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MULTIPLIERS FOR (C,κ)-BOUNDED FOURIER EXPANSIONS IN WEIGHTED LOCALLY CONVEX SPACES AND APPROXIMATION

J. Junggeburth*

Dedicated to Professor Alberto González Domínguez on the occasion of his seventieth birthday.

I. INTRODUCTION.

In [11] some first extensions of the multiplier theory as developed in Banach spaces in [5] and [17] were presented for locally convex spaces. In view of the applications these considerations were essentially restricted to order-preserving operators. In the mean time, however, we observed that some of the given and other examples are also valid in a non-order-preserving setting. In this general frame a multiplier theory for arbitrary multiplier operators has interesting new applications, in particular to weighted locally convex spaces. Motivated by these viewpoints we therefore continue our investigations in [11], this time for general multipliers in locally convex spaces.

In the applications we treat projective and inductive limits, essentially of weighted locally convex Hausdorff spaces. The Fourier series are defined via classical orthogonal systems such as the trigonometric system, Laguerre-, Hermite- or ultraspherical polynomials.

After giving some definitions and general results in Section II, first of all in Section III multipliers are defined. Then some classical inequalities of approximation theory are extended to locally convex spaces and the saturation problem for approximation processes of multiplier operators is treated. In Section IV we derive a multiplier criterion via the (C^{κ})-condition (4.3).

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Finally Section V gives some nontrivial new applications of this criterion in (weighted) locally convex spaces for Fourier expansions by trigonometric and Laguerre polynomials. By similar considerations further examples concerning Hermite and ultraspherical expansions could be worked out.

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II. PRELIMINARIES.

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{P}$ denote the set of all, of all non-negative and of all positive integers, respectively. Furthermore, let $\mathbb{R}$ and $\mathbb{R}^+$ be the set of all real and of all positive real numbers. In the following $(X, \{p_r\}_{r\in J})$, J being an arbitrary index set, will always denote a locally convex Hausdorff space whose topology T is generated by a family of filtrating seminorms, or to be short, by a system of seminorms $\{p_r\}_{r\in J}$. Let $[X]$ be the class of all continuous linear operators of X into itself. A family $\{T(\rho)\}_{\rho>0} \subset [X]$ is called an (equicontinuous) approximation process on $(X, \{p_r\})$, if for each $r \in J$ there exists $t \in J$ and a constant $M(r,t) > 0$ such that

$$p_r(T(\rho)f) \leq M(r,t)p_t(f) \qquad (f \in X,\ \rho > 0) \tag{2.1}$$

$$\lim_{\rho\to\infty} p_r(T(\rho)f - f) = 0 \qquad (f \in X) \tag{2.2}$$

Let $(X,\{p\})$ and $(Y,\{q\})$ be two locally convex spaces such that Y is continuously embedded in X. Let X be complete. With $\mathcal{K} := \prod_{q\in\{q\}} \mathbb{R}^+$, the completion of Y relative to X is defined by (cf.[3])

$$Y^{\sim X} := \bigcup_{R\in\mathcal{K}} \overline{S(R)}^X , \qquad S(R) := \bigcap_{q\in\{q\}} S_q(0;R_q) ,$$

where $\overline{S(R)}^X$ denotes the closure of $S(R)$ in the X-topology and $S_q(0;\varepsilon) := \{h \in Y;\ q(h) < \varepsilon\}$. Let $\{f_\beta\}_{\beta\in\mathbf{D}}$ be a net in Y with directed domain $\mathbf{D}$ and $N_f(Y)$ the class of all nets in Y which converge to f in X. Then for every $q \in \{q\}$

(2.3) $$q(f) := \inf \{\sup_{\beta\varepsilon D} q(f_\beta);\ \{f_\beta\}_{\beta\varepsilon D} \in N_f^b\} \qquad (f \in Y^{\sim X})$$

is a seminorm on $Y^{\sim X}$ with

$$N_f^b := \{\{f_\beta\}_{\beta\varepsilon D} \in N_f(Y);\ \{f_\beta\}_{\beta\varepsilon D} \text{ is bounded in } Y\} \qquad (f \in X).$$

