bien ordenado; precisa claramente lo que demuestra y lo que se deja sin demostrar. Trata en forma completa algunos problemas y otros análogos en forma más sucinta pero precisando bien los puntos que hubieran debido ser más desarrollados.

Se trata de un texto claramente orientado hacia las aplicaciones a la física y la ingeniería. Con la salvedad hecha del tratamiento no algebráico de algunos problemas, es muy recomendable para los alumnos de esas carreras.

Manuel Balanzat

A. I. MARKUSHEVICH: *Theory of functions of a complex variables,* tres volúmenes. Serie Selected Russian Publications in the Mathematical Sceinces. Prentice Hall, Englewood Cliffs, N. J., 1965.

Es esta una obra muy bien traducida del original ruso por R. A. Silverman, quien es además editor de la serie a la que pertenece el título, y que ha añadido además una gran cantidad —casi cuatrocientos— de ejercicios, muchos de ellos de la excelente colección de Volkovyskii, Lunts y Aramanovich (publicada por Addison Wesley).

Se trata de un libro poco usual. Dividido en tres volúmenes, de los cuales sólo han aparecido dos, presenta en forma extremadamente detallada la teoría de funciones de variable compleja y algunas de sus ramificaciones. El tratamiento es riguroso y relativamente moderno, e incluye temas poco frecuentes en obras del mismo nivel.

Podemos resumir el contenido de los dos primeros volúmenes del siguiente modo: el primero incluye varios capítulos dedicados a algunos conceptos básicos: funciones de variable compleja, límites, continuidad, conexión, curvas, dominios, homeomorfismos (se prueba aquí, usando el teorema de la curva cerrada de Jordan, que fuera enunciado poco antes, que las funciones contínuas y biunívocas, definidas en un dominio del plano complejo, son homeomorfismos).

A continuación se entra de lleno en la teoría, en cinco capítulos destinados al estudio de las funciones elementales, incluyendo las multiformes. Por último, la tercera parte del volúmen trata la integración en el campo complejo y la teoría de las series de potencias, e incluye, entre otros temas poco frecuentes en libros de esta clase, un capítulo sobre métodos para desarrollar funciones en serie de Taylor, y algunas secciones dedicadas a las familias normales, donde se prueban los teoremas de Montel y Vitali, y se tratan algunas de sus aplicaciones a las funciones definidas por medio de integrales.

El segundo volumen trata las series de Laurent, el cálculo de residuos, las funciones armónicas y sub-armónicas, y las funciones enteras y meromorfas. Entre los temas poco usuales que se incluyen, pueden mencionarse las series de Dirichlet, teoría de interpolación, funciones univalentes y otros.

El tercer volumen incluirá, según se indica: representación conforme, teoría de aproximación, funciones periódicas y elípticas, superficies de Riemann y prolongación analítica.

En suma, se trata de un gran aporte a la literatura del tema, muy recomendable sobre todo como obra de consulta, y que será de utilidad a todos aquéllos cuyo campo de estudios tenga relación con la teoría de funciones.

G. Hansen

A. DELACHET, *Cálculo Diferencial e Integral*. Editorial Tecnos S. A. Madrid, 1966, 141 páginas. Traducción castellana de Miguel Truyol.

La obra original corresponde a un volumen de la conocida colección francesa *Que sais-je?* No se trata, por tanto, de una introducción elemental al cálculo infinitesimal, si no de una exposición de diversos temas, elegidos un poco al azar, de dicho cálculo, algunos tratados con cierto detalle (por ejemplo el estudio de las funciones $g(x)$ definidas por la ecuación funcional $g(g)(x) = f(x) =$ función dada, con ciertas condiciones) y otros con solamente el enunciado de los teoremas fundamentales.

Los capítulos del libro son los siguientes: 1. Funciones de variables reales (límites, continuidad, funciones de variación acotada); 2. Funciones derivables (de una y varias variables); 3. Concepto de integral (según Riemann); 4. Nociones sobre series y productos infinitos numéricos; 5. Funciones definidas por series e integrales.

La exposición es en general clara y la traducción cuidadosa, de manera que el librito puede ser muy útil como complemento de los cursos regulares de Análisis usuales en los primeros años universitarios, sea tan solo —como dice el autor en el prólogo— para llamar la atención de los estudiantes sobre ciertos temas delicados que muchas veces se ven con demasiada prisa y que, sin embargo, conviene conocer bien, tanto en su forma externa, que debe precisarse, como en su sentido intrínseco, que debe profundizarse.

L. A. Santaló

E. T. COPSON, *Asymptotic Expansions.* Cambridge Tracts in Mathematics and Mathematical Physics. Nº 55. Cambridge University Press, 120 págs.

