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INDUCED SHEAVES AND GROTHENDIECK TOPOLOGIES

by Juan José Martínez

INTRODUCTION. The theory of sheaves, as it is exposed in the classical book of R. Godement {2}, has been generalized in sucessive stops. Ending this process, M. Artin introduced the notion of Grothendieck topology and developed the fundamental part of the theory in a functorial way (cf. {1}). Although, the concept of Grothendieck topology seems to be insufficient to relate certain aspects of the theory of sheaves; for example, the notion of subspace (not necessarily open !) is omited and so, induced sheaves and relative cohomology must be ignored.

The purpose of this paper is to obtain the essential results about induced sheaves (the concept of topological category enable us to work in this direction; cf. §1). Topological methods play an important role in the problems in question, as Godement shows (cf . {2} , Ch. II, §2.9). Therefore, we are forced to introduce a various kind of axioms, valids -of course- in the classical situation of a topological space. We mention that the results of this paper are useful also in not conventional cases, namely, the "étale" Grothendieck topology for preschemes (cf. {1} , Ch. III).

Results and notations of Artin's seminar ({1} , Ch. I, II) are continuosly used, frequently without specific reference. This results are stated in {1}for sheaves of abelian groups, but all of them could be generalized taking an arbitrary category of values and inserting axioms where necessary.Here, we have followed the abstract formulation (the basic facts about limit of functors, existence of injectives, adjoint situations, derived functors of a composition, etc. are stated in the usual literature; for example, cf. {3}) . Of course, the reader could suppose that all sheaves in this paper are abelian sheaves.

1. TOPOLOGICAL CATEGORIES AND INDUCED SHEAVES. This section is of introductory character. Its aim is to lay down the terminology used throughout this paper and to collect the basic facts. We begin with the following:

DEFINITION 1.1. *A topological category is a triple* (M,T,ϕ) *such that* M *is a category,* T *is a family of Grothendieck topologies* $(T_M)_{M\in ObM}$ ϕ *is a family of morphisms in* M $(\phi_M:M \to X)_{M \in ObM}$ *and the following axioms are satisfied:*

tc1) For all object M *in* $\mathcal{M}$ *Cat* T_M *is a full subcategory of* $\mathcal{M}$. X *is an object of Cat* T_X *and* ϕ_X *is the identity morphism* e_X *of* X.

tc2) The diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "\phi_U"'] & & V \arrow[dl, "\phi_V"] \\ & X & \end{tikzcd}$$

is commutative, for all morphism $f\colon U \to V$ *in Cat* T_X.

tc3) $\mathcal{M}$ *has fibered products of the form* $U \mathbin{{}_{\phi_U}\!\times_{\phi_M}} M$ *(briefly noted* $U \times_X M$*), where* U *is an object of Cat* T_X *and* M *is an object of* $\mathcal{M}$, *such that*

$$U \in \mathrm{Ob}(Cat\ T_X) \Longrightarrow U \times_X M \in \mathrm{Ob}(Cat\ T_M)$$

$$(f_i\colon U_i \to U)_{i\in I} \in Cov\ T_X \Longrightarrow (f_i \times_X e_M\colon U_i \times_X M \to U \times_X M)_{i\in I} \in Cov\ T_M$$

for all object M *of* $\mathcal{M}$.

REMARKS 1.2. i) Axioms tc1 and tc2 tell us that M is an object of Cat T_M , because $X \times_X M = M$ and X is an object of Cat T_X.

ii) Recall that if X is an object in a category $\mathcal{M}$, then is called prefinal (resp. final) iff $\mathrm{Hom}_{\mathcal{M}}(M,X) \neq \emptyset$ (resp. $\mathrm{Hom}_{\mathcal{M}}(M,X)$ is a set of one element), for all $M \in \mathrm{Ob}\mathcal{M}$. If $< X >$ is the discret subcategory of $\mathcal{M}$ associated to X, one easily checks that the following statements are equivalent (cf. {1} Ch. I, §0):

a) $\mathcal{M}$ satisfies axiom L1 and $< X >$ is a final subcategory of $\mathcal{M}$.

b) X is a final object of $\mathcal{M}$.

c) $\mathcal{M}$ satisfies axiom L2 and X is a prefinal object of $\mathcal{M}$ such that $\mathrm{Hom}_{\mathcal{M}}(X,X) = \{e_X\}$.

Clearly, if $(\mathcal{M},T,\phi)$ is a tc (topological category) then X is a prefinal object of $\mathcal{M}$.

Let $\mathcal{M}$ be a category with final object X and let ϕ be the family of morphisms canonically associated to X. If T is a family of topologies satisfying tc1 and tc3, respect to $\mathcal{M}$ and ϕ, then $(\mathcal{M},T,\phi)$ is a tc of the following type:

DEFINITION 1.3. *A topological category* $(\mathcal{M},T,\phi)$ *is called* tc^0 *iff it satisfies:*

tc2') For all morphism $f\colon M \to N$ *in* $\mathcal{M}$ *the diagram*

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\phi_M"'] & & N \arrow[dl, "\phi_N"] \\ & X & \end{tikzcd}$$

is commutative.

Given a tc $C = (M, T, \phi)$, we shall be using a naive nomenclature:

M is called the category of subspaces of C (consistently, an object M in M is called a subspace of C). The object X is referred to as the space of C and so, an object M of M is also called a subspace of X refering to ϕ_M as the inclusion morphism of M in X.

If M is a subspace of M, T_M is called the relative topology of M and Cat T_M is called the category of relative open objects of M. Abusing language, T_X is called the topology of C and Cat T_X is called the category of open subspaces of C.

A morphism f: M $\longrightarrow$ N in M is called a ϕ-morphism iff the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\phi_M"'] & & N \arrow[dl, "\phi_N"] \\ & X & \end{tikzcd}$$

is commutative. We define a category M_ϕ putting:

$$\mathrm{Ob}M_\phi = \mathrm{Ob}M$$

$$\mathrm{Hom}\ M_\phi : \phi\text{-morphisms of } M$$

M_ϕ is a subcategory of M and clearly is a full subcategory (equivalently, is equal to M) iff C is tc^0.

DEFINITION 1.4. *A morphism of topological categories* F: $C \longrightarrow C'$ *is a functor* F: $M \longrightarrow M'$ *such that:*

mtc1) For all object M *of* M F/*Cat* T_M *is a morphism of topologies, of* T_M *in* $T'_{F(M)}$.

mtc2) $F(\phi) = \phi'$ *(i.e.* $F(X) = X'$ *and* $F(\phi_M) = \phi'_{F(M)}$, *for all* $M \in \mathrm{Ob}M$*).*

mtc3) F *preserves fibered products of the form* $U \times_X M$, $U \in \mathrm{Ob}(\mathrm{Cat}\ T_X)$ *and* $M \in \mathrm{Ob}M$.

REMARKS 1.5. i) Now we can talk about the category of small topological categories.

ii) If F: $C \longrightarrow C'$ is a mtc then we have

$$U \in \mathrm{Ob}(\mathrm{Cat}\ T_X) \Longrightarrow F(U) \times_{X'} F(M) \in \mathrm{Ob}(\mathrm{Cat}\ T'_{F(M)})$$

$$(f_i : U_i \longrightarrow U)_{i\in I} \in \mathrm{Cov}\ T_X \Longrightarrow (F(f_i) \times_{X'} e_{F(M)} : F(U_i) \times_{X'} F(M) \longrightarrow$$

$$\rightarrow F(U) \times_{X'} F(M))_{i \in I} \in \mathrm{Cov}\ T'_{F(M)}$$

for all subspace M of C.

iii) A mtc F: $C \longrightarrow C'$ induces for each subspace M of C a morphism of topologies F/Cat T_M: $T_M \longrightarrow T'_{F(M)}$ and so, induces the usual functors (direct and inverse image) between the corresponding categories of presheaves or of sheaves.

Let M be a subspace of a tc C. If A is an arbitrary category, the category of presheaves $P(T_M, A)$ is briefly denoted by P_M, and a presheaf in P_M is called a presheaf over M. Similarly, if A is a category with products, S_M denotes the category of sheaves $S(T_M, A)$, and a sheaf in S_M is called a sheaf over M.

ρ_M: $T_X \longrightarrow T_M$ is the morphism of topologies defined by the assignment of objects $U \longrightarrow U \times_X M$.

A category A will be called:

o) A0 iff it is a complete category (respect to functorial direct limits) with products and zero object.

i) A1 iff it is A0 and abelian.

ii) A2 iff it is A1 and satisfies the Grothendieck axiom A.B.5.

iii)A3 iff it is A2 and has a generator.

Let C be a tc and let A be an A1 category (as category of values).

DEFINITION 1.6. *If* M *is a subspace of* C *and* F *is a sheaf over* X *, then we call* ρ_{M_s} (F) *the sheaf induced by* F *over* M, *and we denote it by* F/M.

DEFINITION 1.7. *If* M *is a subspace of* C *and* α:F $\longrightarrow$ G *is a morphism of sheaves over* X,*then we call* $\rho_{M_s}(\alpha)$ *the morphism induced by* α *over* M, *and we denote it by* α/M.

REMARKS 1.8. i) Since $\rho_X = e_{T_X}$ it is clear that F/X = F and $\alpha/X = \alpha$

ii) Since ρ_{M_s} is a functor it is clear also that $e_F/M = e_{F/M}$ and $(\beta\alpha)/M = (\beta/M)(\alpha/M)$.

iii) Remark that expresions of the type (F/M)/N have no sense here, because the "absolute" topology T_X plays a special role in our developments.

Now we need to prove some previous results. In the next lemma, and only in the next, A may be an arbitrary category.

LEMMA 1.9. *If* f: $K \longrightarrow K'$ *is a morphism of small categories, the following statements are true*:

i) If f *is a full and representative functor, then* $f^P\colon P' \longrightarrow P$ *is full.*

ii) If f *is representative and* $\alpha \in Hom\, P'$ *is such that* $f^P(\alpha)$ *is an isomorphism, then* α *is also an isomorphism.*

Proof i) We want to show that the function $\mathrm{Hom}_{P'}(P_1 , P_2) \longrightarrow$

$\longrightarrow \mathrm{Hom}_P(f^P(P_1) , f^P(P_2))$, $\alpha \longrightarrow f^P(\alpha)$, is surjective. Given

$\beta \in \mathrm{Hom}_P(f^P(P_1) , f^P(P_2))$ we define a morphism $\alpha \in \mathrm{Hom}_{P'}(P_1 , P_2)$ in the following way: since f is representative, given an object V in K' there exists an object U in K such that $f(U) = V$; therefore, we take $\alpha(V) = \beta(U)$. The good definition of α is obtained by the following argument: if U' is an object of K such that $f(U')=V$, since f is a full functor there exists a morphism $m\colon U' \longrightarrow U$ such that $f(m) = e_V$. Now, since β is a morphism of presheaves, we have the commutative diagram

$$\begin{array}{ccc} f^P(P_1)(U) & \xrightarrow{\beta(U)} & f^P(P_2)(U) \\ {\scriptstyle f^P(P_1)(m)}\downarrow & & \downarrow{\scriptstyle f^P(P_2)(m)} \\ f^P(P_1)(U') & \xrightarrow[\beta(U')]{} & f^P(P_2)(U') \end{array}$$

i.e. we have the commutative diagram

$$\begin{array}{ccc} P_1(V) & \xrightarrow{\beta(U)} & P_2(V) \\ {\scriptstyle e_{P_1(V)}}\downarrow & & \downarrow{\scriptstyle e_{P_2(V)}} \\ P_1(V) & \xrightarrow[\beta(U')]{} & P_2(V) \end{array}$$

and so, $\beta(U) = \beta(U')$. It is trivial that $f^P(\alpha) = \beta$.

ii) Given $\alpha \in \mathrm{Hom}\, P'$ and $V \in \mathrm{Ob}K'$, observe that

$V = f(U) \Longrightarrow \alpha(V) = f^P(\alpha)(U)$.

COROLLARY 1.10. *If* $f\colon T \longrightarrow T'$ *is a morphism of topologies, the following statements are true*:

i) If f *is a full and representative functor, then* $f^s: S' \longrightarrow S$ *is full.*

ii) If f *is representative and* $\alpha \in Hom\ S'$ *is such that* $f^s(\alpha)$ *is an isomorphism, then* α *is also an isomorphism.*

Proof : Apply the lemma, taking in mind that $if^s \simeq f^p i'$, where i (resp. i') is the inclusion functor of S (resp. S') in P (resp. P')

COROLLARY 1.11. *If* $f: T \longrightarrow T'$ *is a full and representative morphism of topologies, then* $f_s: S \longrightarrow S'$ *is a representative functor.*

Proof: Since f_s is left adjoint to f^s, there exists a canonical morphism of functors $\Lambda: f_s \circ f^s \longrightarrow e_{S'}$. Now, since f^s is full by 1.10,i, we have that $f^s(\Lambda_{F'}): f^s(f_s \circ f^s(F')) \longrightarrow f^s(F')$ is an isomorphism, for all sheaf F' in S'. Therefore, applying 1.10 , ii , $\Lambda_{F'}: f_s f^s(F') \longrightarrow F'$ is also an isomorphism; and so,given a sheaf F' in S' the sheaf $f^s(F')$ is a preimage by f_s of F'.

The situation above suggest us the following

DEFINITION 1.12. *A tc C is called* tc^1 *iff the morphism* ρ_M *is a full and representative functor, for all subspace* M *of* C.

REMARK 1.13. If C is tc^1 we can apply both corollaries to the morphism $\rho_M: T_X \longrightarrow T_M$. In particular, 1.11 tell us that the restriction functor $./M: S_X \longrightarrow S_M$ is representative, for all subspace M of a tc^1 C.

In order to obtain the classical theorems about " characteristic " sheaves (cf. {2} , Ch. II § 2.9) our first result is

LEMMA 1.14. *Let* M *be a subspace of a* tc^1 C *and let* F *be a sheaf over* X. *If we define the sheaf* F^M *by* $F^M = f^s f_s(F)$ *then* $F^M/M \simeq F/M$. *(* f *is* ρ_M *).*

Proof: Adjointness gives us a canonical morphism of functors $\Delta: e_S \rightarrow f^s f_s$; since f is a full functor (cf. 1.10,i) $f_s(\Delta_F): f_s(F) \longrightarrow f_s(F^M)$ is an isomorphism. Hence, $F/M \simeq F^M/M$.

The technique concerning to open subspaces will be obtained using the following type of categories

DEFINITION 1.15. *A tc C is called tc*^{2} *iff for all open subspace A of C the following conditions are satisfied:*

i) T_A *is a subtopology of* T_X.

ii) If V is open in A, then $V \times_X A = V$ *(i.e.* $e_V \times_X \phi_A : V \times_X A \longrightarrow$ $\longrightarrow V \times_X X = V$ *is an isomorphism).*

LEMMA 1.16. *If A is an open subspace of a tc*^{2} *C and V is open in A, then* $<(V,h_V)>$ *is initial in* I_V^f *, where* $f\colon T_X \longrightarrow T_A$ *is the morphism* ρ_A *and* $h_V\colon V \longrightarrow f(V)$ *is the inverse morphism of* $e_V \times_X \phi_A$

Proof: Let (U,n) be any object in I_V^f . If $p_U\colon f(U) \longrightarrow U$ denotes the first projection, we define a morphism $m\colon V \longrightarrow U$ by $m = p_U n$. We claim that $m\colon (V,h_V) \longrightarrow (U,n)$ is a morphism in I_V^f ; to prove this we are reduced to check that the diagram

$$\begin{tikzcd} f(V) \arrow[rr, "f(m)"] & & f(U) \\ & V \arrow[ul, "h_V"] \arrow[ur, "n"'] & \end{tikzcd}$$

is commutative. In fact, if $g_V\colon f(V) \longrightarrow V$ is the morphism $e_V \times_X \phi_A$ we have the commutative diagram

$$\begin{tikzcd} f(V) \arrow[r, "ng_V"] \arrow[d, "g_V"'] & f(U) \arrow[d, "p_U"] \\ V \arrow[r, "m"'] & U \end{tikzcd}$$

Therefore, recalling that g_V is the first projection of $f(V)$, an uniqueness result on fibered products yields $f(m) = ng_V$. Hence, $f(m)h_V = n$.

The last thing to check is that $\mathrm{End}((V,h_V))$ is a set of one ele - ment. In fact, if $r\colon (V,h_V) \longrightarrow (V,h_V)$ is a morphism in I_V^f , then the diagram

$$\begin{tikzcd} f(V) \arrow[rr, "f(r)"] & & f(V) \\ & V \arrow[ul, "h_V"] \arrow[ur, "h_V"'] & \end{tikzcd}$$

is commutative, i.e. $f(r) = e_{f(V)}$. Therefore, since the diagram

$$\begin{tikzcd} f(V) \arrow[r, "f(r)"] \arrow[d, "g_V"'] & f(V) \arrow[d, "g_V"] \\ V \arrow[r, "r"'] & V \end{tikzcd}$$

is also commutative, results $r = e_V$.

COROLLARY 1.17. *If* A *is a complete category, Cat* T_X *has fibered products and* V *is open in* A, *then the following statements are true:*

i) If P *is a presheaf over* X, *then* $f_p(P)(V) \simeq P(V)$.

ii) If in addition A *has products and* F *is a sheaf over* X, *then* $f_p i(F)$ *is a sheaf over* A*(*i: $S_X \longrightarrow P_X$ *is the inclusion functor).*

iii)If A *is abelian too and* F *is a sheaf over* X, *then* $F/A(V) \simeq F(V)$.

Proof: i) Since Cat T_X has fibered products and f preserves fibered products (because is a morphism of type ρ_M), the category I_V^f satisfies axiom L1* (cf. {1} II, Th. 4.14). Therefore, applying the lemma, we see that (V,h_V) is an initial object in I_V^f, and so

$f_p(P)(V) = \varinjlim P_V \simeq P_V((V,h_V)) = P(V)$.

ii) Applying i, check the definition of sheaf.

iii) It is clear by ii that $F/A \simeq f_p i(F)$. Hence, i yields the desired result.

2. COMPLEMENTED TOPOLOGICAL CATEGORIES AND CLOSED SUBSPACES.

At this point, we need the notion of closed object in a topological category. Since we have the concept of open object, thinking in the closed sets of a topological space it is enough to find a notion replacing the set-theoretic operation of complement. Thus, we give the following

DEFINITION 2.1. *A complemented topological category is a tc* C, *together with a functor* c: $M^* \longrightarrow M$ *such that, if* θ = cX *and* $U_X = \mathrm{Cat}T_X$, *the following axioms are satisfied:*

ctc1) c *is an involution functor (i.e.* $c^* \circ c = e_{M^*}$*).*

ctc2) $\theta \in \mathrm{Ob}U_X$ *and there exists* $(U_i \longrightarrow \theta)_{i \in I} \in \mathrm{Cov}\, T_X$ *such that* $I=\emptyset$

ctc3) $cM \times_X M = \theta$ *and* $\theta \times_X M = \theta$, *for all* $M \in \mathrm{Ob}M$.

REMARKS 2.2. i) $F_X = cU_X$ is called the category of closed sub - spaces of C, and c is called the complement operator of C.

ii) Axiom ctc3 says that the diagrams

$$\begin{tikzcd} \theta \arrow[r, "c(\phi_{cM})"] \arrow[d, "c(\phi_M)"'] & M \arrow[d, "\phi_M"] \\ cM \arrow[r, "\phi_{cM}"'] & X \end{tikzcd} \qquad \begin{tikzcd} \theta \arrow[r, "c(\phi_{cM})"] \arrow[d, "c(\phi_X)=e_\theta"'] & M \arrow[d, "\phi_M"] \\ \theta \arrow[r, "\phi_\theta"'] & X \end{tikzcd}$$

are fibered products. We recall that (in the following proofs) we only need the first condition of ctc3 for closed subspaces, and the second for open subspaces.

iii) ctc1 tell us:

a) the complement of a closed object is open.
(The dual proposition is trivially true). From the definition of θ and ctc2 we obtain:

b) θ is open and closed.