The locally convex spaces X to be considered in the applications are representable as projective or inductive limits (cf. [16]) of locally convex spaces or even Banach spaces. In treating projective limits, we always examine the special case $X \subset X_r$, $r \in J$, with locally convex spaces X_r, and the linear mappings $u_r : X \rightarrow X_r$ are the identity mappings. Furthermore, the system of seminorms $\{p_r\}_{r\varepsilon J}$ on X is usually given by a countable system of norms $\{a_k\}_{k\varepsilon P}$ which are in concordance . If the spaces X_k are complete for all $k \in P$, we obtain the class of the complete, countably normed spaces (Fréchet spaces) (cf. [7]) as a special class of the projective limit. In the same way in our examples of inductive limits the linear mappings $u_r : X_r \longrightarrow X$ are always the restrictions of the identity map from X to the locally convex subspaces $X_r \subset X$. The topology of the inductive limit is then the finest locally convex topology on X which induces on each X_r a coarser topology than the initial topology. Particularly $X = \bigcup_{m=0}^{\infty} X_m$, the inductive limit of a monotone increasing sequence $\{(X_m, T_m)\}_{m\varepsilon P}$ of locally convex spaces, is called the countable inductive limit of the spaces (X_m, T_m), $m \in P$, or sometimes a countable union space.

Let $X = \bigcup_{m=0}^{\infty} X_m$ be the countable inductive limit of a sequence of metrisable, locally convex spaces (X_m, T_m) and let Y be the strict inductive limit of locally convex spaces (Y^k, T^k) with the additional property that each Y^k is closed in Y^{k+1}. Then a family $\{T(\rho)\}_{\rho>0} : X \longrightarrow Y$ of linear operators is equicontinuous iff to each $m \in P$ there exists a $k = k(m) \in P$ such that $\{T(\rho)\} : X_m \longrightarrow Y^k$ is equicontinuous (cf. [16; p. 89], [1]).

Furthermore, each closed linear operator T from a countable inductive limit of B-complete locally convex Baire spaces $\{X_k\}_{k\varepsilon P}$ into itself is known to be continuous (cf. [1]). A corresponding version of the closed graph theorem holds for a linear closed operator T of a barreled, B-complete space X into itself, especially for complete countably normed spaces X (cf.[16; p. 126]).

III. GENERAL THEORY.

III.1 MULTIPLIERS.

Let X be a locally convex (Hausdorff) space whose topology T is generated by a system $\{p_r\}_{r\in J}$ of seminorms. Furthermore, let $\{P_k\}_{k\in\mathbf{P}} \subset [X]$ be a total sequence of mutually orthogonal projections on X, in short a system $\{P_k\}$, i.e., (i) mutually orthogonal: $P_j P_k = \delta_{jk} P_k$, δ_{jk} being Kronecker's symbol, and (ii) total: $P_k f=0$ for all $k \in \mathbf{P}$ implies f=0. Then to each $f \in X$ one may associate its unique Fourier series expansion

$$(3.1) \qquad f \sim \sum_{k=0}^{\infty} P_k f \qquad (f \in X)$$

The sequence $\{P_k\}_{k\in\mathbf{P}}$ is said to be fundamental if the set Π of all polynomials, i.e., the set of all finite linear combinations $\sum_{k=0}^{n} f_k$ with $f_k \in P_k(X)$, is dense in $(X, \{p_r\}_{r\in J})$.

With ω the set of all sequences $\tau = \{\tau_k\}_{k\in\mathbf{P}}$ of scalars, $\tau \in \omega$ is called a multiplier for X (with respect to $\{P_k\}$) if for each $f \in X$ there exists an element $f^\tau \in X$ such that

$$(3.2) \qquad P_k f^\tau = \tau_k P_k f \qquad (k \in \mathbf{P})$$

Since $\{P_k\}$ is total, f^τ is uniquely determined by f. The class of all multipliers τ for X with respect to $\{P_k\}$ is denoted by $M = M(X; \{P_k\})$.