El libro espane en forma clara una serie de métodos, que se hallan dispersos en la literatura, y que sirven para obtener desarrollos asintóticos de funciones dadas por una integral definida, o de funciones analíticas definidas como integral de contorno en el plano complejo. Los métodos son ilustrados con algunas de las más importantes funciones especiales. Supone para su lectura sólo conocimientos básicos de Análisis y de funciones de una variable compleja.

Dada la importancia de las funciones especiales y de su comportamiento asintótico en muchas ramas de la matemática, este texto puede ser útil a matemáticos puros o aplicados, y también a físicos teóricos.

Los siguientes métodos son explicados en detalle: Integración por partes, método de la fase estacionaria, Aproximación de Laplace, Lema de Watson sobre transformada de Laplace, método de las pendientes máximas, método del punto de ensilladura.

Los dos últimos capítulos tratan de la integral de Airy y de desarrollos asintóitcos uniformes respectivamente.

Agnes Panzone

A. P. ROBERTSON, W. ROBERTSON, *Topological vector spaces,* Cambridge, 1964.

Este pequeño libro de apenas 150 págs. lleva el nº 53 en la bien conocida colección "Cambridge Tracts in Mathematics and Mathematical Physics". Pese a su reducida dimensión este volumen ofrece al lector una buena y accesible introducción a la teoría moderna de los espacios vectoriales topológicos. Para su lectura sólo es necesario un conocimiento mínimo de álgebra lineal y de topología general, el cual es desarrollado sucintamente en los dos primeros párrafos del libro. Consta, por otra parte, de ocho capítulos, los cuales, con excepción del último, poseen apéndices que suministran ejemplos y donde pueden encontrarse esquematizados ulteriores desarrollos de los tópicos discutidos.

Su contenido es: Capítulo I: definiciones y propiedades elementales. Espacios vectoriales. Espacios topológicos. Espacios vectoriales topológicos. Capítulo II: dualidad y el teorema de Hahn-Banach. Aplicaciones lineales. Funcionales lineales y el teorema de Hahn-Banach. Dualidad y topología débil. Polares. Subespacios de dimensión finita. Adjunta. Capítulo III: conjuntos acotados. Topología polar. Cjtos. precompactos. Cjtos. compactos. Filtros Completidad. Teorema de Mackey-Arens. Capítulo IV: espacios tonelados y el teorema de Banach-Steinhaus. Espacios tonelados. Topologías en espacios de transformaciones. El doble dual y reflexividad. Capítulo V: límites inductivos y proyectivos. Espacios cocientes. Límites inductivos. Espacios de Mackey. Límites proyectivos. Espacios productos. Sumas directas. Subespacios suplementarios. Capítulo VI: completidad y el teorema del gráfico cerrado. La completación de un espacio localmente convexo. Teoremas del gráfico cerrado y de la transformación abierta. Esp. de Fréchet. Capítulo VII: otros tópicos. Límite inductivo estricto. Transformaciones bilineales y producto tensorial. Teorema de Krein-Milman. Capítulo VIII: aplicaciones lineales compactas. La teoría de F. Riesz. Teoría de dualidad.

Sin duda este texto puede ser ensayado con éxito en un curso semestral de auditorio matemático. Digamos finalmente que la impresión es óptima.

Rafael Panzone

PENNISI, L. L.: *Elements of complex variables,* X + 459 págs. Holt, Rinehart And Winston, New York, 1963.

Se trata de un libro excelente, que será apreciado, sobre todo, como texto para un primer curso de funciones de variable compleja.

El contenido es el siguiente: 1. Números complejos y su representación geométrica; 2. Conjuntos de puntos, sucesiones y aplicaciones; 3. Funciones analíticas (uniformes) de varaible compleja; 4. Funciones elementales; 5. Integración, (la exposición del teorema de Cauchy sigue la línea de Ahlfors: Complex analysis); 6. Series de potencias; 7. Cálculo de residuos; 8. Representación conforme; 9. Aplicación de las funciones analíticas a la teoría de fluídos.

Las características más notables del libro son: la cuidadosa motivación y exposición de los temas tratados, la abundantísima colección de ejemplos, estudiados con todo detalle, las interesantes observaciones que siguen a casi todo resultado expuesto, y en las que se precisa el alcance o se sugieren generalizaciones del os resultados obtenidos, y la excelente colección de ejercicios, complementada, al final del volúmen, con una sección de respuestas y sugerencias.

Para concluir diremos que es un libro que deberá ser tenido en cuenta, y que podemos recomendarlo sin reticencias.