Therefore:

c) X is open and closed.

Using ctc2 and the second condition of ctc3, we see:

d) If M is a subspace of C, then $\theta \in \mathrm{Ob}U_M$ and there exists a cov- ering $(V_i \longrightarrow \theta)_{i \in I}$ in T_M such that $I = \emptyset$.

Recalling that, in a category with zero object, the product of an empty family of objects is the zero object, we obtain:

e) If A is a category with products and zero object and P is a presheaf over a subspace M of C, then

$$P \text{ monopresheaf} \Longrightarrow P(\theta) = 0$$

In particular,

$$P \text{ sheaf} \Longrightarrow P(\theta) = 0$$

(In the sense of {1}, a monopresheaf is a presheaf satisfying (+)).

Now, we have the necessary technique in order to prove one of the crucial results of this paper.

THEOREM 2.3. *If* A *is an* A1 *category and* M *is a subspace of a* tc^1C, *then for any sheaf* F *over* X *we have:*

i) The sheaf F^M *defined by* $F^M = f^s f_s(F)$, *where* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M, *satisfies* $F^M/M \simeq F/M$.

If A *is an* A2 *category*, C *is, in addition,* tc^2 *and has a complement operator* c, *Cat* T_X *has fibered products, and* X *is final in Cat* T_X, *then for any sheaf* F *over* X *we have:*

ii) If M *is a closed subspace of* C, *then* $F^M/cM \simeq 0$; *if the sheaf* F_{cM} *is defined by the exactness of the sequence* $0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M$ *(i.e.* $F_{cM} = Ker\ \Delta_F$*), then* $F_{cM}/cM \simeq F/cM$ *and* $F_{cM}/M \simeq 0$.

Proof: i) it is 1.14, exactly.

ii) We begin with the first statement. Since cM is an open subspace of C, applying 1.17, iii it is enough to show that $F^M(V) = 0$, for any V open in cM. Recalling that f^p preserves sheaves, because f is a morphism of topologies, we see that $F^M \simeq f^p i_M f_s(F)$. Therefore, we have $F^M(V) \simeq f_s(F)(f(V))$; but $f(V) = \theta$, because V is an object of T_{cM} and ρ_M is a representative functor (see axiom ctc3), and $f_s(F)(\theta) = 0$, because $f_s(F)$ is a sheaf (cf. 2.2, iii, e). Hence, $F^M/cM \simeq 0$.

Now, we prove the second statement. Since Cat T_X has fibered products, X is a final object in Cat T_X, A is an A2 category, and $\rho_A\colon T_X \longrightarrow T_A$, where A is any subspace of C, preserves the "spaces" of the topologies and fibered products, then $\rho_{A_s}\colon S_X \longrightarrow S_A$ is an exact functor (cf. {1} II, th. 4.14). Hence, f_s and g_s are exact functors ($f = \rho_M$ and $g = \rho_{cM}$).

In S_X we have the exact sequence

$$0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M$$

Thus, the sequence

$$0 \longrightarrow g_s(F_{cM}) \longrightarrow g_s(F) \longrightarrow g_s(F^M)$$

is exact, or equivalently, is exact the sequence

$$0 \longrightarrow F_{cM}/cM \longrightarrow F/cM \longrightarrow F^M/cM \simeq 0$$

Hence, $F_{cM}/cM \simeq F/cM$.

In a similar way, we obtain the exact sequence

$$0 \longrightarrow F_{cM}/M \longrightarrow F/M \longrightarrow F^M/M$$

Since Δ_F/M is an isomorphism, the exactness of this sequence yields $F_{cM}/M \simeq 0$.

The main purpose of the latter part of this section is to prove that, under certain restrictive conditions, $\Delta_F\colon F \longrightarrow F^M$ is an epimorphism. Until this moment, (C,c) will denote a fixed ctc.

If A is a category with products and zero object, we give the following

DEFINITION 2.4. *If* M *is a subspace of* C *and* F *is a sheaf over* X, *we say that* F *is null outside* M *iff for all open subspace* U *of* C *we have:*

$$U \times_X M = \theta \Longrightarrow F(U) = 0$$

If M is any subspace of C, $S(M)$ will denote the full subcategory of S_X defined by the sheaves null outside M. If $f\colon T_X \longrightarrow T_M$ is a morphism of topologies and A is an $\Lambda 1$ category, $f_\circ\colon S(M) \longrightarrow S_M$ will denote the functor $f_s/S(M)$.

THEOREM 2.5. *(A of type* A1*). If* M *is a subspace of* C *and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M *, then the functor* $f_\circ\colon S(M) \longrightarrow S_M$ *has a right adjoint* $f^\circ\colon S_M \longrightarrow S(M)$.

Proof: Since f^s is right adjoint to f_s, it is enough to show that the image of f^s is a subcategory of $S(M)$. (Then, f° is f^s with $S(M)$ as codomain).

Given a sheaf G over M, notice that $f^s(G) \simeq f^p i_M(G)$, where i_M is the inclusion functor of S_M in P_M , and so, we only need to show that

$$U \times_X M = \theta \Longrightarrow f^p i_M(G)(U) = 0$$

for any open subspace U of C. In fact, we have

$$f^p i_M(G)(U) = i_M(G)(f(U)) = i_M(G)(\theta) = G(\theta) = 0$$

(the last equality is true because G is a sheaf).

Now, we wish to obtain a theorem of equivalence between the categories $S(M)$ and S_M. A similar result of Artin concerning to closed subschemes (cf. {1} III, Th. 2.2), guide us in the generalization process.

LEMMA 2.6. *Let* $f\colon K \longrightarrow K'$ *be a functor and let* V *be an object of* K' *such that for any* $A \in \mathrm{Ob}K$ *and any* $\eta \in Hom_{K'}(V,f(A))$ *there exist*

$U \in ObK$ *and* $m \in Hom_K(U,A)$ *satisfying* $V \xrightarrow[h]{\sim} f(U)$ *and* $f(m)\circ h = n$.
Then, the full subcategory $I^f(V)$ *of* I^f_V *defined by the class* $\{(U,h);$ $U \in ObK,\ h \in Iso_{K'}(V,f(U))\}$ *is initial in* I^f_V.

Proof: Let (A,n) be any object in I^f_V ; applying the hypothesis on V to the morphism $n: V \longrightarrow f(A)$, we can find a morphism in K m : $U \longrightarrow A$ and an isomorphism in K' $V \xrightarrow{h} f(U)$ such that $f(m)h = n$. Therefore, the diagram

$$\begin{tikzcd} f(U) \arrow[rr, "f(m)"] & & f(A) \\ & V \arrow[ul, "h"] \arrow[ur, "n"'] & \end{tikzcd}$$

is commutative and so, $m: (U,h) \longrightarrow (A,n)$ is a morphism in I^f_V .

COROLLARY 2.7. *If* $f: K \longrightarrow K'$ *is a full and representative functor, then* $I^f(V)$ *in* I^f_V *, for all object* V *of* K'.

COROLLARY 2.8. *(A is a complete category). Let* $f: K \longrightarrow K'$ *be a morphism of small categories such that* K *has fibered products and* f *is a full and representative functor which preserves fibered products. Then, any presheaf* P *in* $P(K,A)$ *satisfies* $f_p(P)(V) \simeq \varinjlim p_V/I^f(V)^*$, $V \in ObK'$.

Proof: It is enough to notice that I^f_V satisfies the axiom L1* , because K has fibered products and f preserves fibered products.

DEFINITION 2.9. (C,c) *is called*·

i) ctc^1 *iff* C *is* tc^1 *and for any closed subspace* M *the following conditions are satisfied:*

a) *Any covering in* T_M *is induced by* ρ_M *from a covering in* T_X .
b) *If* U *and* U' *are open subspaces of* C *such that* $U \times_X M \simeq U' \times_X M$ *and* F *is a sheaf null outside* M, *then* $F(U) \simeq F(U')$.

ii) ctc^2 *iff* C *is* tc^2 *and any closed subspace* M *satisfies:*

$$U \times_X M = \theta \Longrightarrow U \times_X cM = U$$

for all open subspace U *of* C.

iii) ctc^3 *iff (C,c) is* ctc^1 *and* ctc^2.

LEMMA 2.10. *(A of type A0). If (C,c) is* ctc^1, *Cat* T_X *has fibered* *products,* F *is a sheaf null outside a closed subspace* M *of* C*, and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M*, then the following statements are true, for any open subspace* U *of* C*:*

i) $f_p i(F)(U \times_X M) \simeq F(U)$.

ii) $f_p i(F)$ *is a sheaf over* M.

iii) $F/M(U \times_X M) \simeq F(U)$.

Proof: i) Since Cat T_X has fibered products, applying 2.9 we see that $f_p i(F)(U \times_X M) \simeq \varinjlim i(F)_{U \times_X M}/I^f(U \times_X M)^*$. Now, since (C,c) satisfies 2.9, i, b it is obvious that the values of the functor $i(F)_{U \times_X M}$ are all isomorphic, because any one is isomorphic to F(U). Hence, $\varinjlim i(F)_{U \times_X M}/I^f(U \times_X M)^* \simeq F(U)$.

ii) Since (C,c) satisfies 2.9, i, a, applying i it follows easily that $f_p i(F)$ is a sheaf (one only needs to check the definition of sheaf).

iii) Because of ii we have $f_s(F) \simeq f_p i(F)$. Therefore, i yields the desired result.

Now, it is almost obvious how to prove:

THEOREM 2.11. *(A of type* A1*). If* (C,c) *is* ctc^1*, Cat* T_X *has fibered products,* M *is a closed subspace of* C*, and* $f\colon T_X \longrightarrow T_M$ *is the morphism* ρ_M*, then the functor* $f_\circ\colon S(M) \longrightarrow S_M$ *is an equivalence of categories, which inverse is* $f^\circ\colon S_M \longrightarrow S(M)$.

Proof: By adjointness (see 2.5), there are natural transformations $\Phi\colon e_{S(M)} \longrightarrow f^\circ f_\circ$ and $\Psi\colon f_\circ f^\circ \longrightarrow e_{S_M}$. It is a straightforward matter, which we leave to the reader, to check that Φ and Ψ are functorial isomorphisms.

COROLLARY 2.12. *If* F *and* F' *are sheaves null outside* M*, then*

$$F/M \simeq F'/M \Longrightarrow F \simeq F'$$

THEOREM 2.13. *(A of type* A1*). If* (C,c) *is* ctc^2*, Cat* T_X *has fibered products and* M *is a closed subspace of* C*, then*

$$F/cM \simeq 0 \Longrightarrow F \text{ is null outside } M$$

for any sheaf F *over* X.

Proof: Let U be an open subspace of C such that $U \times_X M = \theta$; taking in mind that (C,c) is ctc^2 and applying 1.17,iii we see that $F(U) \simeq$ $\simeq F(U \times_X cM) \simeq F/cM(U \times_X cM) = 0$.

COROLLARY 2.14. *If (C,c) is ctc^3 and F and F' are sheaves over X, then*

$$F/M \simeq F'/M \quad , \quad F/cM \simeq 0 \simeq F'/cM \Longrightarrow F \simeq F'$$

Now, we can obtain the desired result:

THEOREM 2.15. *(A of type A2). If (C,c) is ctc^3, Cat T_X has fibered products, X is final in Cat T_X and M is a closed subspace of C, then the following statements are true, for any sheaf F over X :*

i) $\Delta_F : F \longrightarrow F^M$ *is an epimorphism.*

ii) F^M *is uniquely determined by* F.

Proof: i) Recall that f_s and g_s are exact functors (see the proof of 2.3). Let C be the sheaf over X defined by the exactness of the sequence

$$F \xrightarrow{\Delta_F} F^M \longrightarrow C \longrightarrow 0$$

Then, we have the exact sequence

$$g_s(F) \longrightarrow g_s(F^M) \longrightarrow g_s(C) \longrightarrow 0$$

or equivalently

$$F/M \longrightarrow \underset{\overset{\shortparallel}{0}}{F^M/cM} \longrightarrow C/cM \longrightarrow 0$$

and so, $C/cM = 0$.

In a similar way we obtain the exact sequence

$$F/M \longrightarrow F^M/M \longrightarrow C/M \longrightarrow 0$$

Therefore, since Δ_F/M is an isomorphism, we conclude that $C/M = 0$. Now, 2.14 yields that $C = 0$.

ii) If F' is a sheaf over X satisfying $F'/M \simeq F/M$ and $F'/cM \simeq 0$, then $F'/M \simeq F^M/M$ and $F'/cM \simeq 0 \simeq F^M/cM$. Hence, 2.14 yields that $F' \simeq F^M$.

We end this section with a well known result on "characteristic" sheaves.

THEOREM 2.16. *(A of type A2). If (C,c) is* tc^1 *and* tc^2*,* $CatT_X$ *has fibered products, X is final in Cat* T_X *and M is a closed subspace of C, then for any sheaf G over M there exists a sheaf F over X such that F/M ≈ G and F/cM ≈ 0. If (C,c) is* ctc^3*, then F is uniquely determined by G.*

Proof: Since C is tc^1, $./M: S_X \longrightarrow S_M$ is a representative functor (see 1.13) and so, given a sheaf G over M we can find a sheaf H over X such that H/M ≈ G. Then, taking $F = H^M$, 2.3 enable us to conclude that F/M ≈ G and F/cM ≈ 0 .

3. RELATIVE COHOMOLOGIES.

This section is devoted to realize an analysis of the cohomological effects of induced sheaves. Of course, the well known results exposed in the book of Godement (cf. {2} Ch. II, §4.9, §4.10, Th. 5.11.1) are obtained here, employing functorial methods. The compact exposition of cohomological theory presented in the Artin's seminar ({1} Ch. II) is continuosly used. Sheaves and presheaves are considered in this order.

I) COHOMOLOGY OF SHEAVES.

Let A be an A3 category and let C be a tc such that Cat T_X has fibered products and X is final in Cat T_X. (Notice that the hypothesis on Cat T_X yield the exactness of the restriction functors).

We begin introducing the "true" cohomology.

DEFINITION 3.1. *If M is a subspace of C, for each integer n ≥ 0 we define the functor* $H^n_M: (\mathrm{Cat}\ T_X)^* \times S_X \longrightarrow A$ *by:*

$$H^n_M = H^n_{T_M}(. \times_X M\ ,\ ./M)$$

THEOREM 3.2. *The following statements are true:*

i) $U \times_X M \approx U' \times_X M$, $F/M \approx F'/M \Longrightarrow H^n_M(U,F) \approx H^n_M(U',F')$.

ii) $H^n_X \approx H^n_{T_X}$.

iii) $H^*_M(U,\ )$ *is an exact cohomological functor.*

If the functor $./M: S_X \longrightarrow S_M$ *carries injective sheaves into flask sheaves, then* *iv)* $H^n_M(U,\ ) \simeq R^n\, H^\circ_M(U,\ )$.

If C *has a complement operator* c *such that* (C,c) *is ctc*^{1} *and* M *is a closed subspace of* C, *then for any sheaf* G *over* M *we have:*

v) $H^n(U\times_X M, G) \simeq H^n(U, \rho^s_M(G))$.

and for any sheaf F *over* X *we have:*

vi) $H^n_M(U,F) \simeq H^n(U,F^M)$.

If C *has a complement operator* c *such that* (C,c) *is ctc*^{3} *and* M *is a closed subspace of* C, *then for any sheaf* F *over* X *we have:*

vii) If F *is null outside* M, *then* $H^n_M(U,F) \simeq H^n(U,F)$.

viii) There is a cohomological exact sequence of general term

$$H^n(U,F_{cM}) \longrightarrow H^n(U,F) \longrightarrow H^n(U,F^M)$$

Proof: i) and ii) are trivial.

iii) Notice that $H^*_{T_M}(U\times_X M,\ )$ is an exact cohomological functor and $./M\colon S_X \longrightarrow S_M$ is an exact functor.

iv) Since $R^n\, H^\circ_{T_M}(U\times_X M,\ ) \simeq H^n_{T_M}(U\times_X M,\ )$ and f^s, where $f\colon T_X \longrightarrow T_M$ is the morphism ρ_M, is an exact functor, which carries injectives into $H^\circ_{T_M}(U\times_X M,\ )$ - acyclics, the proposition follows easily:

$$R^nH^\circ_M(U,\ ) = R^n(H^\circ_{T_M}(U\times_X M,\ )\circ f_s) \simeq (R^nH^\circ_{T_M}(U\times_X M,\ ))\circ f_s \simeq H^n_{T_M}(U\times_X M,\ )\circ f_s =$$
$$= H^n_M(U,\ ) .$$

v) We claim that f^s is an exact functor; since the diagram

$$\begin{tikzcd} S_M \arrow[rr, "f^s"] \arrow[dr, "f^\circ"'] & & S_X \\ & S(M) \arrow[ur, "j_M"'] & \end{tikzcd}$$

where $j_M\colon S(M) \longrightarrow S_X$ is the inclusion functor, is commutative, it is enough to show that f° and j_M are exact functors. The exactness of f° is clear by reasons of equivalence (see 2.11), and the exactness of j_M follows from the fact that $S(M)$ is closed in S_X under taking kernels and cokernels, as it is easily deduced from the definitions.

The spectral theorem of Artin-Leray, applied to the morphism $f\colon T_X \longrightarrow T_M$, tell us that

$$H^p(U, R^q f^s(G)) \underset{p}{\Longrightarrow} H^n(U\times_X M, G)$$

for any sheaf G over M. Therefore, recalling that

$$q > 0 \Longrightarrow R^q f^s = 0$$

(because f^s is exact), we obtain

$$H^n(U, f^s(G)) \approx H^n(U \times_X M, G).$$

vi) Applying the above result, we have

$$H^n(U, F^M) = H^n(U, f^s f_s(F)) \approx H^n(U \times_X M, f_s(F)) = H^n_M(U, F).$$

vii) If (C,c) is ctc^3 , then for any sheaf F over X we have

$$F \text{ null outside } M \Longrightarrow F \approx F^M$$

In fact, from 2.13 follows that F^M is null outside M and so, since $F/M \approx F^M/M$, 2.12 yields that $F \approx F^M$.

Applying this result and i, we obtain

$$H^n(U, F) \approx H^n(U, F^M)$$

Hence, vi yields the desired result.

viii) By 2.3, ii and 2.15,i the sequence of sheaves over X

$$0 \longrightarrow F_{cM} \longrightarrow F \xrightarrow{\Delta_F} F^M \longrightarrow 0$$

is exact and so, iii yields the desired result.

REMARKS 3.3. i) Notice that the relative (read local) character of the cohomology just defined appears clearly in 3.2,i, 3.2,ii and 3.2,vii.

ii) Of course, the hypothesis on 3.2,iv can not be removed. Sufficient conditions in the classical case are well known (cf. {2} , II §3.3).

iii) Observe that the statement (notations as in 3.2, vi)

$$H^n_{cM}(U, F) \approx H^n(U, F_{cM})$$

is not true, in general. Then, if we introduce the notation:

$${}_{cM}H^n(U, F) = H^n(U, F_{cM})$$

under the hypothesis on 3.2,viii, we obtain an exact cohomological sequence of general term

$${}_{cM}H^n(U, F) \longrightarrow H^n(U, F) \longrightarrow H^n_M(U, F)$$

Now, we focus our attention in the cohomology with presheaves values.