To each multiplier $\tau \in M$ there corresponds a closed linear multiplier operator $T^\tau : X \longrightarrow X$, defined by $T^\tau f = f^\tau$. (In general we don't distinguish between multipliers and the corresponding multiplier operators). The set $M_C = M_C(X; \{P_k\})$ of all $\tau \in M$ for which the operator T^τ is continuous on X, can be identified with a closed subspace of $[X]$, denoted by $[X]_{M_C}$. In general $M_C \subset M$, but if the closed graph theorem holds on X, then $M_C = M$. In this case, to each $r \in J$ there exists $t \in J$ and a constant $B(r,t) > 0$ such that

$$(3.3) \qquad p_r(T^\tau f) \leqslant B(r,t) p_t(f) \qquad (f \in X)$$

and we set

$$(3.4) \qquad \|T^\tau\|_{r,t} := \inf\{B(r,t);\ p_r(T^\tau f) \leqslant B(r,t;\tau) p_t(f),\ f \in X\} := \|\tau\|_{M,r,t}$$

If the seminorms $\{p_r\}_{r\varepsilon J}$ on X are norms as in the case of countably normed spaces, then $\|T^\tau\|_{r,t} = \|T^\tau\|_{[X^r,X^t]}$, where the Banach spaces X^t and X^r are the completions of the locally convex space X under the norms p_t and p_r, respectively.

For an arbitrary $\psi \in \omega$ we define

(3.5) $\quad X^\psi := \{f \in X;$ there exists an $f^\psi \in X$ with $\psi_k P_k f = P_k f^\psi$ for all $k \in \mathbf{P}\}$

Evidently $X^\psi \subset X$, and the linear operator $B^\psi: X^\psi \longrightarrow X$, defined by $B^\psi f = f^\psi$ for $f \in X^\psi$, is closed for each $\psi \in \omega$. Furthermore, $P_k(X) \subset X^\psi$ for each $k \in \mathbf{P}$, so that B^ψ is densely defined if $\{P_k\}$ is fundamental on X. The operators B^ψ are called operators of multiplier-type.
It is easy to see

LEMMA (3.6). (*a*) *Under the system of seminorms* $\{p_r^\psi\}_{r\varepsilon J}$, *defined by* $\quad p_r^\psi(f) := p_r(f) + p_r(B^\psi f) \quad (r \in J,\ f \in X^\psi)$
X^ψ *becomes a locally convex subspace of* X; *the system* $\{p_r^\psi\}_{r\varepsilon J}$ *is filtrating and separating.*

(*b*) *If* $(X, \{p_r\}_{r\varepsilon J})$ *is a complete locally convex space, then* $(X^\psi, \{p_r^\psi\}_{r\varepsilon J})$ *is complete.*

In contrast to the Banach space theory in arbitrary locally convex spaces there here exist unbounded multipliers τ corresponding to a continuous operator T^τ. A simple example is the differential operator $R = -i(d/dx)$ with eigenvalues $\{\lambda_k\}_{k\varepsilon\mathbf{P}} = \{k\}_{k\varepsilon\mathbf{P}}$ which is not a continuous multiplier operator on $C_{2\pi}$ (with respect to the system $\{e^{ikx}\}$) but a bounded one on $\mathcal{D}_{2\pi}$, the locally convex space of 2π-periodic infinitely differentiable test functions.

III.2 INEQUALITIES OF JACKSON-, BERNSTEIN- AND ZAMANSKY-TYPE AND SATURATION.

In the following some fundamental inequalities in approximation theory will be extended to locally convex spaces, and with these means the saturation problem for multiplier operators in locally convex spaces will be treated.

Let $\phi(\rho)$ be a positive, monotonely decreasing function on $(0,\infty)$

with $\lim_{\rho\to\infty} \phi(\rho) = 0$.

THEOREM (3.7). *Let* $\{T(\rho)\}_{\rho>0}$ *be a family of multiplier operators on* $(X,\{p_r\})$ *corresponding to* $\{\tau(\rho)\} \subset M$, *and let* $\psi \in \omega$. *Furthermore, let the family of multiplier operators* $\{L(\rho)\}_{\rho>0}$, *given via* $\{\lambda(\rho)\} \subset M_C$, *be equicontinuous on* X *with respect to* ρ. *Then the condition*

$$(3.8) \qquad \phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k\lambda_k(\rho) \qquad (k \in \mathbf{P})$$

implies that to each $r \in J$ *there exists* $t \in J$ *and a constant* $B(r,t) > 0$ *such that the Jackson-type inequality*

$$(3.9) \qquad \phi^{-1}(\rho)p_r(T(\rho)f - f) \leqslant B(r,t)p_t(B^{\psi}f) \qquad (f \in X^{\psi})$$