G. Hansen

HEINS, Maurice: *Selected topixs in the classical theory of functions of a complex variable;* XI + 160 págs. Holt, Rinehart And Winston, New York, 1962.

Es este un muy buen libro sobre la teoría geométrica de funciones, y tiene por objeto presentar, en forma relativamente moderna, algunos de los resultados más importantes de la teoría clásica de funciones, como el teorema de representación conforme de Riemann, el teorema de Fatou de límites radiales, el teorema de Phragmén-Lindelöf, el teorema grande Picard, y otros.

El contenido del volúmen es el siguiente: 1. Preliminares; 2. Propiedades de cubrimiento de las funciones meromorfas; 3. El teorema de Picard; 4. Funciones armónicas y sub-armónicas; 5. Aplicaciones; 6. Comportamiento en el contorno de la transformación de Riemann para regiones de Jordan simplemente conexas, y un apéndice, que trata el teorema de Riesz de representación de funcionales de C, el teorema de Lebesgue de derivación de funciones monótonas, y una forma débil del teorema de Jordan.

Los prerrequisitos para leer el libro son: un primer curso de funciones de variable compleja, buen conocimiento de análisis real elemental, y algún conocimiento de topología. No se supone conocida la integral de Lebesque, y los elementos de la misma que son necesarios son desarrollados en el texto. La razón de esto, según el propio autor, no es dar la impresión de que se puede llegar bastante lejos sin conocimientos profundos de análisis real, sino más bien la de proveer una adecuada motivación al desarrollo de la teoría de Lebesque, mostrando áreas del análisis donde tal teoría se hace indispensable.

El libro exige una activa partciipación del lector, ya que algunos temas están desarrollados en sucesiones de ejercicios, provistas de indicaciones que los hacen accesibles. Los temas están tratados cuidadosamente, y con abundantes referencias que complementan el texto.

Se trata, en resumen, de un libro muy adecuado para todos aquéllos que deseen ampliar sus conocimientos en el campo de la teoría de funciones, y que puede ser un complemento muy apropiado al libro de Pennisi comentado en este mismo número.

G. Hansen

ARSAC, J.: *Fourier transforms and the theory of distributions,* traducido del francés por A. Nussbaum y G. C. Heim; XV 318 pág. Prentice Hall, Inc., Englewood Cliffs, N. J., 1966.

Se trata de un libro escrito por un físico para físicos. Por tal motivo no deben buscarse en él ni el tratamiento riguroso ni la sutileza de detalles propia de las obras especializadas; más aún: numerosos resultados aparecen sin demostración. Como el propio autor declara, ha evitado en la exposición "los puntos finos de la teoría de funciones".

En cambio, pueden hallarse en él abundantes aplicaciones, lo cual presupone buenos conocimientos de física por parte del lector, particularmente de óptica y radioastronomía.

El resultado es un libro de estilo simple y directo, que podrá seguramente ser aprovechado por una audiencia bastante amplia.

Está dividido en cuatro partes. La primera trata las bases matemáticas, en cinco capítulos titulados: 1. Recapitulación matemática; 2. Transformadas de Fourier de funciones sumables; 3. Transformadas de Fourier de funciones de cuadrado sumable; 4. Teoría elemental de distribuciones y definición de sus transformadas de Fourier; 5. Transformadas de Fourier y distribuciones en espacios multidimensionales.

La segunda parte, titulada "Ejemplos de aplicación de la transformada de Fourier", abarca los siguientes capítulos: 6. Difracción en el infinito; 7. Impedancias complejas y transformadas de Fourier en el plano complejo; 8. La transformada de Fourier en física matemática. La tercera parte, filtros lineales, contiene tres capítulos, titulados: 9. Propiedades generales de las funciones cuya transformada de Fourier cubre un intervalo de longitud finita; 10. Teoría de aproxmiación por funciones cuya transformada de Fourier tiene soporte compacto: aplicaciones al estudio del poder separador; 11. Transformada de Fourier de funciones aleatorias, función de autocorrelación y distribución espectral de la energía.

Cierra el volúmen la cuarta parte, que contiene un solo capítulo dedicado a los métodos numéricos.

G. Hansen

Este número de la Revista de la Unión Matemática Argentina y de la Asociación Física Argentina se ha publicado con la contribución del Consejo Nacional de Investigaciones Científicas y Técnicas. Tal contribución no significa que el Consejo asuma responsabilidad alguna por el contenido del mismo.