DEFINITION 3.4. *If* M *is a subspace of* C, *for each integer* $n \geq 0$ *we define the functor* $H^n_M: S_X \longrightarrow P_M$ *by:*

$$H^n_M = H^n_{T_M}(./M)$$

THEOREM 3.5. *The following statements are true:*

i) $F/M \simeq F'/M \Longrightarrow H^n_M(F) \simeq H^n_M(F')$.

ii) $H^n_X \simeq H^n_{T_X}$

iii) H^*_M *is an exact cohomological functor.*

If the functor $./M: S_X \longrightarrow S_M$ *carries injective sheaves into flask sheaves, then*

iv) $H^n_M \simeq F^n H^\circ_M$

Without assumptions, we have for any sheaf F *over* X:

v) $H^n_M(F)\circ\rho_M \simeq H^n_M(\,,F)$.

If C *has a complement operator* c *such that* (C,c) *is* ctc^1 *and* M *is a closed subspace of* C, *then for any sheaf* F *over* X *we have:*

vi) $H^n_M(F)\circ\rho_M \simeq H^n(F^M)$.

If (C,c) *is* ctc^3, *we also have:*

vii) If F *is null outside* M, *then* $H^n_M(F)\circ\rho_M \simeq H^n(F)$.

viii) There is an exact cohomological sequence of general term

$$H^n(F_{cM}) \longrightarrow H^n(F) \longrightarrow H^n(F^M)$$

Proof: i), ii), iii) and iv) can be obtained as in 3.2.

v) Knowing that $H^n_{T_M}(G) \simeq H^n_{T_M}(\,,G)$, for any sheaf G over M, the proposition follows easily:

$$H^n_M(F)(U \times_X M) = H^n_{T_M}(F/M)(U \times_X M) \simeq H^n_{TM}(U \times_X M, F/M) = H^n_M(U,F).$$

vi) Applying v and 3.2,vi, we obtain:

$$H^n_M(F)(U \times_X M) \simeq H^n_M(U,F) \simeq H^n(U,F^M) \simeq H^n(F^M)(U).$$

vii) The statement in question can be obtained as vi, applying now v and 3.2,vii. Also, it can be proved in the following way: since (C,c) is ctc[3], for any sheaf F over X we have

$$F \text{ null outside } M \Longrightarrow F \simeq F^M$$

Hence, $H^n(F) \simeq H^n(F^M)$ and so, the proposition follows from vi.

viii) It can be obtained as 3.2,vii.

II) COHOMOLOGY OF PRESHEAVES.

Let A be an A1 category and let C be an arbitrary tc. First, we consider the cohomology of a covering. In order to conserve a spectral result and to obtain a new one, we adopte the following

DEFINITION 3.6. *If* M *is a subspace of* C *and* K_U *is the category of coverings of an open subspace* U *of* C*, for each integer* $n \geq 0$ *we define the functor* $H^n_M\colon K^*_U \times P_M \longrightarrow A$ *by*

$$H^n_M = H^n_{T_M}(\cdot \times_X M, \cdot)$$

THEOREM 3.7. *The following statements are true:*

i) $(U_i \to U)_{i\in I} \simeq (U'_j \longrightarrow U)_{j\in J} \Longrightarrow H^n_M((U_i \longrightarrow U)_{i\in I}, \;) \simeq H^n_M((U'_j \longrightarrow U)_{j\in J}, \;)$

ii) $H^n_X \simeq H^n_{T_X}$

iii) $H^*_M((U_i \to U)_{i\in I}, \;)$ *is an exact cohomological functor.*

If A *is an A3 category, then we have:*

iv) $H^n_M((U_i \longrightarrow U)_{i\in I}, \;) \simeq R^n H^\circ_M((U_i \longrightarrow U)_{i\in I}, \;)$.

v) $H^p_M((U_i \longrightarrow U)_{i\in I}, H^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$.

If A *is an A1 category,* C *has a complement operator* c *such that* (C,c) *ctc*[1]*, Cat* T_X *has fibered products and* M *is a closed subspace of* C *, then for any sheaf* F *over* X *we have:*

vi) If F *is null outside* M*, then* $H^n_M((U_i \longrightarrow U)_{i\in I}, F/M) \simeq H^n_M((U_i \to U)_{i\in I}, F)$

If A *is an A3 category and* X *is final in Cat* T_X*, we also have:*

vii) If F *is null outside* M*, then*

$$H^p_M((U_i \longrightarrow U)_{i\in I}, \rho_{M_p}(H^q(F))) \underset{p}{\Longrightarrow} H^n(U,F)$$

Proof. i) and ii) are trivial.

iii) Notice that $H^*_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I}, \;)$ is an exact cohomological functor.

iv) Since A is an A3 category, $H^n_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I}, \ ) \simeq$
$\simeq R^n H^\circ_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I}, \ )$.

v) Since A is an A3 category, the cohomologies of sheaves are defined and we have

$$H^p_{T_M}((U_i \times_X M \longrightarrow U \times_X M)_{i\in I}, \mathcal{H}^q_{T_M}(F/M)) \underset{p}{\Longrightarrow} H^n_{T_M}(U \times_X M, F/M)$$

vi) It follows easily from 2.10,iii, by a direct analysis of the complex which gives the cohomology.

vii) Observe that $S(M)$ has injectives, because it is equivalent to the category of sheaves over M (see 2.11) (since A is an A3 category, S_M has injectives). Also, observe that the functor j_M carries injectives into flasks, because $f^s = j_M \circ f^\circ$, where $f\colon T_X \longrightarrow T_M$ is the morphism ρ_M, and f^s has this property.

Now, consider the (two) functors given by the commutative diagram

$$\begin{tikzcd} S(M) \arrow[r] \arrow[d, "j_M"'] & P_M \arrow[r, "H^\circ_M((U_i \longrightarrow U)_{i\in I}, \ )"] & A \\ S_X \arrow[r, "i"'] & P_X \arrow[u, "f_p"'] & \end{tikzcd}$$

Let us evaluate its derived functors. By iv, we have

$$R^p H^\circ_M((U_i \longrightarrow U)_{i\in I}, \ ) \simeq H^p_M((U_i \longrightarrow U)_{i\in I}, \ )$$

Recalling that f_p is an exact functor, by the hypothesis on Cat T_X, and that j_M is an exact functor, which carries injectives into i-acyclics, we obtain

$$R^q(f_p i j_M) \simeq f_p \circ R^q(i j_M) \simeq f_p \circ (R^q i) \circ j_M = f'_p \circ \mathcal{H}^q \circ j_M$$

Both results elucidate the first member of the spectral convergence in question. Concerning to the second member, 2.10,i implies

$$H^\circ_M((U_i \longrightarrow U)_{i\in I}, \ ) \circ (f_p i j_M) \simeq H^\circ((U_i \longrightarrow U)_{i\in I}, \ ) \circ (i j_M)$$

Therefore, recalling that $H^\circ((U_i \longrightarrow U)_{i\in I}, \ ) \circ i \simeq \Gamma_U$, we obtain

$$H^\circ_M((U_i \longrightarrow U)_{i\in I}, \ ) \circ (f_p i j_M) \simeq \Gamma_U \circ j_M$$

and so, since j_M is an exact functor which carries injectives into Γ_U-acyclics, we have

$$R^n(H^\circ_M((U_i \longrightarrow U)_{i\in I}, \ ) \circ (f_p i j_M)) \simeq R^n(\Gamma_U j_M) \simeq (R^n \Gamma_M) \circ j_M =$$
$$= H^n(U, \ ) \circ j_M$$

We introduce the limit cohomology of presheaves by a more general procedure than the one used by Artin in {1} . Of course, both definitions agree in the case that the category of values is A3.

Let A be an A1 category and let T be an arbitrary topology. If U is an object of Cat T and K is a subcategory of K_U, for each integer $n \geq 0$ we define the functor $H^n_T(K, \ ): P(T,A) \longrightarrow A$ by:

$$H^n_T(\ ,P) = \varinjlim H^n_T(\ ,P) \circ k^* \ , \quad P \in \mathrm{Ob}P \quad ,$$

where $k: K \longrightarrow K_U$ is the inclusion functor (notice that $H^n_T(\ ,P) : K^*_U \longrightarrow A$).

It is straightforward, to check the following propositions:

i) If A is an A2 category and K is filtrant, then $H^*_T(K, \ )$ is an exact cohomological functor.

ii) If A is an A3 category and K is filtrant, then $H^n_T(K, \ ) \simeq R^n H^\circ_T(K, \ )$. (Concerning to i, the usual statement about the exactness of the limit is required; and for ii, the proposition i tell us that it is enough to show that $H^n_T(K, \ )$ vanishes on injectives, if $n > 0$).

Notice that all the other results of {1}, concerning to limit cohomology, are preserved by our definition.

The Čech cohomology of presheaves is introduced following {1}.

Now, we focus our attention in the relative limit cohomology.

DEFINITION 3.8. *If* M *is a subspace of* C *and* K *is a subcategory of* K_U *for each integer* $n \geq 0$ *we define the functor* $H^n_M(K, \ ): P_M \longrightarrow A$ *by*

$$H^n_M(K, \ ) = H^n_{T_M}(K \times_X M, \ )$$

(Notice that $K \times_X M$ is a subcategory of $K_U \times_X M$, which is a subcategory of $K_{U \times_X M}$).

THEOREM 3.9. *The following statements are true:*

i) $K \times_X M \simeq K' \times_X M \Longrightarrow H^n_M(K, \ ) \simeq H^n_M(K', \ )$.

ii) $H^n_X(K, \ ) \simeq H^n_{T_X}(K, \ )$.

If $k: K \longrightarrow K_U$ *is the inclusion functor, then for any presheaf* P *over* M *we have*

iii) $H^n_M(K,P) \simeq \varinjlim H^n_M(\ ,P) \circ k^*$

If K^* *is filtrant, then:*

iv) $H^\circ_M(K, \ ) \circ i_M \simeq \Gamma_{U \times_X M}$

If A *is an A2 category and* K^* *is filtrant, then:*

v) $H^*_M(K,\ )$ *is an exact cohomological functor.*

If A *is an A3 category and* K^* *is filtrant, then:*

vi) $H^n_M(K,\ ) \approx R^n H^\circ_M(K,\ )$.

vii) $H^p_M(K, H^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$

If A *is an A1 category,* K^* *is filtrant,* C *has a complement operator* c *such that* (C,c) *is ctc*[1]*,* $Cat\ T_X$ *has fibered products and* M *is a closed subspace of* C*, then for any sheaf* F *over* X *we have:*

viii)If F *is null outside* M*, then* $H^n_M(K,F/M) \approx H^n(K,F)$.

If A *is an A3 category and* X *is final in* $Cat\ T_X$*, then we also have:*

ix) If F *is null outside* M*, then* $H^p_M(K,\rho_M H^q(F)) \underset{p}{\Longrightarrow} H^n(U,F)$.

Proof: i) and ii) are trivial.
iii) If $k_M\colon K\times_X M \longrightarrow K_{U\times_X M}$ is the inclusion functor, by definition we have for any presheaf P over M

$$H^n_{T_M}(K\times_X M,P) \approx \varinjlim H^n_{T_M}(\ ,P)\circ k^*_M$$

and it is clear that

$$\varinjlim H^n_{T_M}(\ ,P)\circ k^*_M \approx \varinjlim H^n_{T_M}(.\times_X M,P)\circ k^*$$

iv) Notice that

$$K^* \text{ filtrant} \Longrightarrow K\times_X M^* \text{ filtrant}$$

and so, we have

$$H^\circ_{T_M}(K\times_X M,\ )\circ i_M \approx \Gamma_{U\times_X M}$$

v) Since $K\times_X M^*$ is filtrant, then $H^*_{T_M}(K\times_X M,\ )$ is an exact cohomological functor.

vi) Recalling that K^* is filtrant, we have

$$H^n_{T_M}(K\times_X M,\ ) \approx R^n H^\circ_{T_M}(K\times_X M,\ )$$

vii) By the same reasons, we have the spectral convergence

$$H^p_{T_M}(K\times_X M\ ,\ H^q_{T_M}(F/M)) \underset{p}{\Longrightarrow} H^n_{T_M}(U\times_X M,F/M).$$

viii) Applie 3.7, vi and pass to the limit over K^*_U, using the proposition iii.

ix) It can be obtained as 3.7,vii.

We end this section introducing the relative Čech cohomology of pre sheaves. The definition is not the expected one, because the natural definition do not preserves the relative character (see 3.11,i). However, in the special case 3.11,xi both procedures agree.

DEFINITION 3.10. *If* M *is a subspace of* C *and* U *is open, for each integer* $n \geq 0$ *we define the functor* $\check{H}^n_M(U,\ ): P_M \longrightarrow A$ *by:*

$$\check{H}^n_M(U,\ ) = \check{H}^n_{T_M}(U \times_X M,\ )$$

THEOREM 3.11. *The following statements are true:*

i) $U \times_X M \simeq U' \times_X M \Longrightarrow \check{H}^n_M(U,\ ) \simeq \check{H}^n_M(U',\ )$

ii) $\check{H}^n_X(U,\ ) \simeq \check{H}^n_{T_X}(U,\ )$

iii) $\check{H}^n_M(U,P) \simeq \varinjlim H^n_M(\ ,P)$

iv) $\check{H}^\circ_M(U,\ ) \circ i_M \simeq \Gamma_{U \times_X M}$

If A *is an A2 category, then:*

v) $\check{H}^*_M(U,\ )$ *is an exact cohomological functor.*

If A *is an A3 category, then:*

vi) $\check{H}^n_M(U,\ ) \simeq R^n\check{H}^\circ_M(U,\ )$

vii) $\check{H}^p_M(U,H^q_M(F)) \underset{p}{\Longrightarrow} H^n_M(U,F)$

viii) $\check{H}^1_M(U,F/M) \simeq H^1_M(U,F)$, $\check{H}^2_M(U,F/M) \underset{\sim}{\subset} H^2_M(U,F)$

If A *is an A1 category,* C *has a complement operator* c *such that* (C,c) *is ctc*[1]*, Cat* T_X *has fibered products and* M *is a closed subspace of* C*, then for any sheaf* F *over* X *we have:*

ix) *If* F *is null outside* M*, then* $\check{H}^n_M(U,F/M) \simeq \check{H}^n(U,F)$

If A *is an A2 category and* X *is final in Cat* T_X*, then we also have:*

x) *If* F *is null outside* M*, then* $\check{H}^p_M(U,\rho_{M_p}(H^q(F))) \underset{p}{\Longrightarrow} H^n(U,F)$.

If A *is an A1 category and* M *is a subspace of* C *such that any covering of* $U \times_X M$ *is induced by* ρ_M *from a covering of* U*, then we have:*

xi) $\check{H}^n_M(U,\ ) \simeq H^n_M(K_U,\ )$.

Proof: i), ii), iii), iv), v), vi) and vii) can be obtained as the homologous propositions of 3.9.

viii) It follows immediately from

$$\check{H}^1_{T_M}(U \times_X M, F/M) \simeq H^1_{T_M}(U \times_X M, F/M) \ , \quad \check{H}^2_{T_M}(U \times_X M, F/M) \underset{\sim}{\subset} H^2_{T_M}(U \times_X M, F/M).$$

It should be pointed out that viii could be obtained from vii just as in the absolute cohomology case.

ix) and x) can be obtained as 3.9, viii and 3.9,ix, resp.

xi) The hypothesis on M tell us that $K_U \times_X M \approx K_{U\times_X M}$. Hence, we have

$$H^n_{T_M}(K_{U\times_X M}, \ ) \approx H^n_{T_M}(K_U \times_X M, \ )$$

or equivalently

$$\check{H}^n_M(U, \ ) \approx H^n_M(K_U, \ )$$

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NOTE ON GALOIS EXTENSION OVER THE CENTER

by Manabu Harada

In {2} S.U.Chase, D.K.Harrison and A.Rosenberg obtained a Galois Theory for strongly Galois extensions of commutative rings (CHR-Galois). This was generalized to non commutative rings by F.R. Demeyer {3} , T.Kanzaki {6} , H.F.Kreimer {8} and others. Recently, O.E.Villamayor and D.Zelinsky obtained in {11} a weak Galois theory of commutative rings in order to study the strong one from a different point of view.

In the first section of this short paper we shall use similar arguments to those of {11} to show that if an algebra Λ over a commutative ring R is a strongly Galois extension of R, then Λ and its center C are weakly Galois extensions over C and R, respectively . If Λ is a weakly Galois extension over C, Λ is the sum of all C-modules J_σ (see below or {10} for the definition of J_σ). By means of some properties of the J_σ's, we shall study in section 2, a Galois theory over the center, the argument being similar to that of {7} and {5}, Theorem 1.

The author would like to express his thanks to Professor O.E.Villamayor for inviting him to Universidad de Buenos Aires and giving an opportunity to see his and Zelinsky's preprint of {11}.

1. GALOIS EXTENSION OVER R.

Let R be a commutative ring with identity, Λ an algebra over R, C the center of Λ and G a finite group of automorphisms of Λ. We say that Λ is a Galois extension with respect to G of its G-fixed ring Λ^G if there exist elements x_i , y_i i=1,2,...,n, in Λ such that $\sum_{\sigma\varepsilon G}\sigma(x_i)y_i = \delta_{\sigma,1}$. We note that if $\Lambda^G = R$, then Λ is a finitely generated and separable R-algebra by {9} , Lemma 2.

Let Γ be an R-subalgebra of Λ and G the group of all automorphisms of Λ leaving invariant the elements of Γ.We quote here the definition of weakly Galois extension of {11}: Λ is said to be a (right) weakly Galois extension of Γ if the following two conditions are satisfied:

a) Λ is a finitely generated projective right Γ-module.

b) $G\Lambda_\ell = \mathrm{Hom}_\Gamma(\Lambda,\Lambda)$, where $\sum g_i x_{i\ell}(\lambda) = \sum g_i(x_i\lambda)$ for $g_i \varepsilon G, x_i, \lambda \varepsilon \Lambda$.

LEMMA 1. *Let* Λ *be a strongly Galois extension of* R *with group* G. *If* R *has no proper idempotents, then* Λ *and its center* C *are weakly*

Galois extensions of C *and* R, *respectively.*

Proof: Since Λ is a finitely generated R-module, there exist mutually orthogonal primitive central idempotents e_i such that $\sum_{i=1}^{n} e_i = 1$. Then $\mathrm{Hom}_C(\Lambda,\Lambda) = \sum_i \oplus \mathrm{Hom}_{Ce_i}(\Lambda e_i, \Lambda e_i)$ and $\mathrm{Hom}_R(C,C) = \sum_i \oplus \mathrm{Hom}_{Re_i}(Ce_i, Ce_i)$. Let $T_i = \{g|\varepsilon G, g(e_i)=e_i\}$. We can easily see that Λe_i is a strongly Galois extension of Re_i with group T_i (see {3}). Furthermore, Λe_i and Ce_i are strongly Galois extensions over Ce_i and Re_i with respect to H_i and T_i/H_i by {3} and {9}, where $H_i = \{g|\varepsilon T_i, g(d)=c\ \forall\ c\varepsilon Ce_i\}$. Hence $\mathrm{Hom}_{Ce_i}(\Lambda e_i, \Lambda e_i) = H_i(\Lambda e_i)_\ell$ and $\mathrm{Hom}_{Re_i}(Ce_i, Ce_i) = (T_i/H_i)(Ce_i)_\ell$. Now, we put $H_i^* = \{h|\varepsilon G, h|\Lambda e_i = h' \text{ for some } h'\varepsilon H_i, h|\Lambda e_j = I_{\Lambda e_j} \text{ for } i \neq j\}$. Then $H_i^* \subseteq H = \{h|\varepsilon G, h|C = I_C\}$. Therefore, $\mathrm{Hom}_C(\Lambda,\Lambda) = H\Lambda_\ell$. Similarly we obtain $\mathrm{Hom}_R(C,C) = G'C_\ell$, where G' is the group of all automorphisms of C over R. Condition a) follows from {1}, since Λ is separable over R.