holds. On the other hand, the condition

$$(3.10) \qquad \psi_k\tau_k(\rho) = \phi^{-1}(\rho)\lambda_k(\rho) \qquad (k \in \mathbf{P})$$

implies the Bernstein-type inequality

$$(3.11) \qquad p_r(B^{\psi}T(\rho)f) \leqslant B(r,t)\phi^{-1}(\rho)p_t(f) \qquad (f \in X)$$

and

$$(3.12) \qquad \psi_k\tau_k(\rho) = \phi^{-1}(\rho)\lambda_k(\rho)\{\tau_k(\rho) - 1\} \qquad (k \in \mathbf{P})$$

the Zamansky-type inequality

$$(3.13) \qquad p_r(B^{\psi}T(\rho)f) \leqslant B(r,t)\phi^{-1}(\rho)p_t(T(\rho)f - f) \qquad (f \in X)$$

The proofs are easy and follow analogously as those in Banach spaces (cf. [6]). Indeed, (3.8) immediately implies (3.9) since

$$(3.14) \qquad \phi^{-1}(\rho)\{T(\rho)f - f\} = L(\rho)B^{\psi}f \qquad (f \in X^{\psi}, \rho > 0)$$

In a similar way one may treat Bohr-type inequalities (cf. [8]) and the comparison problem (cf. [11], [12]).
Let us briefly examine the saturation problem for approximation processes $\{T(\rho)\}_{\rho>0}$, defined via multipliers $\{\tau(\rho)\}_{\rho>0}$. Proceeding as in the Banach space frame (cf. [5;II] and [17]), we set for an approximation process $\{T(\rho)\}_{\rho>0} \subset [X]_{M_C}$

$$\bar{\mathbf{T}} := \{k \in \mathbf{P};\ \tau_k(\rho) = 1 \quad \text{for all } \rho > 0\}$$

Under the assumption $\mathbf{T} \neq \mathbf{P}$ we postulate as a sufficient condition for the solution of the saturation problem:

(3.15) *Let* $\{T(\rho)\} \subset [X]_{M_C}$ *be an approximation process on* $(X, \{p_r\}_{r\in J})$ *with associated multipliers* $\{\tau(\rho)\}$. *Let there exist a family* $\{\eta(\rho)\} \subset M_C$, *whose associated multiplier operators* $\{E(\rho)\}$ *form an approximation process on* X, *and a sequence* $\psi \in \omega$ *with* $\psi_k \neq 0$ *if* $k \notin \bar{\mathbf{T}}$ *such that for all* $\rho > 0$ *and* $k \in \mathbf{P}$

$$\phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k \eta_k(\rho)$$

As $\{E(\rho)\}$ is an approximation process, there holds $\lim_{\rho\to\infty} \eta_k(\rho) = 1$ for all $k \in \mathbf{P}$ so that

$$\lim_{\rho\to\infty} \phi^{-1}(\rho)\{\tau_k(\rho) - 1\} = \psi_k \qquad (k \in \mathbf{P}) \tag{3.16}$$

THEOREM (3.17). *If* $p_r(T(\rho)f - f) = 0_r(\phi(\rho))$ *for each* $r \in J$ *then* $f \in \bigcup_{m\in\bar{\mathbf{T}}} P_m(X)$, *and* $T(\rho)f = f$ *for all* $\rho > 0$, *i.e.* f *is an invariant element.*

Proof. As $P_k \in [X]$ and

$$P_k(\phi^{-1}(\rho)\{T(\rho)f - f\}) = \phi^{-1}(\rho)\{\tau_k(\rho) - 1\}P_kf$$

for each $k \in \mathbf{P}$ and $r \in J$ there exists some $t \in J$ such that by (3.16)

$$p_r(\psi_k P_k f) = \lim_{\rho\to\infty} p_r(\phi^{-1}(\rho)\{\tau_k(\rho) - 1\}P_kf) \leq$$
$$\leq B(r,t;k) \lim_{\rho\to\infty} p_t(\phi^{-1}(\rho)\{T(\rho)f - f\}) = 0$$

Thus $\psi_k P_k f = 0$ for all $k \in \mathbf{P}$ which implies $P_kf = 0$ for $k \notin \bar{\mathbf{T}}$, while for $k \in \bar{\mathbf{T}}$ one has $P_kT(\rho)f = P_kf$. Hence $P_kT(\rho)f = P_kf$ for each $k \in \mathbf{P}$, and the theorem is proved.