PUBLICACIONES DE LA U. M. A

Revista de la U. M. A. — Vol. I (1936-1937); Vol. II (1938-1939); Vol. III (1938-1939); Vol. IV (1939); Vol. V (1940); Vol. VI (1940-1941); Vol. VII (1940-1941); Vol. III (1942); Vol. IX (1943); Vol. X (1944-1945).

Revista de la U. M. A. y órgano de la A. F. A. — Vol. XI (1945-1946); Vol. XII (1946-1947); Vol. XIII (1948); Vol. XIV (1949-1950).

Revista de la U. M. A. y de la A. F. A. — Vol. XV (1951-1953); Vol. XVI 1954-1955); Vol. XVII (1955); Vol. XVIII (1959); Vol. XIX (1960-1962); Vol. XX (1962); Vol. XXI (1963); Vol. XXII (1964-1965).

Los volúmenes III, IV, V y VI comprenden los siguientes fascículos separados:

Nº 1. Gino Loria. *Le Matematiche in Ispagna e in Argentina.* — Nº 2. A González Domínguez. *Sobre las series de funciones de Hermite.* — Nº 3. Michel Petrovich. *Remarques arithmétiques sur une équation differentielle du premier ordre.* — Nº 4. A. González Domínguez. *Una nueva demostración del teorema límite del Cálculo de Probabilidades. Condiciones necesarias y suficientes para que una función sea integral de Laplace.* — Nº 5. Nikola Obrechkoff. *Sur la sommation absolue par la transformation d'Euler des séries divergentes.* — Nº 6. Ricado San Juan. *Derivación e integración de series asintólicas.* — Nº 7. Resolución adoptada por la U. M. A. en la cuestión promovida por el Sr. Carlos Biggeri. — Nº 8. F. Amodeo. *Origen y desarrollo de la Geometría Proyectiva.* — Nº 9 Clotilde A. Bula. *Teoría y cálculo de los momentos dobles.* — Nº 10. Clotilde A. Bula. *Cálculo de superficies de frecuencia* — Nº 11. R. Frucht. *Zur Geometria auf einer Fläche mit indefiniter Metrik (Sobre la Geometría de una superficie con métrica indefinida).* — Nº 12. A. González Domínguez. *Sobre una memoria del Prof. J. C. Vignaux.* — Nº 13. E. Toranzos. *Sobre las singularidades de las curvas de Jordan.* — Nº 14. M. Balanzat. *Fórmulas integrales de la intersección de conjuntos.* — Nº 15. G. Knie. *El problema de varios electrones en la mecánica cuantista.* — Nº 16. A. Terracini. *Sobre la existencia de superficies cuyas líneas principales son dadas.* — Nº 17. L. A. Santaló. *Valor medio del número de partes en que una figura convexa es dividida por* n *rectas arbitrarias.* — Nº 18. A. Wintner. *On the iteration of distribution functions in the calculus of probability (Sobre la iteración de funciones de distribución en el cálculo de probabilidades).* — Nº 19. E. Ferrari. *Sobre la paradoja de Bertrand.* — Nº 20. J. Babini. *Sobre algunas propiedades de las derivadas y ciertas primitivas de los polinomios de Legendre.* — Nº 21. R. San Juan. *Un algoritmo de sumación de series divergentes.* — Nº 22. A. Terracini. *Sobre algunos lugares geométricos.* — Nº 23. V. y A. Fraile y C. Crespo. *El lugar geométrico y lugares de puntos áreas en el plano.* — Nº 24. R. Frucht. *Coronas de grupos y sus subgrupos, con una aplicación a los determinantes.* — Nº 25. E. R. Raimondi. *Un problema de probabilidades geométricas sobre los conjuntos de triángulos*

En 1942 la U. M. A. ha iniciado la publicación de una nueva serie de "Memorias y monografías" de las que han aparecido hasta ahora las siguientes:

Vol. I; Nº 1. — Guillermo Knia, *Mecánica ondulatoria en el espacio curvo.* Nº 2. — Guido Beck, *El espacio físico.* Nº 3. — Julio Rey Pastor, *Integrales parciales de las funciones de dos variables en intervalo infinito.* Nº 4. — Julio Rey Pastor, *Los últimos teoremas geométricos de Poincaré y sus aplicaciones.* Homenaje póstumo al Prof. G. D. Birkhoff.

Vol. II; Nº 1. — Yanny Frenkel, *Criterios de bicompacidad y de H-completidad de un espacio topológico accesible de Frechet-Riesz.* Nº 2. — Georges Valiron, *Fonctions entières.*

Vol. III; Nº 1. — F. S. Bertomeu y C. A. Mallmann, *Funcionamiento de un generador en cascadas de alta tensión.*

Además han aparecido tres cuadernos de *Miscelánea Matemática.*