THEOREM 1. *Let* Λ *be a strongly Galois extension of* R *with finite group* G. *Then* Λ *and its center* C *are weakly Galois extensions of* C *and* R, *respectively.*

Proof: We shall use the same notation and argument of {11}. Since Λ is $\Lambda \otimes \Lambda^*$-projective, $C_x = (\mathrm{Hom}_{\Lambda^\ell}(\Lambda,\Lambda))_x = \mathrm{Hom}_{\Lambda_x^\ell}(\Lambda_\alpha,\Lambda_\alpha)$ (cf. {11}, (2.7)). Furthermore, R_x has no proper idempotents by {11}, (2.13). Hence, $H(x)(\Lambda_x)_\ell = \mathrm{Hom}_{C_x}(\Lambda_x,\Lambda_x)$ and $G'(x)(C_x)_\ell = \mathrm{Hom}_{Rx}(C_x,C_x)$, where $H(x)$, $G'(x)$ are as above in Λ_x and C_x. Since C is R-finitely generated, all elements of $H(x)$ and $G'(x)$ are induced by elements of H and G', respectively (by {11}, (2.14)). Hence $(H\Lambda\ell)_x = H(x)(\Lambda_x)_\ell = \mathrm{Hom}_{Cx}(\Lambda_x,\Lambda_x) = (\mathrm{Hom}_C(\Lambda,\Lambda))_x$ since Λ is C-finitely generated and projective. Therefore, $H\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$. Similarly $G'C_\ell = \mathrm{Hom}_R(C,C)$.

We shall give latter an example in which a strongly Galois extension of R is not a strongly Galois extension over its center with respect to the corresponding subgroup.

2. GALOIS EXTENSION OVER CENTER.

In this section we always assume that Λ is a separable algebra over its center C. If $H\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$, then for any element x in Λ^H, x_ℓ belongs to the center of $\mathrm{Hom}_C(\Lambda,\Lambda) = C$; hence $\Lambda^H = C$. We shall study some properties are treated in {7} and {10}. For $\sigma \varepsilon H$ let $J_\sigma = \{x|\varepsilon\Lambda, yx = x\sigma(y) \text{ for all } y\varepsilon\Lambda\} = \mathrm{Hom}_{\Lambda^\ell}(\Lambda,\Lambda\sigma)$, where $\Lambda\sigma$ is the same module as Λ as left Λ-module and the operation of

Λ as right Λ-module is defined by $x*y = x\sigma(y)$. Furthermore, $\Lambda_\sigma = \Lambda \otimes_C J_\sigma$ and $\Lambda = \Lambda J_\sigma$ (see {10}). For any element $x_\ell\sigma \in J_{\sigma\ell}\sigma \subseteq \Lambda_\ell H(x_\ell\sigma)(y) = x_{\sigma(y)} = yx_r$ for every $y \in \Lambda$. Hence, $(J_\sigma)_\ell^\sigma = (J_\sigma)_r$ and $\Lambda_\ell\sigma = \Lambda_\ell J_{\sigma\ell}\sigma = \Lambda_\ell(J_\sigma)_r = \Lambda_\ell \otimes_C (J_\sigma)_r$ since $\Lambda_\ell\Lambda_r = \Lambda_\ell \otimes_C \Lambda_r$ by {1}.

PROPOSITION 2. *Let* Λ *be separable over its center* C *and* S *a subset of* H. *Then* $S\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$ *if and only if* $\Lambda = \sum_{\sigma\in S} J_\sigma$.

Proof: $S\Lambda_\ell = \sum_{\sigma\in S}\Lambda_\ell\sigma = \sum\Lambda_\ell \otimes (J_\sigma)_r$. Since C is a C-direct summand of Λ and $\mathrm{Hom}_C(\Lambda,\Lambda) = \Lambda_\ell \otimes_C \Lambda_r$, the proposition follows.

COROLLARY. Λ *is a weakly Galois extension if and only if* $\Lambda = \sum_{\sigma\in S} J_\sigma$, *where* S *is a finite subset of* H. *Furthermore,* Λ *is generated by units as* C*-module if and only if* $S\Lambda_\ell = \mathrm{Hom}_C(\Lambda,\Lambda)$ *and the elements of* S *are inner-automorphisms.*

Proof: It is clear.

Let S be a subset of H. We call S strongly distinct if there exists a family of elements $\{x_i^{(\sigma)}, y_i^{(\sigma)}\}_{i=1}^{n=n(\sigma)}$, $\sigma \in S$ such that

$$\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)} = \delta_{\tau,\sigma} \quad \text{for all } \sigma,\tau \in S.$$

It is clear that this condition is equivalent with the existence of Galois generators if S is a group.

THERORЕM 3. *Let* S, J_σ *be as above and* $\Gamma = \sum_{\sigma\in S} J_\sigma$. *If* S *is strongly distinct, then* $\Gamma = \sum_{\sigma\in S}\oplus J_\sigma$. *Conversely, if* $\Gamma = \sum_{\sigma\in S}\oplus J_\sigma$ *and* Γ *is a direct summand of* Λ *as* C*-module, then* S *is strongly distinct.*

Proof: Assume that $\Gamma = \sum_\sigma \oplus J_\sigma$ and Γ is a direct summand of Λ as C-module. $\Lambda_\ell \otimes \Lambda_r = \mathrm{Hom}_C(\Lambda,\Lambda)$, since Λ is C-separable. Let p_σ be a projection of Λ onto J_σ. Then $p_\sigma \in \mathrm{Hom}_C(\Lambda,\Lambda)$. Hence, there exist elements $\{x_i^{(\sigma)}, y_i^{(\sigma)}\}_{i=1}^{n=n(\sigma)}$ in Λ such that $\sum x_{i\ell}^{(\sigma)} \otimes y_{ir}^{(\sigma)} = p_\alpha$. Therefore, $0 = p_\sigma(J_\tau) = J_\tau\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)}$ for $\sigma\neq\tau$. Since $\Lambda J_\tau = \Lambda$, $\sum\tau(x_i^{(\sigma)})y_i^{(\sigma)} = 0$. Similarly, $J_\sigma(I - \sum\sigma(x_i^{(\sigma)})y_i^{(\sigma)}) = 0$. Hence, $I = \sum\sigma(x_i^{(\sigma)})y_i^{(\sigma)}$. Conversely, assume S is strongly distinct. We assume $0 = \sum_{\sigma\in S} z_\sigma$, $z_\sigma \in J_\sigma$. Then $0 = \sum_i\sum_\tau x_i^{(\sigma)} z_\tau y_i^{(\sigma)} = \sum_\sigma z_\sigma(\sum_i \tau(x_i^{(\sigma)})y_i^{(\sigma)}) = z_\sigma$. Hence, $\Gamma = \sum\oplus J_\sigma$.

LEMMA 2. *Let* $A \supset B$ *be* R*-algebras. If* B *is* R*-separable, then* $V_A(B)$

is a direct summand of A *as two-sided* $V_A(B)$*-module, where* $V_A(B) = \{a | \varepsilon A\ ,\ ba = ab$ *for all* $b \varepsilon B\}$.

Proof: We consider A as a left $B \otimes_R B^*$-module. Since B is R-separable, there exist elements $\{x_i, y_i\}_i$ in Λ such that $\sum x_i y_i = 1$ and $\sum bx_i \otimes y_i = \sum x_i \otimes y_i b$ for all $b \varepsilon B$. Now we define a map $\phi: A \longrightarrow A$ by setting $\phi(a) = \sum(x_i \otimes y_i^*)a = \sum x_i a y_i$ for $a \varepsilon A$. From the above relations of $\{x_i, y_i\}$ we obtain $\phi(A) \subseteq V_A(B)$ and $\phi | V_A(B) = I_{V_A(B)}$. Furthermore, $\phi(aa') = \sum_i x_i aa' y_i = (\sum x_i a y_i)a' = \phi(a)a'$ for $a' \varepsilon V_A(B)$. Similarly $\phi(a'a) = a'\phi(a)$. Hence, ϕ is $V_A(B)$ - $V_A(B)$ homomorphism. Since $V_A(B)$ is $V_A(B)$ - projective, $A = V_A(B) \oplus \ker \phi$.

PROPOSITION 4. *Let* Λ *be a central separable* C*-algebra and* Γ *a separable subalgebra* ($\Gamma \supseteq C$)*. Then* Γ *is a direct summand of* Λ *as a two-sided* Γ*-module.*

Proof: We know from {6}, Theorem 2 that $\Gamma = V_\Lambda(V_\Lambda(\Gamma))$ and $V_\Lambda(\Gamma)$ is C-separable. Hence the proposition follows from Lemma 2.

From now we assume that the subset S of H is a finite group G.

PROPOSITION 5. *Let* Λ *be a central separable* C*-algebra and* G *a finite subgroup of the group of* C*-automorphism of* Λ*; let* $\Gamma = \sum_{\sigma\varepsilon G} J_\sigma$. *Then the following statements are equivalent:*

1) $\Gamma = \sum_{\sigma\varepsilon G} \oplus J_\sigma$ *and* $|G|$ *is a unit in* C.

2) $\Gamma = \sum_{\sigma\varepsilon G} \oplus J_\sigma$ *and* Γ *is* C*-separable.*

3) Λ *is a strongly Galois extension of* Λ^G *and* Λ^G *is* C*-separable.*

Proof: 1) $\longleftrightarrow$ 2) It is clear from ({4} Lemma 4) by localization of C.

2) $\longrightarrow$ 3) Since Γ is C-separable, Γ is a direct summand of Λ as Γ-module by Proposition 4. Hence, G is strongly distinct by Theorem 3.

3) $\longrightarrow$ 1) $|G|$ is unit in C by {5} , Proposition 5,and the rest is clear.

LEMMA 3. $J_\sigma J_\tau = J_{\tau\sigma}$ *for any* σ, τ .

Proof: Let m be a maximal ideal in C. Then $(J_\sigma)_m = \mathrm{Hom}_{\Lambda_m} \ell(\Lambda_m, \Lambda_m \sigma) = C_m u_\sigma$, where $\sigma(y) = u_\sigma^{-1} y u_\sigma$. Hence, $(J_\sigma J_\tau)_m = C_m u_\sigma u_\tau$ and $u_{\tau\sigma}(u_\sigma u_\tau)^{-1}$,

$u_\sigma u_\tau u_{\tau\sigma}^{-1}$ belong to C_m. Therefore, $J_\sigma J_\tau = J_{\tau\sigma}$.

PROPOSITION 6. *Let* G *be a finite subgroup of* H *and assume* Λ *is a strongly Galois extension of* Λ^G*; let* Ω *be a separable* C*-algebra between* Λ *and* Λ^G. *Then the following statements are equivalent:*

1) $\Omega = \Lambda^H$ *for some subgroup* H *of* G.

2) $V_\Lambda(\Omega) = \sum_{\sigma\varepsilon S} J_\sigma$ *for some subset* S *of* G.

3) There exist elements $\{x_i \varepsilon \Omega, y_i \varepsilon \Lambda\}$ *such that* $\sum x_i y_i = I$ *and* $\sum \rho(x_i) y_i = 0$ *for* $\rho|\Omega \neq I_\Omega$, $\rho \varepsilon G$. *({8}, Proposition 3.5).*

Proof: Since $V_\Lambda(\Omega) \subseteq \sum_{\sigma\varepsilon G} \oplus J_\sigma$, S is a subgroup from Lemma 3. Furthermore, $\mathrm{Hom}_{\Omega_r}(\Lambda,\Lambda) = \Lambda_\ell \otimes_C V_\Lambda(\Omega)_r$ by {6} , Theorem 2, since Ω is C-separable and therefore 1) and 2) are equivalent.

1) $\longrightarrow$ 3) Let $G = \bigcup_i \rho_i H$. Then $\Gamma = \sum_{\sigma\varepsilon G} \oplus J_\sigma = \sum_i \Gamma_H J_{\rho_i}$ and Γ is a direct summand of Λ as Γ-module, where $\Gamma_H = \sum_{\sigma\varepsilon H} \oplus J_\sigma$. Let p be a projection of Λ onto $\Gamma_H J_i = J_H$. Then $p \varepsilon \mathrm{Hom}_{\Gamma H_\ell}(\Lambda,\Lambda) = (\Lambda^H)_\ell \otimes \Lambda_r$. Hence, there exist $\{x_i \varepsilon \Lambda^H , y_i \varepsilon \Lambda\}$ such that $\sum x_{i\ell} \otimes y_{ir} = p$.

3) $\longrightarrow$ 2) Put $H = \{\sigma | \varepsilon G, \sigma|\Omega = I_\Omega\}$. Then $V_\Lambda(\Omega) \supseteq \sum_{\sigma\varepsilon H} J_\sigma$. Let $y \varepsilon V_\Lambda(\Omega)$ and $y = x_{\rho_1} + x_{\rho_2} + \ldots$, where $x_{\rho_i} \varepsilon \Gamma_H J_{\rho_i}$. Then $y = yI = y\sum x_i y_i = \sum x_i y y_i = \sum_j \sum_i x_i x_{\rho_j} y_i = \sum x_{\rho_j} \sum \rho_j(x_i) y_i = x_{\rho_1}$. Hence $V_\Lambda(\Omega) = \sum_{\sigma\varepsilon H} J_\sigma$.

Finally, we shall give an example of a strongly Galois extension Λ of R, such that Λ is not a strongly Galois extension over its center with respect to its subgroup. However Λ is a strongly Galois extension over its center with respect to a suitable group.

Let G_2 be a cyclic group of order 2 and Q the field of rational numbers. Put $G = G_2 \times G_2$ and $K = Q(\sqrt{2})$. Then $L = K \otimes_Q K$ is a strongly Galois extension of Q with respect to G by {9} ,Proposition 1 . Let g and h be the inner-automorphisms of Q_2 induced by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

respectively. Then $(g) \times (h) \simeq G$ and Q_2 is a strongly Galois extension of Q with respect to G, since $\frac{1}{2}\{e_{11}, e_{11}, e_{22}, e_{22}, e_{21}, e_{12}, e_{12}, e_{21}\}$ is a family of Galois generators. Put $\Lambda = Q_2 \oplus L$. Then Λ is a strongly Galois extension of Q with group G if we define $g(a+b) = g(a)+g(b)$

for $g \in G$, $a \in Q$ and $b \in L$. It is clear that the fixing group of its center is equal to G. But if we define $g(a + b) = g(a) + b$, then Λ is a strongly Galois extension of its center with respect to G.

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A MEAN VALUE THEOREM AND DARBOUX'S PROPERTY FOR THE DERIVATIVE OF AN ADDITIVE SET FUNCTION WITH RESPECT TO A MEASURE ON E^n

by R. J. Easton, and S. G. Wayment

I) INTRODUCTION: During a recent investigation of existence and equality almos everywhere of the cross partial derivatives f_{xy} and f_{yx}, a somewhat different derivative Df for a function f(x,y) was used {4} . This derivative Df is also defined for a function f of n-variables. The purpose of this paper is to establish a mean value theorem and a Darboux property for the function Df, and to generalize these results to the derivative T'* as defined on pp. 268-271 of {8}. A similar result was obtained by L. Misik in {7}, but the technique is much more cumbersome. The method of proof is the same as that given in {5}. In the case n=2 the technique of proof is used to establish a theorem concerning the equality of the three derivatives. For simplicity the proofs and definitions will be given for n=2.

II) THE DERIVATIVE Df: Let $R = [a,b;c,d] = \{(x,y) \mid x\epsilon[a,b], y\epsilon[c,d]\}$. If f(x,y) is a function whose domain contains R, then the f-area of R is denoted by $F(R) = f(b,d) - f(a,d) - f(b,c) + f(a,c)$. The ordinary area of R will be denoted by A(R). A rectangle $R = [a,b;c,d]$ is said to be of order M, if $M \geq 1$ and $1/M \leq (d-c)/(b-a) \leq M$. One then defines the upper and lower derivatives of order M at a point (x,y) to be $\overline{\lim}$ and $\underline{\lim}$ respectively of ratios of f-areas to ordinary areas of rectangles or order M which contain (x,y) and whose areas converge to zero. Then f is said to have a derivative of order M, $D_Mf(x,y) = D_Mf(P)$, at $P = (x,y)$ if the upper and lower derivatives of order M are equal. The function f is said to be two non-decreasing if the f-area of each sub-rectangle of R is non-negative. It follows {2}, that if f is of bounded variation in the sense of Hardy, then except for a set of measure zero, $D_Nf = D_Mf$ for each $N \geq M \geq 1$. The common value is denoted by Df. It also follows {4}, that f_{xy} and f_{yx} each exist except possibly on a set of measure zero.

III) THE MEAN VALUE THEOREM:

THEOREM 1: *If D_Mf exists at each point P of a closed rectangle R of order M and f is continuous at each point of R, then there exists a point $Q \epsilon R$ such that $D_Mf(Q) = F(R)/A(R)$.*

Proof: Suppose $R_1 = R = [a,b;c,d]$ and divide R_1 into four rectangles using the lines $x = a+h/2$, $y = b+k/2$ where $h = b-a$ and $k = d-c$. Denote the rectangles, beginning in the lower left hand corner and proceeding counterclockwise, by $R_{11}, R_{12}, R_{13}, R_{14}$, and observe that each of the four rectangles is similar to R_1. It follows that

$$\textstyle\sum_{i=1}^{4} F(R_{1i}) = F(R_1) \text{ and } \sum_{i=1}^{4} A(R_{1i}) = A(R_1)$$

and hence there must exist a j and a k such that

$$F(R_{1j}) \geq (1/4)F(R_1) \text{ and } F(R_{1k}) \leq (1/4)F(R_1).$$

We now proceed to find a rectangle R_2 of order M with sides parallel to the sides of R_1 such that $R_2 \subseteq R_1$, $F(R_2) = (1/4)F(R_1)$, and $A(R_2) = (1/4)A(R_1)$. If $F(R_{1i}) = (1/4)F(R_1)$ for some i, then choose $R_2 = R_{1i}$. Suppose equality does not hold for any i and consider the case j=3 and k=1. The other cases would follow in a similar manner. Let $\alpha = h/k$ and define the auxiliary function

$$g(t)=f(a+t+h/2,b+\alpha t+k/2)-f(a+t,b+\alpha t+k/2)-f(a+t+h/2,b+\alpha t)+f(a+t,b+\alpha t).$$

Then $g(0) = F(R_{11})$, $g(h/2) = F(R_{13})$, and g is a continuous function of t for $0 \leq t \leq h/2$. The ordinary intermediate value theorem for a function of one variable guarantees the existence of a $t_0 \varepsilon (0,h/2)$ such that $g(t_0) = (1/4)F(R_1)$. This value t_0 defines R_2 and we note that $F(R_2)/A(R_2) = F(R_1)/A(R_1)$. In the sequel we shall refer to the above selection process for determining R_2 as the sliding technique. We proceed inductively to define a nested sequence of closed rectangles $\{R_i\}$, each of order M with sides parallel to R_1, such that

i) $F(R_{i+1}) = (1/4)F(R_i)$ and

ii) $A(R_{i+1}) = (1/4)A(R_i)$.

By the nested interval theorem there exists exactly one point $Q \varepsilon \bigcap R_i$, and

$$D_M f(Q) = \lim_{i\to\infty} F(R_i)/A(R_i) = F(R_1)/A(R_1).$$

We shall say that the set function F has property I provided the auxiliary function g(t) has the intermediate value property along the lines x = constant, y = constant, and $y = \pm\alpha x$. We have the somewhat stronger result.