If, in addition, the set

$$F[X; T(\rho)] := \{f \in X;\ p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) = 0_r(1) \quad \text{for } \rho\to\infty \text{ and each } r \in J\} \tag{3.18}$$

contains a noninvariant element, then the approximation process

$\{T(\rho)\}$ is saturated in X with order $\phi(\rho)$, and $F[X; T(\rho)]$ is called its Favard or saturation class. Such a noninvariant element always exists as for each $k \notin \overline{T}$, $r \in J$ and $0 \neq h \in P_k(X)$

$$p_r(T(\rho)h - h) = |\tau_k(\rho) - 1| p_r(h)$$

Let us observe that (3.15) implies for each $r \in J$

$$\phi^{-1}(\rho) p_r(T(\rho)f - f) = p_r(B^{\psi}E(\rho)f) \qquad (f \in X, \rho > 0) \tag{3.19}$$

THEOREM (3.20). *Given* $(X, \{p_r\}_{r\varepsilon J})$ *such that the closed graph theorem holds on* X. *If* $\{T(\rho)\}$ *satisfies* (3.15), *then the Favard class of* $\{T(\rho)\}$ *is characterized as* $(X^{\psi})^{\sim X}$ *and the following seminorms are equivalent on* $F[X; T(\rho)]$:

$$\text{(i)}\quad p_r(f) + \sup_{\rho>0} p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) \qquad (r \in J)$$

$$\text{(ii)}\quad \tilde{q}_r^{\psi}(f) \quad \textit{and} \quad \text{(iii)}\quad \sup_{\rho>0} p_r^{\psi}(S(\rho)f) \qquad (r \in J)$$

where $\{S(\rho)\}_{\rho>0} \subset [X]_{M_C}$ *is an approximation process with* $S(\rho)(X) \subset \subset X^{\psi}$.

Proof. (i) $\Leftrightarrow$ (iii): On account of (3.15) one may choose $\{S(\rho)\}_{\rho>0} = \{E(\rho)\}_{\rho>0}$, and the assertion follows immediately by (3.19) and

$$\begin{aligned}\sup_{\rho>0} p_r^{\psi}(E(\rho)f) &\leq B(r,t)p_t(f) + \sup_{\rho>0} p_r(\phi^{-1}(\rho)\{T(\rho)f - f\}) \leq \\ &\leq B(r,t)[p_s(f) + \sup_{\rho>0} p_s(\phi^{-1}(\rho)\{T(\rho)f - f\})] \leq \\ &\leq B(r,t) \sup_{\rho>0} p_s^{\psi}(E(\rho)f)\end{aligned}$$

as $\{p_r\}_{r\varepsilon J}$ is filtrating.

(ii) $\Rightarrow$ (iii): Given $f \in (X^{\psi})^{\sim X}$, there exists a net $\{f_{\beta}\}_{\beta\varepsilon D} \subset X^{\psi}$ such that $p_r^{\psi}(f_{\beta}) \leq R_r$ $(r \in J)$ for some $R_r > 0$ and $\lim_{\beta\varepsilon D} p_r(f_{\beta} - f) = 0$. Obviously $B^{\psi}S(\rho)$ is defined and closed on X, so that $B^{\psi}S(\rho) \in [X]_{M_C}$.
On X^{ψ} we have $B^{\psi}S(\rho) = S(\rho)B^{\psi}$, and therefore

$$\begin{aligned}p_r^{\psi}(S(\rho)f) &= p_r(S(\rho)f) + \lim_{\beta\varepsilon D} p_r(B^{\psi}S(\rho)f_{\beta}) = \\ &= p_r(S(\rho)f) + \lim_{\beta\varepsilon D} p_r(S(\rho)B^{\psi}f_{\beta}) \leq \\ &\leq B(r,t)[p_t(f) + \sup_{\beta\varepsilon D} p_t(B^{\psi}f_{\beta})] \leq \\ &\leq B(r,t) \sup_{\beta\varepsilon D} p_t^{\psi}(f_{\beta})\end{aligned}$$

The left side is independent of the special choice of the net $\{f_\beta\}_{\beta\in D}$, and the right side is independent of ρ; therefore

$$\sup_{\rho>0} p_r^\psi(S(\rho)f) \leqslant B(r,t)\ \inf\{\sup_{\beta\in D} p_t^\psi(f_\beta);\ \{f_\beta\} \in N_f^b\} = B(r,t)\tilde{q}_t^\psi(f).$$

(iii) $\Rightarrow$ (ii): This direction is easily proved by examining the particular net $\{S(\beta)\}_{\beta\in R^+} \subset X^\psi$.