THEOREM 2. *If* $D_M f$ *exists at each point* P *of a closed rectangle* R *of order* M *and the set function* F *has property* I, *then there exists a point* $Q \varepsilon R$ *such that* $D_M f(Q) = F(R)/A(R)$.

The following example shows that theorem 2 is a stronger result.

EXAMPLE 1: Let $f(x,y)$ be defined as follows on the unit square:

$f(x,y) = 1$ if y is rational and

$f(x,y) = 0$ if y is irrational

Then $D_M f$ exists and is zero at each point of the unit square and the set function F has property I.

We now proceed to remove the condition that $D_M f$ exist along the boundary of R.

THEOREM 3. *If* R *is a rectangle of order* M, $D_M f$ *exists at each point* $P \varepsilon$ *int*(R), *and* F *has property* I, *then there exists a point* $Q \varepsilon$ *int*(R) *such that* $D_M f(Q) = F(R)/A(R)$.

Proof: We will use the same notation as in theorem 1 and modify the selection process to obtain, for some k, an $R_k \subset \mathrm{int}(R)$. We consider the following two cases:

1) If $R_2 \neq R_{1i}$ for any i, then R_2 has at most one edge contained in bdry(R).

2) If $R_2 = R_{1i}$ for some i, say i=1, then the following argument allows us to choose R_3 with at most one edge contained in bdry(R).

Divide R_2 into R_{21}, R_{22}, R_{23}, R_{24}, and if $F(R_{21}) = F(R_{22}) = F(R_{23}) =$ $= F(R_{24}) = (1/4)F(R_2)$, choose $R_3 = R_{23} \subset \mathrm{int}(R)$. If $F(R_{21}) \neq (1/4)F(R_2)$ for some i, then the sliding technique gives an R_3 with at most one edge contained in bdry(R). For case (1), suppose the bottom edge of R_2 is contained in $\mathrm{bdry}(R_1)$ and divide R_2 as before. If $F(R_{2i}) =$ $= (1/4)F(R_2)$ for i=3, choose $R_3 = R_{23}$ and if $F(R_{23}) \neq (1/4)F(R_2)$, then the sliding technique will again give an $R_3 \subset \mathrm{int}(R)$.

IV) A DARBOUX PROPERTY:

THEOREM 4: *Let* $\mathcal{O}$ *be a connected open set in* E^2. *Suppose* $D_M f$ *exists at each point of* $\mathcal{O}$ *and that* F *has property* I. *Let* $P, Q \varepsilon \mathcal{O}$ *and suppose* $D_M f(P) = \alpha$, $D_M f(Q) = \beta$, $\alpha < \beta$, *and* $\lambda \varepsilon (\alpha,\beta)$. *If* $\overline{PQ}$ *is an arc which is contained in* $\mathcal{O}$ *with endpoints* P *and* Q, *then for each* $\varepsilon > 0$ *there exists a point* $S \varepsilon \mathcal{O}$ *such that the distance* $d(S,\overline{PQ})$ *from* S *to* $\overline{PQ}$ *is less than* ε *and* $D_M f(S) = \lambda$.

Proof: Construct a polygonal arc $\hat{PQ}$ from P to Q consisting of horizontal and vertical straight line segments such that each point of $\hat{PQ}$ is within $\min(\xi/2,\varepsilon/2)$ of $\overline{PQ}$, where $\xi = d(\overline{PQ},\mathrm{bdry}\ \mathcal{O})$. Let $u =$

$= \min\left[\varepsilon/2, (\lambda-\alpha)/2\ ,\ (\beta-\lambda)/2\right]$. There exist rectangles R_1 and R_2 of order M centered at P and Q respectively with edges parallel to the coordinate axes and having the same base and height, such that $|F(R_1)/A(R_1)-\alpha| < u$, $|F(R_2)/A(R_2)-\beta| < u$, and $\mathrm{diam}(R_1) = \mathrm{diam}(R_2) \leq \min(\varepsilon/2, \xi/2)$.

Since F has property I the sliding technique allows us to obtain a rectangle R or order M, with sides parallel to the coordinate axes, centered at a point of $\hat{PQ}$ such that $\mathrm{diam}(R) = \mathrm{diam}(R_1)$ and $F(R)/A(R) = \lambda$. Theorem 1 implies the existence of a point $S \in R$ such that $D_M f(S) = \lambda$ and $d(S,\overline{PQ}) < \varepsilon$.

REMARK. Let u be Lebesgue measure on E^n and let T be any absolutely continuous measure with respect to u. Further suppose that the derivative $T'{}^*(x)$, as defined in {8}, exists at every point in an interval $R_n \subset E^n$. The above technique may be used to establish a mean value theorem and a Darboux property for this derivative. These results also hold if T is an additive set function defined on at least the closed intervals in E^n and has the intermediate value property along straight lines in the appropriate directions, u is a translational invariant measure which is finite on regular rectangles, and $T'{}^*(x)$ exists at every point in Int (R). A further generalization is given in Section VI of this paper.

V) A THEOREM ON THE EQUALITY OF THE DERIVATIVES f_{xy}, f_{yx}, and Df.

It is well known that if $f_{xy}(x,y)$ exists at each point of an open set $\mathcal{O}$ and R is a closed rectangle with $R \subset \mathcal{O}$, then there exists a point $P \in$ int (R) such that $f_{xy}(P) = F(R)/A(R)$. Example 1 shows that there are functions for which $f_{xy}(x,y)$ and Df exist on a rectangle and f_{yx} fails to exist at any point. Also, the example can be modified by defining $f(x,y) = 2$ whenever x and y are rational , $f(x,y) = 1$ if exactly one of x or y is rational, and zero otherwise to give a function such that Df exists on a rectangle while both f_{xy} and f_{yx} fail to exist at any point.

THEOREM 5. *If* f_{xy} *and* $D_M f$ *exist on an open set* $\mathcal{O}$ *and (a) the function* f_{xy} *is continuous or (b) the related set function* F *has the intermediate value property along straight lines in the appropriate directions and* $D_M f$ *is continuous, then* $f_{xy}(P) = D_M f(P)$ *for each* $P \in \mathcal{O}$.

Proof: Suppose f_{xy} is continuous. Let $\{R_i\}$ be a sequence of nested rectangles of order M contained in 0 and closing down on P. Then $D_M f(P) = \lim_{i\to\infty} F(R_i)/A(R_i)$. For each i there exists a point $P_i \in R_i$ such that $f_{xy}(P_i) = F(R_i)/A(R_i)$ and the continuity of f_{xy} gives the desired result.

Suppose that $D_M f$ is continuous and that the set function F has the intermediate value property and hence theorem 2 applies. Let $P = (x_o,y_o) \in \mathcal{O}$. If $\varepsilon > 0$ then there exists t_1 such that $0 < t_1 < \varepsilon$ and $|[f_x(x_o,y_o+t_1)-f_x(x_o,y_o-t_1)]/2t_1-f_{xy}(x_o,y_o)| < \varepsilon/3$. There exists $t_2 > 0$ such that $t_2 = nt_1$ for some integer n and so that $\left|\left[f(x_o+t_2,y_o+t_1)-f(x_o-t_2,y_o+t_1)\right]/\left[2t_2\right]-f_x(x_o,y_o+t_1)\right|/\left[2t_1\right] < \varepsilon/3$ and $\left|\left[f(x_o+t_2,y_o-t_1)-f(x_o-t_2,y_o-t_1)\right]/\left[2t_2\right]-f_x(x_o,y_o-t_1)\right|/\left[2t_1\right] < \varepsilon/3$. We can now divide rectangle $R = [x_o-t_2,x_o+t_2;y_o-t_1,y_o+t_1]$ into n squares and conclude from the sliding technique that there exists a square R' such that $R' \subset R$ and $F(R)/A(R) = F(R')/A(R')$. Hence there is a point $P' \in R' \subset R$ such that $D_M f(P') = F(R)/A(R)$ and hence $|D_M f(P')-$ $- f_{xy}(P_o)| < \varepsilon$ and $d(P',P) < \varepsilon$. Continuity implies the desired result.

VI) A FURTHER GENERALIZATION:

We shall say that the additive set function F has property C provided the auxiliary function g(t) as defined in theorem 1 is continuous along the lines x=constant, y=constant, and $y = \pm\alpha x$. Note that property C implies property I.

THEOREM 6. *Suppose that* S *and* T *are additive set functions defined on rectangles,* u *is a translational invariant measure, and* $S'*(p)$ *and* $T'*(p)$ *exist at each point* $p \in$ *Int* (R) *and* $T'*(p) \neq 0$ *for any* p. *Then there exists a point* $q \in$ *Int* (R_o) *so that*

$$\frac{S(R_o)}{T(R_o)} = \frac{S'*(q)}{T'*(q)}$$

Proof: Let $U(R) = S(R_o)T(R)-T(R_o)S(R)$. Then $U(R_o) = 0$ and U has property I. Hence there exists a point $q \in$ Int (R_o) so that $U'*(q) = 0 = S(R_o)T'*(q)-T(R_o)S'*(q)$. This holds without the condition that $T'*(p) \neq 0$ for $p \in R_o$. The result now follows.

The method of proof in the preceding theorem allows one to remove the condition of translational invariant u. Suppose S and T are additive set functions defined on rectangles and having property C. Define $dS(p)/dT$ to be the limit of the ratio of the S area to the T area of regular rectangles as the diameters of the rectangles tend to zero.

THEOREM 7. *If* dS/dT *exists at each point* p *in Int* (R_o), $T(R) \neq 0$ *for* $R \subset R_o$, *and* S *and* T *have property* C, *then there is a point* $q \in$ *Int* (R_o) *so that* $dS(q)/dT = S(R_o)/T(R_o)$.

Proof: Define U as in theorem 6. Then use the procedures of theorems 1 and 3 to define a nested sequence $\{R_i\}$ of rectangles closing down on $q \in \text{Int } R_o$ and such that $U(R_i) = 0$. Then $S(R_o)/T(R_o) = S(R_i)/T(R_i)$ and the result follows.

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NOTES ON COMARGINAL PROBABILITY MEASURES

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1. INTRODUCTION.

By $S = (\Omega,A,B,C,P_1,P_2)$ we shall denote a system consisting of two σ-algebras B, C of subsets of Ω, the generated σ-algebra $A=\tau(B,C)$, and probabilities P_1, P_2 defined on B,C respectively, which are compatible in the sense that $P_1 = P_2 = P$ on the intersection σ-algebra $D = B \cap C$.

A probability Q on A such that its restrictions to B, C are P_1,P_2, respectively, is said to be *comarginal* with P_1, P_2, or briefly comarginal.

A probability Q on A, not neccesarily comarginal, is called *commutative* if on A-measurable Q-integrable functions:

$$(1) \qquad E_Q^B \cdot E_Q^C = E_Q^C \cdot E_Q^B = E_Q^D \qquad [Q] \quad ,$$

where E_Q^G denotes the conditional expectation operator with respect to the σ-algebra $G \subset A$ and to the measure Q.

The main problem we consider here is to search under what conditions a system S admits a comarginal and commutative measure Q. For such a measure we can assert its uniqueness. Owing to this fact and to the following example we shall call Q the *(generalized) product measure* on S.

Given the probability spaces (Ω_1,B',P_1') , (Ω_2,C',P_2') the system S formed by $\Omega = \Omega_1 \times \Omega_2$, $B = \Pi_1^{-1}(B')$, $C = \Pi_2^{-2}(C')$, $A = \tau(B,C)$, $P_i = P_i' \Pi_i$, (Π_i = projection on Ω_i), i=1,2, admits the comarginal and commutative measure $Q = P_1' \times P_2'$. The relation of commutation (1) is here Fubini's theorem.

In this example $D = \{\phi,\Omega\}$; the opposite extreme case, when $D = B$ (or $D = C$), gives us also an example very trivial for a product measure: $Q = P_2$ (or $Q = P_1$).

Intermediate cases can be given as follows:

Let $\Omega = \Omega_x \times \Omega_y \times \Omega_z$ and probabilities dx, dy, dz be given on σ-algebras B_x, B_y, B_z of Ω_x, Ω_y, Ω_z respectively. We define $B = \{\phi,\Omega_x\} \otimes B_y \otimes B_z$, $C = B_x \otimes \{\phi,\Omega_y\} \otimes B_z$ and the probabilities $dP_1 = dy\, dz$, $dP_2 = dx\, dz$. Here D is isomorphic to B_z and the system $(\Omega,A,B,C,P_1 P_2)$ has the product measure $dQ = dx\, dy\, dz$.

The preceding situation always appears in a Markov process. Assume

$\{x_i\}_{i=1,2..}$ is a Markov chain and $\mathcal{B} = \tau(x_1,\dots x_n)$, $\mathcal{C} = \tau(x_n,x_{n+1}\dots)$ and $\mathcal{D} = \tau(x_n)$. The product measure Q turns out to be the probability associated with the process, and the relation of commutation (1) can be rewritten as expressing the conditional independence of past ($\mathcal{B}$) and future ($\mathcal{C}$), given the present ($\mathcal{D}$). (cf. {1}, §14, or {4}, part A, ch. II).

For further examples we refer to {1} , §3.

2. COMARGINAL MEASURES AND BILINEAR FORMS.

We shall say that T: $\mathcal{B} \times \mathcal{C} \longrightarrow [0,+\infty)$ is a (positive) *bilinear form* on the system $S = (\Omega,A,\mathcal{B},\mathcal{C},P_1,P_2)$ if T(B,C) is additive in each variable separately; and T will be said to be *compatible* (with P_1,P_2) if $T(B,\Omega) = P_1(B)$, $T(\Omega,C) = P_2(C)$, for any $B \in \mathcal{B}$, $C \in \mathcal{C}$.

To each finitely additive measure Q on the algebra $A_o = \mathcal{B} \vee \mathcal{C}$ comarginal with P_1, P_2 we associate the compatible bilinear form T(B,C)= = Q(BC), which verifies: T(B,C) = 0 if BC = $\emptyset$. Conversely:

THEOREM 1. *Every compatible bilinear form* T *on* $\mathcal{C}$ *such that* T(B,C)=0 *whenever* BC = $\emptyset$, *defines on* A_o *a unique finitely additive comarginal probability given by* Q(B.C) = T(B,C).

Proof: We define Q(BC) = T(B,C). If BC = B'C', then BC=(BB')(CC')= = B_1C_1. To prove that T(B,C) = T(B',C') it is enough to prove that $T(B,C) = T(B_1,C_1)$. It follows from $T(B,C) = T(B_1,C) + T(B-B_1,C) = T(B_1,C)$, since $(B-B_1)C = \emptyset$, and from $T(B_1,C) = T(B_1,C_1)$. Hence Q is well defined on sets of the form BC , $B \in \mathcal{B}$, $C \in \mathcal{C}$.

Let $BC = \sum_\alpha B_\alpha C_\alpha$, where α runs on a finite family of indices. In order to prove that Q can be (uniquely) extended to A_o as a finitely additive measure it is enough to prove that $Q(BC) = \sum_\alpha Q(B_\alpha C_\alpha)$.

Let $\{B_i\}$ be the partition of B defined by the B_α's and $\{C_j\}$ that defined by the C_α's on C. We can assume from the beginning that $B_\alpha \subset B$, $C_\alpha \subset C$. Since $B_\alpha = \sum_m B_{\alpha,m}$, $C_\alpha = \sum_n C_{\alpha,n}$, denoting by $B_{\alpha,n}$ ($C_{\alpha,n}$) the sets of the mentioned partition of B (C) included in B_α (C_α), we have:

$$\sum_{i,j} B_iC_j = BC = \sum_\alpha B_\alpha C_\alpha = \sum_\alpha (\sum_m B_{\alpha,m})(\sum_n C_{\alpha,n}) = \sum_\alpha (\sum_{m,n} B_{\alpha,m}C_{\alpha,n})$$

This means that in the first and last sums appear the same non-void terms. Therefore, from the bilinearity of T we get

$$Q(BC) = T(B,C) = \sum_{i,j} T(B_i,C_j) = \sum_\alpha (\sum_{m,n} T(B_{\alpha,m},C_{\alpha,n})) = \sum_\alpha T(\sum_m B_{\alpha,m},\sum_n C_{\alpha,n}) =$$

$= \sum_\alpha T(B_\alpha, C_\alpha) = \sum_\alpha Q(B_\alpha . C_\alpha)$ Q.E.D.

REMARKS. 1) It is not true, in general, that Q is σ-additive. In fact, let Ω be the triangle on the plane defined by $x > 0$, $y > 0$, $x + y < 1$. $\mathcal{B}$ ($\mathcal{C}$) the Borel sets of Ω depending only of $x(y)$. For $B \in \mathcal{B}$ ($C \in \mathcal{C}$) we define $P_1(B) = m(B')$, $(P_2(C) = m(C'))$, where $B'(C')$ denotes the projection of B (C) on the x (y) axis and m the Lebesgue measure on (0,1).
The bilinear form $T(B,C) = \int_0^1 1_{B'}(t)\, 1_{C'}(1-t)dt$ is compatible with P_1, P_2 but, as it is easy to see, Ω can be put as a countable sum of rectangles B.C for which $T(B,C) = 0$.

2) Let us observe that if T is bilinear and comarginal then T is also σ-bilinear; i.e. σ-additive in each variable. In fact,

$T(\sum_{j=0}^\infty B_j, C) = T(\sum_{i=1}^n B_j, C) + T(\sum_{n+1}^\infty B_j, C)$, then

$|T(\sum_{j=1}^\infty B_j, C) - \sum_{i=1}^n T(B_j, C)| \leq \sum_{n+1}^\infty T(B_j, \Omega) = \sum_{n+1}^\infty P_1(B_j) \longrightarrow 0$ if $n \longrightarrow \infty$.

Then the proof of the theorem remains true if we assume that α runs on a countable family of indices such that the B_α's and the C'_α s define, respectively, countable partitions of the spaces Ω.

For example, we can assert the σ-additivity of Q if A is defined by a countable partition of Ω.

3) In theorem 1 we can assume $T(B_o, C_o) = 0$ if $B_oC_o = \emptyset$ for $B_o \in \mathcal{B}_o \subset \mathcal{B}$, $C_o \in \mathcal{C}_o \subset \mathcal{C}$, where $\mathcal{B}_o, \mathcal{C}_o$ are collections of sets with the *approximation property*: $P_1(B) = \sup_{\substack{B_o \subset B \\ B_o \in \mathcal{B}_o}} P_1(B_o)$, $P_2(C) = \sup_{\substack{C_o \subset C \\ C_o \in \mathcal{C}_o}} P_2(C_o)$, for $B \in \mathcal{B}$, $C \in \mathcal{C}$.
In fact, if $B_o \subset B$, $C_o \subset C$, and $B.C = \emptyset$: $T(B,C) = T(B-B_o+B_o, C-C_o+C_o) =$
$= T(B-B_o, C) + T(B_o, C-C_o) + T(B_o, C_o) \leq P_1(B-B_o) + P_2(C-C_o)$. The last member can be done arbitrarily small, hence $T(B,C) = 0$.

4) The σ-additivity of Q follows under the following hypothesis

1) $K_\mathcal{B} \subset \mathcal{B}$, $K_\mathcal{C} \subset \mathcal{C}$ are *semi-compact classes* (i.e. every countable family of $K_\mathcal{B}$ ($K_\mathcal{C}$) with an empty intersection has a finite subfamily which also intersects in the empty set) verifying the approximation property (as defined above).

2) $K_\mathcal{B}.K_\mathcal{C} = \{K.L ; K \in K_\mathcal{B}, L \in K_\mathcal{C}\}$ is a semicompact class of sets. In fact, the class L of finite unions of sets of $K_\mathcal{B}.K_\mathcal{C}$ enjoys the property of approximation in $\mathcal{B} \vee \mathcal{C}$, since, as it was shown above, for $K \subset B$, $L \subset C$ we have $Q(BC - KL) \leq P_1(B - K) + P_2(C - L)$, then $Q(BC - KL)$ can be done arbitrarily small. L being compact

and with the approximation property the σ-additivity of Q follows from a theorem of Alexandrov (cf {4}, pp. 47).

3. ∇-COMMUTATIVE SYSTEMS.

For any probability space (Ω, A, P) we define the measurable hull of $X \subset \Omega$ as a set $A \in A$ containing X except by a set of P-outer measure zero and with minimal P-measure. Of course, the measurable hull is defined except on a null set of A, and it provides a well defined element of the Boolean measure algebra: $A/[P]$. If $B \subset A$ is a σ-subalgebra of A the measurable hull of $X \in A$ with respect to B coincides with $\{E_P^B 1_X > 0\}$, $[P]$.

For a system $S = (\Omega, A, B, C, P_1, P_2)$ we shall designate $\nabla^1 X$, $\nabla^2 X$, ∇X the measurable hulls of $X \subset \Omega$ with respect to B, C, D and to the measures P_1, P_2 , P respectively.

If Q is a comarginal measure on S and E, F , G denote E_Q^B , E_Q^C , E_Q^D respectively, we can see that the condition $EF = FE = G[Q]$ (on bounded A- measurable functions) is equivalent to $Ef = Gf\ [P_1]$ (on C-measurable functions) and also to $Ff = Gf\ [P_2]$ (on B-measurable functions) (cf {1} , § 1).

For a comarginal measure Q on S, the condition $EF = FE = G$ implies $\nabla^1\nabla^2 X = \nabla^2\nabla^1 X = \nabla X \quad [Q]$ for $X \in A$. This condition is equivalent to $\nabla^2 B = \nabla B\ [P_2]$, for any $B \in B$, and also to $\nabla^1 C = \nabla C \quad [P_1]$, for any $C \in C$. (cf. {1} , §5).

We note that any of these last conditions can be introduced even if we do not assume that a comarginal measure Q is known. Then we adopt the following definition:

We shall say that the system S ∇-*commutes* if $\nabla^2 B = \nabla B \quad [P_2]$, $\forall\, B \in B$.

From the above considerations it follows:

In order that there exists a comarginal and commutative measure on $S = (\Omega, A, B, C, P_1, P_2)$ *it is necessary that* S *be a* ∇-*commutative system.*

An independent proof will be given in next theorem 2.

A ∇-commutative system is said to be *simple* if $D/[P]$ is the Boolean algebra {0,1} .

4. THE FINITELY ADDITIVE MEASURE ASSOCIATED TO A ∇-COMMUTATIVE SYSTEM.

Given the system $S = (\Omega, A, B, C, P_1, P_2)$ we observe that the conditional expectation operator G can be calculated on B (C)-measurable functions, with respect to P_1 (P_2) even if there is no comarginal measure. Hence we can define the compatible bilinear form:

$$(2) \qquad T(B,C) = \int G1_B . G1_C \, dP \ .$$

THEOREM 2. *In order that* $Q(BC) = T(B,C)$ *defines a finitely additive comarginal measure on* S *it is neccesary and sufficient that* S *be a* ∇-*commutative system.*

In this case, if Q *is a probability on* A *(i.e. if* Q *is* σ-*additive),* Q *is the unique commutative comarginal measure on* S*. (uniqueness of the product measure).*

Proof: By theorem 1, to prove that ∇-commutativity implies that Q is a finitely additive measure, we have to show that $B.C = \emptyset$ implies $T(B,C) = 0$. From $B.C = \emptyset$ we get $\nabla^2 B.C = \nabla B.C = \emptyset[P_2]$; i.e. $\{G1_B > 0\}.C = \emptyset[P_2]$. Then $T(B,C) = \int_C G1_B . dP_2 = 0$. Conversely, if Q is a finitely additive and comarginal measure, S ∇-commutes. In fact, $\nabla^2 B \subset \nabla B[P_2]$ and on the other hand from $P_2^*(B - \nabla^2 B) = 0$ we have $B - \nabla^2 B \subset C' \in C$ with $P_2(C') = 0$. Hence $Q(B - \nabla^2 B) \leq Q(C') = P_2(C') = 0$. Then,
$Q(B - \nabla^2 B) = \int_{\complement \nabla^2 B} G1_B . dP_2 = 0$.
This means $\{G1_B > 0\}.\complement\nabla^2 B = \emptyset \ [P_2]$ which implies $\nabla B \subset \nabla^2 B[P_2]$.
If Q, as defined above, is a measure, and E, F the conditional expectations with respect to B, C :

$$(3) \qquad Q(B.C) = \int G1_B . G1_C \, dP = \int_C G1_B \, dQ = \int_C F1_B \, dQ$$

Then $G1_B = F1_B \ , \ \forall \ B \in B$. This implies the commutation of Q. Another comarginal commutative measure Q' must verify (3), but since

$$\int_C G1_B \, dQ' = \int_C F1_B dQ' = Q'(B.C) = Q(B.C) \ ,$$

Q and Q' coincide on $B \vee C$, and therefore on A. QED.

REMARK: The condition $\nabla^2 B = \nabla B \ [P_2]$, $B \in B$, defining a ∇-commutative system implies the symmetric one $\nabla^1 C = \nabla C \ [P_1]$, $C \in C$.

In fact, the first one implies that $\int G1_B . G1_C \, dP$ defines a finitely additive measure, and from this we derive $\nabla^1 C = \nabla C \ [P_1]$ as it was done with $\nabla^2 B = \nabla B \ [P_2]$ in the proof of theorem 2. Now, we obtain easily

$\nabla^2\nabla^1 X = \nabla^1\nabla^2 X = \nabla X$ [P] for each $X \in \mathcal{B} \vee \mathcal{C}$.

THEOREM 3. *i) If the ∇-commutative system S is simple (i.e. $\mathcal{D}/[P]\cdot =$ $= \{0,1\}$) then* $Q(B.C) = P_1(B).P_2(C)$. *In particular for a system S obtained from a cartesian product (as described in the introduction),* $Q = P_1' \times P_2'$.

ii) If, conversely, $Q(B.C) = P_1(B).P_2(C)$ *defines a finitely additive measure on* $\mathcal{B} \vee \mathcal{C}$, *$S$ is a simple ∇-commutative system.*

Proof: i) It follows from theorem 2. and the fact that $G1_B = P_1(B).1_\Omega$, $G1_C = P_2(C).1_\Omega$.

ii) If $D \in \mathcal{D}$ we have $Q(D) = P(D) = Q(D.D) = P(D)^2$, then $P(D) = 0$ or 1. $G1_B$, $G1_C$ are computed like in i), then $\int G1_B.G1_C = P_1(B).P_2(C)$. From theorem 2) it follows that S is a ∇-commutative system.

We shall say that a system S is *complete* with respect to a comarginal probability P defined on $\mathcal{A}$ if every P-null set of $\mathcal{A}$ belongs to $\mathcal{D}$.

THEOREM 4. *If the system S is complete with respect to a comarginal probability* P *and* R *is a commutative probability on S equivalent to* P *then the product measure* Q *exists and it is equivalent to* P.

Proof: Assume $f = \frac{dR}{dP}$. By hypothesis $F_R = G_R$ on $\mathcal{B}$-measurable functions and $E_R = G_R$ on $\mathcal{C}$-measurable functions. We have (c.f. { 1 } §2):

$$E_R(h) = E(f.h)\Big/E(f) \qquad ,$$

$$F_R(g) = F(f.g)\Big/F(f) \quad , \quad G_R(m) = G(f.m)\Big/G(f) \qquad ,$$

where E, F, G denote here the conditional expectation operators with respect to the measure P and $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ respectively.

In {1} , th. 2 , §10 , it is proved that the probability measures equivalent to R that also commute are characterized as those whose Radon-Nikodym derivatives with respect to R are of the form: g.h, where g (h) is a $\mathcal{B}(\mathcal{C})$-measurable function (both positive and finite [R]).

Let us consider the functions:

$$g = \frac{+\sqrt{Gf}}{Ef} \qquad , \qquad h = \frac{+\sqrt{Gf}}{Ff}$$

Then, $g.E(h.f) = Gf.E(f/Ff)\,/Ef = Gf.E_R(1/Ff) = Gf.G_R(1/Ff) = G(f/Ff) =$ $= GF(f/Ff) = 1$. Analogously $h.F(g.f) = 1$. From $\int_B ghf\, dP = \int_B gE(hf)\, dP =$

$= \int_B 1\ dP = P(B)$ and $\int_C ghf\ dP = P(C)$

we see that the probability Q defined by

$$Q(A) = \int_A gh.f\ dP = \int_A g\ h\ dR$$

is comarginal with P and since $\frac{dQ}{dR} = g.h$, it commutes and obviously $Q \sim R$. QED.

REMARK: If the system S is complete with respect to P any other comarginal measure R is absolutely continuous with respect to P, since $P(A) = 0$ implies $A \in \mathcal{D}$ and then $R(A) = P(A) = 0$. Then, if a product measure Q exists on S, we have $Q \ll P$. In spite of the fact that for $B \in \mathcal{B}$, $C \in \mathcal{C}$, $Q(B.C) = 0$ implies $P(B.C) = 0$ (since $\int G1_B G1_C dP = 0$ implies $P_2(\nabla B.C) = 0$ and then $P(B.C) = 0$) we have not $Q \sim P$, in general. Let us see the following example:

Let Ω be the product of X and Y, $X = Y = (0,1)$, A = the Borel sets of $X \times Y$, $\mathcal{B}$ ($\mathcal{C}$) the Borel sets independent of y (x), P the probability on A equal to $\frac{m \times m}{2}$ (m the Lebesgue measure on (0,1)) plus a measure of total mass 1/2 concentrated on the diagonal and uniformely distributed there.

The product measure $Q = m \times m$ is not equivalent to P.

If $D \in \mathcal{D}$ defines an atom of the σ-algebra $\mathcal{D}/[P]$ it can be seen that $S_D = (D, A \cap D, \mathcal{B} \cap D, \mathcal{C} \cap D, P_1/P_1(D), P_2/P_2(D))$ is a simple ∇-commutative system, whenever S is a ∇-commutative system. Moreover, if Q is the (product) finitely additive measure on S defined above , $\frac{Q(B.C.D)}{Q(D)}$ is the product measure on S_D as it is easily seen. Since S_D is simple: $\frac{Q(B.C.D)}{Q(D)} = \frac{P_1(B.D)}{P(D)} \cdot \frac{P_2(C.D)}{P(D)}$, and then

$$Q(B.C.D) = P_1(B.D).P_2(C.D)/P(D) \tag{4}$$

THEOREM 5. *If in a ∇-commutative system S, A is defined by a countable partition of Ω, there exists the product measure Q. If $\{D_i\}$ is the partition defining $\mathcal{D}$, then Q is defined by:*

$$Q(B.C) = \sum_i \frac{P_1(BD_i).P_2(CD_i)}{P(D_i)} \tag{5}$$

Proof: The σ-additivity of Q follows from the second remark in §2. We have $Q(B.C) = \sum_i Q(B.C.D_i)$, and applying (4) we obtain the equality (5).

5. THE BOOLEAN MEASURE STRUCTURE OF A ∇-COMMUTATIVE SYSTEM

From a given probability space $(\Omega,\mathcal{A},P)$ and a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, we get the (measure) Boolean algebra $A = \mathcal{A}/[P]$ quotient of $\mathcal{A}$ mod. P-null sets and the subalgebra $B = \mathcal{B}/[P]$. To the operation of taking the $\mathcal{B}$-measurable hull in $\mathcal{A}$ corresponds in A a so called *monadic operator* (c.f. {2}).

Let A be a Boolean algebra with a subalgebra B such that for each $a \in A$ there exists an element $b \in B$ which is the least element of B verifying $b \geq a$; we set $b = \nabla a$, and we call ∇ the monadic operator in A related to B.

A monadic operator verifies the following properties: $\nabla 0 = 0$, $\nabla(a \vee b) = \nabla a \vee \nabla b$, $\nabla\nabla a = \nabla a$, $\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$.

The algebraic system (A,B,∇) is called a *monadic algebra*. Let us consider a ∇-commutative system $S = (\Omega,\mathcal{A},\mathcal{B},\mathcal{C},P_1,P_2)$ with a product measure Q. By passing to the quotient mod. $[Q]$ we get the Boolean (measure) algebras A,B,C,D, which are the images of $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ resp.; B,C,D are subalgebras of A and $D = B \cap C$. If we denote by $\mathcal{A}_o = \mathcal{B} \vee \mathcal{C}$ then $A_o = B \vee C = \mathcal{A}_o/[Q]$, where $B \vee C$ is the Boolean algebra generated by B and C.

If ∇^1,∇^2,∇ designate the monadic operators in A corresponding to the measurable hull operations in $\mathcal{A}$ denoted before with the same symbols we have $\nabla^1\nabla^2 a = \nabla^2\nabla^1 a = \nabla a$ for any $a \in A$. The same hold if we restrict our-selves to elements $a \in A_o = B \vee C$. So we have an instance of what is called a biadic algebra (c.f. {2}).

The algebraic system $(A_o,B,C,\nabla^1,\nabla^2)$ is called a ***biadic algebra*** if (A_o,B,∇^1), (A_o,C,∇^2) are monadic algebras, $A_o = B \vee C$ and $\nabla^1\nabla^2 = \nabla^2\nabla^1$. It is easy to see that in this case $\nabla = \nabla^1\nabla^2$ defines the monadic operator related to $D = B \cap C$.

We can say that the underlying Boolean structure of a ∇-commutative system is a biadic algebra. We have seen this when a product measure Q is given in the system and it is easy to see that the same is true even if Q does not admit a σ-additive extension from $\mathcal{A}_o$ to $\mathcal{A}$.

In particular, if S is a simple ∇-commutative system, we obtain a simple biadic algebra $(A_o,B,C,\nabla^1,\nabla^2)$ i.e. it verifies $D = B \cap C = \{0,1\}$. For such simple algebras we have $A_o = B \oplus C$, direct sum of B, C which means that $A_o = B \vee C$ and, for $b \in B$, $c \in C$, $b \wedge c = 0$ implies $b = 0$ or $c = 0$ (in fact, if $b \wedge c = 0$ then $0 = \nabla(b \wedge c) = \nabla^1\nabla^2(b \wedge c) = \nabla^1(\nabla^2 b \wedge c) = \nabla^1(\nabla^2\nabla^1 b \wedge c) = \nabla^1(\nabla^1\nabla^2 b \wedge c) = \nabla b \wedge \nabla^1 c = \nabla b \wedge \nabla c$, therefore $\nabla b = 0$ or $\nabla c = 0$, so $b = 0$ or $c = 0$) (The converse also holds: if $A_o = B \oplus C$, $(A_o,B,C,\nabla^1,\nabla^2)$ is a simple biadic algebra).

In simple ∇-commutative systems, for example the systems obtained from cartesian products, what really matters from the point of view of the theory of measure Boolean algebras are the algebras B, C and A_o. Explicitly, if A'_o, B', C' are obtained from another simple ∇-commutative system S', then if we have Boolean isomorphisms $B \simeq B'$ and $C \simeq C'$ we get $A_o \simeq A'_o$. This is due to the fact that A_o, A'_o are direct sums. In fact, the direct sum $A_o = B \oplus C$ has the property of extension of homomorphisms: if $B \longrightarrow \bar{A}$ and $C \longrightarrow \bar{A}$ are Boolean homomorphisms with range a Boolean algebra $\bar{A}$, there exists one and only one extension of them to a homomorphism: $A_o = B \oplus C \longrightarrow \bar{A}$. (c.f. {5}).

On the other hand, if (B,P_1) (C,P_2) are given Boolean measure algebras we can construct at least a simple ∇-commutative system S for which the associated biadic algebra is precisely $(B \oplus C,B,C,\nabla^1,\nabla^2)$. In fact, it is well known that the Stone space of $B \oplus C$ is the cartesian product of the Stone spaces of B and C, $S(B \oplus C) = S(B) \times S(C)$ (Precisely the algebra of clopens of $S(B) \times S(C)$ is used to define $B \oplus C$). We set on the clopens of $S(B)$ and $S(C)$ the measures P_1 and P_2 in the obvious way and we extend P_1, P_2 to the σ-algebras generated by clopens associated to elements of B, C respectively. The product of the probability spaces so obtained gives us the required system S.

For general ∇-commutative systems we can prove analogous results.

To a ∇-commutative system $S = (\Omega,A,B,C,P_1,P_2)$ we have associated a biadic algebra (A_o,B,C). Moreover B,C are measure Boolean algebras with the probabilities P_1, P_2 defined on B, C, respectively, coinciding in $D = B \cap C$. Let us call $M(S) = (A_o,B,C,P_1,P_2)$ this Boolean measure structure associated with S. We shall say that the ∇-commutative systems S, S' have the same Boolean measure structure, $M(S) \equiv M(S')$, if under a unique Boolean isomorphism $A_o \simeq A'_o$, $B \simeq B'$, $C \simeq C'$ and $D \simeq D'$; and the probabilities P_i, P'_i $(i=1,2)$ correspond under the isomorphism.

THEOREM 6. *1) Given two probability Boolean algebras* (B,P_1), (C,P_2) *and sub-σ-algebras* $D \subset B$, $D' \subset C$ *such that* $D \simeq D'$ *under a fixed isomorphism preserving the measures* $P_1|_D$, $P_2|_{D'}$, *there exists a ∇-commutative system S such that* $M(S) = (\bar{A}_o,\bar{B},\bar{C},\bar{P}_1,\bar{P}_2)$, *where* $\bar{B} \simeq B$, $\bar{C} \simeq C$ *are measure preserving isomorphisms (with respect to* P_i, $\bar{P}_i$ $i=1,2$*) which restricted to* $\bar{D} = \bar{B} \cap \bar{C}$ *are isomorphisms* $\bar{D} \simeq D$, $\bar{D} \simeq D'$ *commuting with the given one* $D \simeq D'$.

2) Such a ∇*-commutative system admits a product measure* Q.

3) If S' *is another* ∇*-commutative system verifying the properties of* S *in (1), then* $M(S) \equiv M(S')$.

The proof of 1) and 3) are based on an algebraic theorem concerning biadic algebras that we give next.

We shall write the complete proof of theorem 6 in §7.

THEOREM 7. *1) Given two monadic algebras* (B,D,∇^1) *and* (C,D',∇^2) *and a fixed isomorphism:* $D \simeq D'$ *there exists a biadic algebra* $(\bar{A},\bar{B},\bar{C})$ *such that* $\bar{B} \simeq B$, $\bar{C} \simeq C$ *are isomorphisms which restricted to* $\bar{D} = \bar{B} \cap \bar{C}$ *give isomorphisms* $\bar{D} \simeq D$, $\bar{D} \simeq D'$ *commuting with the given one:* $D \simeq D'$.

2) If there is another biadic algebra $(\tilde{A},\tilde{B},\tilde{C})$ *with the same properties, then the isomorphisms* $\bar{B} \simeq \tilde{B}$, $\bar{C} \simeq \tilde{C}$ *obtained through the isomorphisms of* $\bar{B}$, $\bar{C}$ *with* B *and* C, *have a unique common extension to an isomorphism* $\bar{A} \simeq \tilde{A}$.