IV. A MULTIPLIER CRITERION FOR CESÀRO BOUNDED FOURIER EXPANSIONS.

In the applications the problem arises whether a given sequence $\eta = \{\eta_k\}_{k\in P} \in \omega$ is a multiplier with respect to a given orthogonal system $\{P_k\}_{k\in P} \subset [X]$ in a locally convex space $(X,\{p_r\}_{r\in J})$. In this section we obtain a first criterion for subclasses of $M(X;\{P_k\})$ by the uniform boundedness of the (C,κ)-means; these are just the classes $bv_{\kappa+1}$, well known in the literature for some time, particularly in connection with the theory of divergent series.

In the locally convex space $(X,\{p_r\})$ with the system of projections $\{P_k\}_{k\in P}$ let the (C,κ)-means for $\kappa \geqslant 0$ be defined by

$$(C,\kappa)_n f := (A_n^\kappa)^{-1} \sum_{k=0}^{n} A_{n-k}^{\kappa} P_k f \qquad (f \in X, n \in P) \tag{4.1}$$

$$A_n^\kappa := \binom{n+\kappa}{n} = \frac{\Gamma(n+\kappa+1)}{\Gamma(n+1)\Gamma(\kappa+1)} \tag{4.2}$$

DEFINITION (4.3). ("*The* (C^κ)-*condition*"). *Let* $\kappa \geqslant 0$ *and* $(X,\{p_r\})$ *be complete. The pair* $(X,\{p_r\}_{r\in J})$, $\{P_k\}$ *satisfies the* (C^κ)-*condition, if for each* $r \in J$ *there exists* $t \in J$ *and a constant* $C(r,t;\kappa) > 0$ *such that*

$$p_r((C,\kappa)_n f) \leqslant C(r,t;\kappa)p_t(f) \qquad (f \in X, n \in P)$$

If (4.3) is satisfied for a fixed $\kappa \geqslant 0$, then it follows that for all $\beta > \kappa$

$$p_r((C,\beta)_n f) \leqslant C(r,t;\kappa)p_t(f) \qquad (f \in X, n \in P)$$

To derive an appropriate multiplier criterion we introduce the sequence spaces $bv_{\kappa+1}$ as subspaces of ℓ^∞, the set of all bounded sequences, by

(4.4) $bv_{\kappa+1} := \{\eta \in \ell^\infty; \|\eta\|_{bv_{\kappa+1}} = \sum_{k=0}^{\infty} A_k^\kappa |\Delta^{\kappa+1}\eta_k| + \lim_{k\to\infty} |\eta_k| < \infty\}$

where the (fractional) difference operator Δ^β is defined via

(4.5) $$\Delta^\beta \eta_k = \sum_{l=0}^{\infty} A_l^{-\beta-1} \eta_{k+l}$$

With $\beta \geqslant 0$ and $\eta \in \ell^\infty$ the series (4.5) converges absolutely. We still remark that $\lim_{k\to\infty} \eta_k = \eta_\infty$ exists for $\eta \in bv_{\kappa+1}$ and

(4.6) $$bv_{\kappa+1} \subset bv_{\gamma+1} \qquad 0 \leqslant \gamma < \kappa$$

Furthermore, for each $\eta \in bv_{\kappa+1}$

(4.7) $$\eta_n - \eta_\infty = \sum_{k=0}^{\infty} A_k^\kappa \Delta^{\kappa+1} \eta_{k+n} \qquad (n \in \mathbf{P})$$

For these fundamentals see [17; Sec.3] and the literature cited there.