Proof: By identifying the isomorphic algebras D, D' through the given isomorphism we can , without loss of generality, consider only the case that B and C are extensions of the same algebra D. So we have the monadic algebras (B,D,∇^1) and (C,D,∇^2).

A filter $F \subset B$ corresponds in $X = S(B)$ with the set X_F of ultrafilters of B that contains F, this is a closed subset that represents with the relative topology the quotient algebra B/F (c.f. {5}). In other words, $X_F = S(B/F)$, and in such a way that if $\hat{b}$ is the clopen set that represents $b \in B$, then $\hat{b} \cap X_F$ is the clopen set that represents the class in B/F containing b. Given an ultrafilter U in D, let us denote with (U) the filter generated in B by U. Then $\{X_{(U)}\}$ is a partition of X. In fact, if $m \in X$ corresponds to the ultrafilter M and $U = M \cap D$, then $m \in X_{(U)}$; if $m \in X_{(U)} \cap X_{(U')}$, then $M \cap D \supset U, U'$, which implies $U = U'$.

Moreover, the monadic operator ∇^1 corresponds with saturation with respect to the partition $\{X_{(U)}\}$; i.e. if $a \in B$, $\widehat{\nabla^1 a} = \text{sat}\,\hat{a} =$
$= \bigcup\{X_{(U)}\ ;\ \hat{a} \cap X_{(U)} \neq \emptyset\}$.

We include the proof of this well-known fact (c.f. {2}) for the sake of completeness. If $d \in D$, then $d \in U$ iff $X_{(U)} \subset \hat{d}$, and it is equivalent to $X_{(U)} \cap \hat{d} \neq \emptyset$, as it is easy to see. In consequence, $\hat{a} \cap X_{(U)} \neq \emptyset$ iff $\nabla^1 a \supset X_{(U)}$, which is also equivalent to $\widehat{\nabla^1 a} \in U$. In fact, $\nabla^1 a \in U$ implies $\hat{a} \cap X_{(U)} \neq \emptyset$, since otherwise $X_{(U)} \subset \complement\hat{a} = \widehat{\complement a}$, i.e. $\complement a \in (U)$, then for some $d \in U$, $d \leq \complement a$, and hence $d \wedge \nabla^1 a = 0$, which contradicts the fact that U is a proper filter of D.

Let us observe that given a set X, a partition Π of X and a algebra B of subsets of X stable under the sat operation with respect to Π, if D is the algebra of saturated sets of B, we have a monadic algebra (B,D,sat.)

Let us suppose now that an extension A of the algebras B, C exists, such that (A,B,C) is a biadic algebra; it is easy to verify that the ∇ operators defined by B, C on A coincide on B, C with the previously given. We call $\nabla = \nabla^1\nabla^2 = \nabla^2\nabla^1$ to the monadic operator defined by D.

Since $B \oplus C$ applies homomorphically onto A, preserving the identity mappings on B and C, $A = B\oplus C/F$ for a filter F in $B \oplus C$. Then the Stone space of A is a closed subset T of $S(B\oplus C) = X \times Y$.

Calling $\{Y_{(U)}\}$ the partition of Y associated to the ultrafilters U of D we have, after elimination of superfluous parentheses:

$$(*) \qquad T = \textstyle\sum_U X_U \times Y_U$$

In fact, if $M \in T$ corresponds to an ultrafilter $\bar{M}$ of A, and $M = (M',M'')$ ($M' \in X$, $M'' \in Y$), considering the homomorphism mentioned above it follows that $M' = B.\bar{M}$, $M'' = C.\bar{M}$. Therefore, $M'.D = M''.D = U$ is an ultrafilter of D. Therefore, $M \in X_U \times Y_U$. Conversely, if $(M',M'') \in X_U \times Y_U$, $M'.D = M''.D = U$. To see that $(M',M'') \in T$ it suffices to see that there is an ultrafilter $\bar{M}$ of A which is a simultaneous extension of M' and M''. It is enough to verify that if $b \in M'$, $c \in M''$ then $b \wedge c \neq 0$. But $\nabla b \wedge \nabla c = \nabla(b \wedge c) \in U$ and therefore is not zero, which implies $b \wedge c \neq 0$.

Let us consider now the following partitions of T:

1) $\{T_x\}$, $x \in X$, $T_x = (\{x\} \times Y) \cap T$ 2) $\{T_y\}$, $y \in Y$, $T_y = (X \times \{y\}) \cap T$. 3) $\{T_U\}$, U ultrafilter of D, $T_U = X_U \times Y_U$.

They define the sat operators corresponding to ∇^1, ∇^2, ∇ of the biadic algebra (A,B,C). This is immediate for ∇^1, ∇^2. For ∇ we observe that the monadic algebra (A,D) is represented by T and the partition associated with the ultrafilters U of D, which is precisely $\{X_U \times Y_U\}$ as it was shown in the proof of (*).

If we start with another extension A' of B and C such that (A',B,C) is a biadic algebra we get again the same set T representing S(A') because the second member of (*) depends only on (B,D) and (C,D). Then $A \simeq A'$ by a unique common extension of the identity isomorphisms of B and C. This proves 2) (except for isomorphic identifications).

The fact that the definitions of T, the partitions 1), 2), 3), and the clopens corresponding to elements of B and C depend only on (B,D) and (C,D) allow us to construct a biadic algebra of sets (A',B',C') such that $B \simeq B'$, $C \simeq C'$ and these isomorphisms when restricted to D give an isomorphism $D \simeq B' \cap C' = D'$. That will prove (except for isomorphic identifications) the first part of the theorem.

Let us define A' as the algebra of sets of the form $a' = \hat{a} \cap T$, where $\hat{a}$ is a clopen in $S(B) \times S(C) = X \times Y$. The algebras $B' = \{(\hat{b}\times Y) \cap T\}_{b\in B}$, $C' = \{(X\times\hat{c}) \cap T\}_{c\in C}$ generate A'.

We define for $a' \in A'$, $\nabla^1 a'$ as the saturated set with respect to $\{T_x\}$, and $\nabla^2 a'$ that obtained from $\{T_y\}$. We must show that $\nabla^1 a' \in B'$ to show that (A',B') is monadic, the same for ∇^2, and that ∇^1, ∇^2 commute. To this end, let us prove:

$$(6) \qquad \nabla^1 c' = (\widehat{\nabla c} \times Y) \cap T \quad , \text{ for } c' \in C'.$$

$\nabla^1 c' = \nabla^1 (\sum^* X_U \times (\hat{c} \cap Y_U))$, where $\hat{c}$ is the projection of c' on Y, and the star means that the sum is extended to those U such that $\hat{c} \cap Y_U \neq \emptyset$. Therefore, $\nabla^1 c' = \sum^*(\nabla^1 (X_U \times (\hat{c} \cap Y_U)) = [(\sum^* X_U) \times Y] \cap T$. Then, since $\hat{c} \cap Y_U \neq \emptyset$ is equivalent to $\widehat{\nabla c} \supset Y_U$, i.e. to $\nabla c \in U$, we get $\sum^* X_U = \widehat{\nabla c}$, which proves the formula.

To show that $\nabla^1 a' \in B'$, for every $a' \in A'$, it suffices to see it for $a' = b' \cap c'$, $b' \in B'$, $c' \in C'$; $\nabla^1 (b' \cap c') = b' \cap \nabla^1 c' =$

$= [(\hat{b} \cap \widehat{\nabla c}) \times Y] \cap T \in B'$.

If $b' = c'$, then from (6) $b' = (\hat{d}_1 \times Y) \cap T = (X \times \hat{d}_2) \cap T$, which implies $\hat{d}_1 = \hat{d}_2$. Therefore, $D' = B' \cap C'$ is defined as those sets of A' such that are saturated with respect to $\{T_U\}$ and project on the same ele - ment of D. Therefore, (6) means that $\nabla^1 c' \in D'$, which is equivalent to the commutation of ∇^1 , ∇^2 (c.f. {1}).

6. AN APPROXIMATION PROCESS.

The generalized product measure Q, when it exists, and in general the finitely additive product measure Q, associated with a ∇-commutative system S, can be obtained as a limit of simpler measures in the way described in the next theorem. We need the following preliminary re sult:

PROPOSITION 1. *i) If* F *is a finite part of* (A,D,∇), *a monadic alge bra, and* $a_1,\dots,a_r$, *the atoms of the Boolean subalgebra generated by* F, *then* $a_1,\dots,a_r$, $\nabla a_1,\dots,\nabla a_r$ *generate a subalgebra* A_o *which is the*

the least one containing F *and stable for* ∇, *(i.e.,* $\nabla A_o \subset A_o$*).*

ii) Assume A_o *is a finite subalgebra of the biadic algebra* $(A,B,C,\nabla^1,\nabla^2)$, *stable for* $\nabla = \nabla^1\nabla^2$, $D = B \cap C$. *If* $B_o = B \wedge A_o$, $C_o = C \cap A_o$ *then there exist operators* ∇_o^1, ∇_o^2 , *such that* $(A_o, B_o, C_o, \nabla_o^1, \nabla_o^2)$ *is a biadic algebra.*

iii) Suppose that $(A_\lambda)_{\lambda\in\Lambda}$ *is the family of all the finite subalgebras of* A *stable for* ∇, *ordered by inclusion, and generated by their elements belonging to* B *or* C. *Then:*

1) *every* $(A_\lambda, B_\lambda = A_\lambda \cap B, C_\lambda = A_\lambda \cap C, \nabla_\lambda^1, \nabla_\lambda^2)$ *is a biadic algebra,*

2) $(A_\lambda)_{\lambda\in\Lambda}$ *is filtering, if ordered by inclusion,*

3) $\bigcup_\lambda A_\lambda = A$.

Proof: i) It is evident that every subalgebra containing F and stable for ∇ must contain the a_i's and the ∇a_i's. Then, it suffices to prove that A_o is stable for ∇. Every atom of A_o is of the form $a = a_i \wedge \bigwedge_j \nabla a_j$ where j runs on some indices $1,\dots,r$. Therefore, $\nabla a = \nabla a_i \wedge \bigwedge_j \nabla a_j \in A_o$.

To finish the proof it suffices to observe that every element of A_o is a union of atoms and that ∇ distributes over the union.

ii) ∇_o^i exists because A_o is finite. $a \in A_o$ implies $\nabla a = \nabla^2\nabla^1 a \leq \nabla_o^2\nabla_o^1 a \leq \nabla_o a = \nabla a$, where ∇_o denotes the operator relative to $D \cap A_o$. Then $\nabla_o^2\nabla_o^1 = \nabla_o$ and analogously, $\nabla_o^1\nabla_o^2 = \nabla_o$.

iii) It follows from i) and ii) and the observation that every $a \in B \vee C$ belongs to a finite subalgebra generated by a finite set $F \cup G$ with $F \subset B$, $G \subset C$.

Given the ∇-commutative system $S = (\Omega, A, B, C, P_1, P_2)$ let $M(S) = (A_o, B, C, P_1, P_2)$ be the associated Boolean measure structure. We apply proposition 1 to the biadic algebra (A_o, B, C) to get the filtering family $(A_\lambda)_\lambda$ described in iii). For each λ we select a representative $S_\alpha = (\Omega, A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$ of $(A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$, that means: B_λ, C_λ are finite subalgebras of B, C such that $M(S_\alpha) = (A_\lambda, B_\lambda, C_\lambda, P_1, P_2)$, $A_\lambda = B_\lambda \vee C_\lambda$. Here $\alpha = A_\lambda$, and we assume the α's ordered by inclusion of the A_λ's. Ussing theorem 5 we know that the measure Q_α associated to S_α is defined by

$$Q_\alpha(B.C) = \sum_j P_1(BD_j)\ P_2(CD_j)/P(D_j)$$

where the sum is on the atoms of $\mathcal{D}_\lambda$ (it defines a comarginal commutative measure on S_α). Denote with G_α the conditional expectation operator relative to $\mathcal{D}_\lambda$ and the probability Q_α in the system S_α.Then

THEOREM 8. *In a ∇-commutative system S, it holds:*

i) For $B \in \mathcal{B}$, $C \in \mathcal{C}$, $G_\alpha 1_B \longrightarrow G1_B$ *and* $G_\alpha 1_C \longrightarrow G1_C$, *when* $\alpha\uparrow$, *uniformely a.e.* [P].

ii) $Q(A) = \lim_\alpha Q_\alpha(A)$, *for any* $A \in \mathcal{B} \vee \mathcal{C}$.

Proof: ii) follows from i) and

$$Q_\alpha(B.C) = \int G_\alpha 1_B . G_\alpha 1_C \, dP \longrightarrow \int G1_B . G1_C \, dP = Q(B.C)$$

i) For $B \in \mathcal{B}_\lambda$, $\alpha = A_\lambda$, we have $G_\alpha 1_B = \sum_j (P_1(BD_j)/P(D_j))\, 1_{D_j}$, where the sum is on the atoms of $\mathcal{D}_\lambda$, since $Q_\alpha(BD_j) = \int_{D_j} G_\alpha 1_B \, dP =$

$= P(BD_j)/P(D_j) \cdot Q_\alpha(D_j) = P_1(BD_j)$ $(P_1 = Q_\alpha$ on $\mathcal{B}_\lambda)$.

Given $G1_B$, let us divide the set of real numbers on intervals $I_i = [m_i, M_i)$ of length $\varepsilon > 0$. Only for a finite number of them $D_i = (G1_B)^{-1}(I_i) \neq \emptyset$ [P]. Consider any finite system S_α such that $D_i \in \mathcal{D}_\lambda$ for all those D_i. Call $\{D_{ij}\}$ the family of atoms of $\mathcal{D}_\lambda$ contained in D_i.

Since, $$m_i P(D_{ij}) \leq \int_{D_{ij}} G\,1_B \, dP \leq M_i P(D_{ij}) \quad ,$$

we obtain: $m_i P(D_{ij}) \leq P_1(BD_{ij}) \leq M_i P(D_{ij})$. Therefore on D_{ij}:

$|G\,1_B - P_1(BD_{ij})/P(D_{ij})| \leq \varepsilon$, a.e [P]. Then

$$|\textstyle\sum_i \sum_j P_1(BD_{ij})/P(D_{ij})\, 1_{D_{ij}} - G\,1_B| \leq \varepsilon \qquad \text{a.e } [P] \qquad \text{QED.}$$

REMARK: If a comarginal probability P exists on A we have $G_\alpha 1_B = E_P(1_B | \mathcal{D}_\lambda)$. Using a result from martingale theory due to Helms {3}, we know that the G_α's form a uniformly integrable martingale converging in L^1 to $G\,1_B$, which implies ii).

7. PRODUCT MEASURE

For special cases of ∇-commutative systems we can assert that a product measure exists.

One of these cases, the discrete case, was considered in theorem 5. Using remark 4) of §2 we have also:

THEOREM 9. *If the ∇-commutative system* $S = (\Omega, A, \mathcal{B}, \mathcal{C}, P_1, P_2)$ *is such* that there are semi-compact classes $K_B \subset B$, $K_C \subset C$ with the property of approximation and $K_B.K_C = \{K.L \; ; \; K \in K_B \; , \; L \in K_C\}$ is also semi-compact , then $Q(B.C) = \int G1_B.G1_C dP$ defines, when extended to A, a probality, i.e. the product measure on S.

Another case in which we can assert the existence of product measure is referred in theorem 6, the proof of which we give now:

Proof of theorem 6. We will suppose, like in the proof of theorem 7, that D is identified to D' through the given isomorphism. By theorem 7 we have an extension algebra A_o of B, C such that (A_o,B,C) is biadic and $B \wedge C = D$. Then we have in the Stone space T of A_o the algebras of clopens $\hat{A}_o$, $\hat{B}$, $\hat{C}$ relative to A_o, B , C. We set $B = \tau(\hat{B})$, $C = \tau(\hat{C})$ and $A = \tau(\hat{A}_o) = \tau(B,C)$. We define in $\hat{B}$, $\hat{C}$ the measures P_1, P_2 given in B, C in the canonical way and extend them to B, C.

Let us prove that $S = (T,A,B,C,P_1,P_2)$ is a ∇-commutative system. If $\hat{b} \in \hat{B}$ and $\hat{c} \in \hat{C}$ are such that $\hat{b}.\hat{c} = \emptyset$, i.e. $b \wedge c = 0$ in A ; we have $0 = \nabla^2(b \wedge c) = \nabla^2 b \wedge c = \nabla^2\nabla^1 b \wedge c = \nabla b \wedge c$. Then $\nabla\hat{b}.\hat{c} = \emptyset$, where $\nabla\hat{b} \in \hat{B} \cap \hat{C}$ contains $\hat{b}$. Hence $\{G1_{\hat{b}} > 0\} \subset \nabla\hat{b}$ $[P]$, and then

$$\int G1_{\hat{b}}.G1_{\hat{c}} \, dP = \int G1_{\hat{b}}.1_{\hat{c}} \, dP_2 \leq P_2(\nabla\hat{b}.\hat{c}) = 0 \quad ,$$

where G is defined on $B(C)$-measurable functions is the expectation operator relative on $\mathcal{D} = B \cap C$.

But $\hat{B} \subset B$ and $\hat{C} \subset C$ have the approximation property. Using remark 3 of §2 we can assert that $Q(B.C) = \int G1_B.G1_C \, dP$ is a finitely additive comarginal measure on S, and from theorem 2, S is a ∇-commutative system.

Q being finitely additive on $B \vee C \supset \hat{A}_o$ is a fortiori σ-additive on the algebra of clopens $\hat{A}_o$ and then can be extended uniquely to $A = \tau(\hat{A}_o)$.

This proves 1) and 2) of the theorem (except for identifications) 3) follows immediately from theorem 7.

Finally we shall prove:

THEOREM 10. *If* $S = (\Omega,A,B,C,P_1,P_2)$ *is a* ∇*-commutative system with the property:*

(P) Ω *is the only set of* B *containing a set* $C \in C$ *with* $P_2(C) > 0$, *then* $Q(B.C) = P_1(B).P_2(C)$ *can be extended to a probability on* A. *((P) implies that* S *is simple).*

Proof: Let us define the "section" in x of a "rectangle" B.C by $(B.C)_x = C$ if $x \in B$ and $\emptyset$ otherwise.

In order to prove the σ-additivity of $Q = P_1.P_2$ it is enough to prove that $\sum_\alpha B_\alpha.C_\alpha = \Omega$ implies $\sum_\alpha P_1(B_\alpha).P_2(C_\alpha) = 1$. For each pair $\alpha \neq \beta$ of indices the set $N_{\alpha,\beta} = \{x \; ; \; (B_\alpha C_\alpha)_x.(B_\beta C_\beta)_x \neq \emptyset \; [P_2]\}$ is contained in a set of $\mathcal{B}$ of P_1-measure zero. In fact, from $B_\alpha C_\alpha B_\beta C_\beta = \emptyset$ we have either $P_1(B_\alpha.B_\beta) = 0$ or $P_2(C_\alpha.C_\beta) = 0$. In the first case, it follows from $N_{\alpha\beta} \subset B_\alpha B_\beta$, in the second one, since $(B_\alpha C_\alpha)_x.(B_\beta C_\beta)_x \subset C_\alpha.C_\beta$ for every x, we have $N_{\alpha\beta} = \emptyset$.

Then except for a set B_o of $\mathcal{B}$ of P_1-measure zero $(\supset \bigcup_{\alpha\neq\beta} N_{\alpha\beta})$

(7) $P_2(\bigcup_\alpha (B_\alpha.C_\alpha)_x) = \sum_\alpha P_2((B_\alpha.C_\alpha)_x) = \sum_\alpha P_2(C_\alpha).1_{B_\alpha}(x)$.

Let b_x be the set of the partition of Ω defined by the sets B_α such that $b_x \ni x$

$$b_x \subset \bigcup_\alpha (B_\alpha.C_\alpha)_x \tag{8}$$

In fact, let $y \in b_x$ and suppose $y \in B_\alpha.C_\alpha$, then $B_\alpha \supset b_x \ni x$ and this implies $(B_\alpha.C_\alpha)_x = C_\alpha$. Since $y \in C_\alpha$ we have $y \in \bigcup_\alpha (B_\alpha.C_\alpha)_x$. This proves (8).

From $\mathcal{B} \ni b_x \subset \bigcup_\alpha (B_\alpha.C_\alpha)_x \in \mathcal{C}$, and the assumed property (P) we have $P_2(\bigcup_\alpha (B_\alpha.C_\alpha)_x) = 1$ for every x.

Hence, by virtue of (7) we have $\forall x \notin B_o$: $\sum_\alpha P_2(C_\alpha)1_{B_\alpha}(x) = 1$. i.e. $\sum_\alpha P_2(C_\alpha).1_{B_\alpha} = 1_\Omega \; [P_1]$. By integration with respect to P_1 :

$$\sum_\alpha P_2(C_\alpha).P_1(B_\alpha) = 1 \qquad \text{QED.}$$

Remark: Condition (P) is equivalent to: (P*) if $C \in \mathcal{C}$ contains $B \neq \emptyset$, $B \in \mathcal{B}$, then $C = \Omega$ a.e. $[P_2]$.

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SÔBRE O PÔSTO DE UM MÓDULO (1)

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O objetivo desta nota é o de generalizar um resultado sôbre formas lineares que se encontra em {2} . No que segue, todo anel é suposto comutativo e com elemento unidade.

1. PRELIMINARES.

Sejam A um anel e M um A-módulo. Designaremos por M^* o dual de M e, se A fôr un anel de integridade, por t(M) o sub-A-módulo de torção de M. É conhecido que t(M) é o núcleo da aplicação natural $M \longrightarrow M \otimes_A K$, onde K é o corpo de frações de A.

Seja $A \longrightarrow K$ um homomorfismo de anéis de A num corpo K que transforma elemento unidade em elemento unidade. Então K pode ser munido, de uma maneira evidente, de uma estrutura de A-módulo. Para todo A-módulo M, o K-*pôsto* de M é definido como sendo a dimensão do K-espaço vectorial $M \otimes_A K$ e notaremos $r_K(M) = [M \otimes_A K : K]$. Se A fôr um anel de integridade en K o corpo de frações de A, falaremos simplesmente do *pôsto* de M e indicaremos com $r(M) = [M \otimes_A K : K]$.

LEMA 1. *Sejam* A *um anel e* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *uma sequência exata de* A-*módulos. Temos* $r_K(M) \leq r_K(M') + r_K(M'')$ *e se* $\mathrm{Tor}_1^A(M'',K) = 0$, *então* $r_K(M) = r_K(M') + r_K(M'')$.

Com efeito, é suficiente ver que se tem a sequência exata de K-espaços vectoriais

$$\cdots\cdots \longrightarrow \mathrm{Tor}_1^A(M'',K) \longrightarrow M' \otimes_A K \longrightarrow M \otimes_A K \longrightarrow M'' \otimes_A K \longrightarrow 0$$

COROLÁRIO. *Se* A *fôr um anel de integridade e* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *uma sequência exata de* A-*módulos, então* $r(M) = r(M') + r(M'')$.

Isto resulta do fato de que K, corpo de frações de A, é um A-módulo plano.

Observemos finalmente que se A é um anel de integridade e $a \neq 0$ um ideal de A, então $r(a) = 1$. Com efeito, é claro que $r(a) \leq 1$. Supondo que $r(a) = 0$, deduziríamos que $a \otimes_A K = 0$, onde K é o corpo

(1) Trabalho realizado com auxílio de FAPESP Proc. Matemática 66/038.

de frações de A, e como a está contido em $a \otimes_A K$, seguiria que $a=0$.

2. MÓDULOS DE PÔSTO n.

TEOREMA. *Sejam* A *um anel de integridade,* M *um A-módulo e* $n \geq 1$ *um inteiro. As seguintes condições são equivalentes:*

i) *existe uma família livre* $(x_i)_{1\leq j\leq n}$ *de elementos de* M *tal que* $\mathrm{Ann}(M/(x_1,\ldots,x_n)A) \neq 0$;

ii) *existe uma família* $(f_i)_{1\leq i\leq n}$ *de formas linares sôbre* M *e uma família* $(x_i)_{1\leq i\leq n}$ *de elementos de* M *tais que*

$f_1(x_1) = \ldots = f_n(x_n) \neq 0$, $f_i(x_j) = 0$ *se* $i \neq j$ *e*

$f_i(x_i)y = \sum_{j=1}^n f_j(y)x_j$ *para todo* $y \in M$ *e para* $i = 1,\ldots,n$;

iii) $r(M) = n$;

iv) *existe uma família* $(f_i)_{1\leq i\leq n}$ *de formas linares sôbre* M*, duas a duas distintas, tais que:* $t(M) = \bigcap_{i=1}^n \mathrm{Ker}\,(f_i)$ *e, para todo* j, $1 \leq j \leq n$, $\bigcap_{i\neq j} \mathrm{Ker}(f_i) \neq t(M)$

(i) $\Longrightarrow$ (ii). Como $\mathrm{Ann}(M/(x_1,\ldots,x_n)A) \neq 0$, existe um elemento $a\in A$, $a \neq 0$ tal que $aM \subset (x_1,\ldots,x_n)A$. Logo, para todo $y \in M$, se tem $ay = \sum_{i=1}^n f_i(y)x_i$ onde $f_i(y) \in A$ para todo i e como a família $(x_i)_{1\leq i\leq n}$ é livre, f_i é uma aplicação linear de M em A para todo i.Com efeito, $(a - f_i(x_i))x_i - \sum_{j\neq i} f_j(x_i)x_j = 0$ implica que $f_i(x_i) = a \neq 0$ para todo i e $f_j(x_i) = 0$ para $i \neq j$. Temos assim $f_i(x_i)y = \sum_{j=1}^n f_j(y)x_j$ para todo y em M.

De outro lado, tomando y_1 e y_2 em M, se tem $\sum_{i=1}^n f_i(y_1 - y_2)x_i =$ $= f_i(x_i)(y_1 + y_2) = f_i(x_i)y_1 + f_i(x_i)y_2 = \sum_{i=1}^n f_i(y_1)x_i +$ $+ \sum_{i=1}^n f_i(y_2)x_i$, logo $f_i(y_1 + y_2) = f_i(y_1) + f_i(y_2)$ para todo i. Ànalogamente, se $c \in A$ e $y \in M$, deduzimos que $f_i(cy) = c.f_i(y)$ para todo i.

(ii) $\Longrightarrow$ (i) Seja $\sum_{j=1}^n a_jx_j = 0$ onde os a_j estão em A. Resulta, da igualdade precedente, que $0 = f_i(\sum_{j=1}^n a_jx_j) = a_if_i(x_i)$ e como $f_i(x_i) \neq 0$ e A é um anel de integridade, então $a_i = 0$ para todo i. Supondo que $\mathrm{Ann}(M/(x_1,\ldots,x_n)A) = 0$ e reduzindo a relação $f_i(x_i)y=$

$\sum_{j=1}^{n} f_j(y)x_j$ para todo $y \in M$, módulo $(x_1,\ldots,x_n)A$, deduziríamos $f_i(x_i) \in \mathrm{Ann}(M/(x_1,\ldots,x_n)A)$ de onde, $f_i(x_i) = 0$ para todo i.

(i) $\Longrightarrow$ (iii). Como A é um anel de integridade é a família $(x_i)_{1\le i\le n}$ é livre, a sequência exata de A-módulos $0 \longrightarrow (x_1,\ldots,x_n)A \longrightarrow M \to$ $\to M/(x_1,\ldots,x_n)A \longrightarrow 0$ nos dá $r(M)=r((x_1,\ldots,x_n)A)+r(M/(x_1,\ldots,x_n)A)=$ $n + r(M/(x_1,\ldots,x_n)A)$. Além disso como $\mathrm{Ann}(M/(x_1,\ldots,x_n)A) \neq 0$, então $(M/(x_1,\ldots,x_n)A) \otimes_A K = t(M/(x_1,\ldots,x_n)A) \otimes_A K = 0$. Logo, $r(M) = n$.

(iii) $\Longrightarrow$ (i). Se $r(M) = n$, é claro que o K-espaço vectorial $M \otimes_A K$ tem sempre uma base do tipo $(x_i \otimes 1)_{1\le i\le n}$ e isto implica que a família $(x_i)_{1\le i\le n}$ é livre. De outro lado, para todo $y \in M$, $y \neq 0$, podemos escrever $y \otimes 1 = \sum_{i=1}^{n}(a_i/b_i)(x_i \otimes 1)$ onde os a_i/b_i estão em K. Pondo $b = \prod_{i=1}^{n} b_i$ e $c_i = a_i\prod_{i\neq j} b_j$, a relação precedente é equivalente à relação $(by - \sum_{i=1}^{n} c_ix_i) \otimes 1 = 0$. Isto implica (cf. {1}, cap 2, prop. 4) que $by - \sum_{i=1}^{n} c_ix_i$ não é livre em M, logo que existe um elemento $c \in A$, $c \neq 0$ tal que $c(by - \sum_{i=1}^{n} c_ix_i) = 0$. Logo, $cb \in \mathrm{Ann}(M/(x_1,\ldots,x_n)A)$ e $cb \neq 0$, uma vez que A é um anel de integridade.

(iv) $\Longrightarrow$ (iii). Com efeito, para todo j, $1 \le j \le n$, existe um elemento $x'_j \in \bigcap_{i\neq j}\mathrm{Ker}(f_i) - t(M)$ tal que $x'_j \notin \mathrm{Ker}(f_j)$. As hipóteses feitas implicam que a família $(x'_j)_{1\le j\le n}$ é livre. Pondo $x_i =$ $= f_1(x'_1)\ldots\ldots\widehat{f_i(x'_i)}\ldots\ldots f_n(x'_n)x'_i$ para todo i, segue-se que a família $(x_i)_{1\le i\le n}$ é livre, que $f_j(x_i) = 0$ se $i \neq j$ e que $f_1(x_1) = f_2(x_2)\ldots$ $= f_n(x_n) \neq 0$. Seja $f\colon M \longrightarrow \bigoplus_{i=1}^{n} f_i(M)$ a aplicação A-linear definida por $f(x) = (f_i(x))_{1\le i\le n}$ para todo x em M e $J = \mathrm{Im}(f)$. A sequência exata de A-módulos $0 \longrightarrow t(M) \longrightarrow M \longrightarrow J \longrightarrow 0$ nos dá $r(M) = r(J)$. Consideremos a família $(e_i)_{1\le i\le n}$ de elementos de J definida por $e_j = (f_i(x_j))_{1\le i\le n}$. Mostremos que a família $(e_i)_{1\le i\le n}$ é livre maximal. Com efeito, a relação $0 = \sum_{i=1}^{n} c_ie_i = (c_if_i(x_i))_{1\le i\le n}$, onde os c_i estão em A, implica que $c_if_i(x_i) = 0$, logo que $c_i = 0$ para todo i. De outro lado, se $y \in J$ existe un elemento $x \in M$ tal que $y=(f_i(x))_{1\le i\le n}$ Se tomarmos $c = -f_1(x_1) = -f_2(x_2) = \ldots\ldots = -f_n(x_n) \neq 0$, temos a re lação não trivial $cy + \sum_{i=1}^{n} f_i(x)e_i = 0$ entre os e_i e y. Isto nos mostra que a família $(e_i)_{1\leq i\leq n}$ é livre maximal, logo que $r(J) = n$.

(ii) $\Longrightarrow$ (iv). Trivial.

OBSERVAÇÕES.

(1) Sejam A um anel de integridade, M um A-módulo e $n \geq 1$ um inteiro. Se uma qualquer das condições equivalentes do teorema precedente fôr verificada, então M não pode ser um módulo de torção. Com efeito, se $M = t(M)$, então $r(M) = 0$, o que é absurdo, uma vez que $r(M) = n \geq 1$.

(2) No caso $n = 1$, $f_1 = f$, em (iv) $\Longrightarrow$ (iii) se tem a sequência exata de A-módulos $0 \longrightarrow t(M) \longrightarrow M \longrightarrow f(M) \longrightarrow 0$ e portanto, as condições $t(M) \neq M$ e $f \neq 0$ são equivalentes.

3. A x-CONDIÇÃO

1. Em seguida, vamos mostrar como o teorema acima se relaciona com a noção de x-condição introduzida em {2}. Sejam A um anel de integridade, M um A-módulo e f uma forma linear sôbre M. Se existir um x em M tal que $f(x) \neq 0$ e $f(x)y = f(y)x$ para todo $y \in M$, diremos que a forma linear f obedece à x-*condição*.

EXEMPLOS.

1) Se f é uma forma linear injectiva sôbre M, então f obedece à x-condição, com $x \in M - \{0\}$, x qualquer. Recìprocamente a x-condição não implica que f seja injectiva, mas sòmente que $\mathrm{Ker}(\psi) = 0{:}f(x)A$. Vemos assin que se $0{:}f(x)A = 0$ as duas noções coincidem.

2) Daremos aquí um exemplo de forma linear que obedece à x-condição mas que não é injectiva. Para isto, sejam $A = Z$ o anel dos inteiros racionais, $M = Z \times Z/(2)$ (produto direto) e f a forma linear sôbre M definida por $f(m,\bar{n}) = 2m$ para todo $(m,\bar{n}) \in M$. É claro que f é uma linear sôbre M e se consideramos o elemento $x = (1,\bar{0}) \in M, f(x)\neq 0$. De outro lado, para todo $y = (m,\bar{n}) \in M$ se tem $f(x)y = 2(m,\bar{n}) = (2m,\bar{0})$ e $f(y)x = 2m(1,\bar{0}) = (2m,\bar{0})$. Assim f obedece a $(1,\bar{0})$-condição. O núcleo de f é o sub-A-módulo de M formado pelos pares $(0,\bar{0})$ e $(0,\bar{1})$.

3) Consideremos o anel $A = Z$, o A-módulo $M = Z \times Z$ (produto direto) e a forma linear f sôbre M definida por $f(m,n) = m + n$, para todo $(m,n) \in M$. Suponhamos que exista um $x = (p,q) \in M$ tal que x obedeça à x-condição. Para todo $y = (m,n) \in M$ se deve ter $(p+q)(m,n)=$

$= (m+n)(p,q)$, isto é $np = mq$. Ora, uma tal relação é impossível para todo $(m,n) \in M$, exceto se $p = q = 0$, o que daria $x = 0$ e portanto $f(x) = 0$.

2. Indiquemos com $M^*(x)$ o sub-A-módulo de M^* das formas lineares que obedecem à x-condição. É lógico que podemos excluir sempre o caso $x=0$, uma vez que $M^*(0) = \emptyset$. Obtem-se assim:

$$\bigcup_{x \in M} M^*(x) \subset M^* \quad , \quad \text{logo} \quad \langle \bigcup_{x \in M} M^*(x) \rangle \subset M^*$$

onde indicamos com $\langle S \rangle$, se S é um sub-conjunto de M, o sub-A-módulo de M gerado por S. Veremos, no N° 4, que em geral:

$$\langle \bigcup_{x \in M} M^*(x) \rangle \subsetneqq M^*$$

isto é, não é possível "aproximar" toda forma linear por formas lineares que obedecem a x-condições, x em M.

3. Do teorema precedente, resulta uma caracterização dos módulos munidos de uma forma linear que satisfaz a uma x-condição.

COROLÁRIO. *Sejam* A *um anel de integridade e* M *um* A-*módulo. As condições seguintes são equivalentes:*

(1) *existe um elemento* x *em* M *tal que* $\mathrm{Ann}(\{x\}) = 0$ *e* $\mathrm{Ann}(M/Ax) \neq 0$.

(2) *existe uma forma linear* f *sôbre* M *que obedece à* x-*condição.*

(3) $r(M) = 1$.

(4) M *não é um módulo de torção e existe uma forma linear* f *sôbre* M *tal que* $\mathrm{Ker}(f) = t(M)$.

4. CONTRA-EXEMPLO A FÓRMULA $\langle \bigcup_{x \in M} M^*(x) \rangle = M^*$.

Suponhamos que a fórmula acima seja verdadeira. Toda forma linear $f \in M^*$ se escreve $f = \sum_i f_i$ (soma finita), onde $f_i \in M^*(x_i)$ para todo i. Logo $f_i(x_i)y = f_i(y)x_i$ para todo $y \in M$. Isto nos mostra que $f_j(x_i)y - f_j(y)x_i \in \mathrm{Ker}(f_i) = t(M)$ (pelo corolário) isto é, existe um elemento $c \in A$, $c \neq 0$ tal que $f_j(cx_i)y = f_j(y)(cx_i)$ para todo y. Logo $f_j \in M^*(cx_i)$ e portanto $f \in M^*(cx_i)$. Mas, o exemplo 3) do N°3 nos mostra que, em geral, isto não é possível.

NOTA: La traducción de "pôsto" es "rango".

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Este volumen constituye una excelente y apropiada exposición de las aplicaciones de procesos estocásticos a la biología y la medicina. La presentación de los temas se encuentra completamente al día.

La organización en general se divide en tres largos capítulos. Los dos primeros tratan la teoría de procesos estocásticos de parámetro discreto y continuo respectivamente los cuales constituyen una introducción al último capítulo dedicado a las aplicaciones, que son muy variadas. Como dato ilustrativo cabe citar a modo de ejemplo los estudios de los modelos de población, migraciones epidemia, modelo de Karlin-McGregor en los procesos estocásticos de la evolución, modelo estocástico de la contracción muscular y hasta un muy reciente modelo de la carie dentaria debida a Lu (1966).

En síntesis, es una obra muy completa y muy competente en el tema, tal vez una de las primeras en su tipo. El lector se enfrenta con una larga bibliografía.

E. Marchi.

THEORY OF RANDOM FUNCTIONS, por A. Blanc-Lapierre y R. Fortet, Vol. I, Gordon and Breach, New York-London-París, 1967, 454 pgs., 29,50 dólares.

Este libro es una traducción al inglés por J. Gani de la primera mitad del bien conocido texto "Théorie des fonctions aléatoires". Además de un apéndice sobre las nociones matemáticas básicas necesarias para desarrollar el material expuesto contiene: Cap. I. Introducción práctica al estudio de las funciones aleatorias. Cap. II. Axiomas, conceptos básicos y teoremas fundamentales de la teoría de la probabilidad. Cap. III. Introducción general a las funciones aleatorias. Cap. IV. Introducción general a los procesos estocásticos: funciones aleatorias con incrementos independientes. Cap. V. Funciones aleatorias derivadas de procesos de Poisson. Cap. VI. Procesos de Markov. Cap. VII. Cadenas de Markov. Funcionales aditivas de un proceso de Markov.

La encuadernación y la impresión son excelentes y el volumen II-que completaría la traducción del original francés - es prometido.

R. Panzone.

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En 1942 la U. M. A. ha iniciado la publicación de una nueva serie de "Memorias y monografías" de las que han aparecido hasta ahora las siguientes:

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Vol. II; Nº 1. — YANNY FRENKEL, *Criterios de bicompacidad y de H-completidad de un espacio topológico accesible de Fréchet-Riesz.* Nº 2. — GEORGES VALIRON, *Fonctions entières.*

Vol. III; Nº 1. — E. S. BERTOMEU y C. A. MALLMANN, *Funcionamiento de un generador en cascadas de alta tensión.*

Además han aparecido tres cuadernos de *Miscelánea Matemática.*

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