THEOREM (4.8). *Let* $(X,\{p_r\}_{r\in J})$, $\{P_k\}$ *satisfy the* (C^κ)*-condition* (4.3) *for some* $\kappa \geqslant 0$. *Then* $bv_{\kappa+1}$ *is continuously embedded in* $M_C(X;\{P_k\})$, *i.e., to each* $r \in J$ *there exists* $t \in J$ *and a constant* $C(r,t;\kappa) > 0$ *such that*

$$\|\eta\|_{M,r,t} \leqslant C(r,t;\kappa)\|\eta\|_{bv_{\kappa+1}} \qquad (\eta \in bv_{\kappa+1})$$

Proof. Analogously to [5; II] or [12] we set up for an arbitrary $f \in X$ and $\eta \in bv_{\kappa+1}$

$$f^\eta := \sum_{k=0}^{\infty} A_k^\kappa \Delta^{\kappa+1}\eta_k (C,\kappa)_k f + \eta_\infty f$$

Then $f^\eta \in X$ since by (4.3) and (4.4) ($\{p_r\}$ being filtrating)

$$p_r(f^\eta) \leqslant C(r,t;\kappa)p_t(f) \sum_{k=0}^{\infty} A_k^\kappa |\Delta^{\kappa+1}\eta_k| + |\eta_\infty| p_r(f) \leqslant$$
$$\leqslant C(r,t;\kappa)\|\eta\|_{bv_{\kappa+1}} \; p_s(f)$$

To prove that $P_n f^\eta = \eta_n P_n f$ for each $n \in \mathbf{P}$ we consider

$$P_n(C,\kappa)_k f = \begin{cases} 0 & \text{if } n > k \\ (A_{k-n}^\kappa / A_k^\kappa) P_n f & \text{if } n \leqslant k \end{cases}$$

and obtain by (4.7) that

$$P_n f^{\eta} = P_n f\{\sum_{k=n}^{\infty} A_k^{\kappa} \frac{A_{k-n}^{\kappa}}{A_k^{\kappa}} \Delta^{\kappa+1}\eta_k + \eta_\infty\} = \eta_n P_n f$$

Concerning sufficient conditions for $\eta \in bv_{\kappa+1}$ we refer to [5; II] and [17; Sec.3 ff].

To give an application, let $(X,\{p_r\})$, $\{P_k\}$ satisfy (4.3) for some $\kappa \geq 0$. Then one may consider the Abel-Cartwright and the Riesz means of (3.1), thus for $\sigma > o$ and $\lambda > 0$

$$(4.9) \qquad W_\sigma(\rho)f \sim \sum_{k=0}^{\infty} w((\tfrac{k}{\rho})^\sigma)P_k f \text{ and } R_{\sigma,\lambda}(\rho)f \sim \sum_{k=0}^{\infty} r_\lambda((\tfrac{k}{\rho})^\sigma)P_k f$$

$$w(x) = e^{-x} \quad \text{and} \quad r_\lambda(x) = \begin{cases} (1-x)^\lambda & 0 \leq x \leq 1 \\ 0 & 1 < x \end{cases}$$

For each $\sigma > 0$, $\lambda \geq \kappa$ one has (cf. [5; II], [17; Sec.4]) that

$$(4.10) \qquad \{w((\tfrac{k}{\rho})^\sigma)\} \quad , \quad \{r_\lambda((\tfrac{k}{\rho})^\sigma)\} \subset bv_{\kappa+1}$$

uniformly in $\rho > 0$. Furthermore,

$$\lim_{\rho\to\infty} w((\tfrac{k}{\rho})^\sigma) = 1 \quad \text{and} \quad \lim_{\rho\to\infty} r_\lambda((\tfrac{k}{\rho})^\sigma) = 1$$

so that convergence of the Abel-Cartwright and the Riesz means follows on Π (cf. Sec. III.1). If Π is dense in X and X barreled, then the families $\{W_\sigma(\rho)\}_{\rho>0}$ and $\{R_{\sigma,\lambda}(\rho)\}_{\rho>0}$ form approximation processes on X.

To determine the Favard class $F[X; W_\sigma(\rho)]$ we examine (3.15) and have for any $\sigma > 0$

$$\psi_k = -k^\sigma, \quad \phi(\rho) = \rho^{-\sigma}, \quad e(x) = -x^{-\sigma}[\exp(-x^\sigma)-1] \ ,$$

$$\eta_k(\rho) = e(k/\rho), \ \lim_{\rho\to\infty} \eta_k(\rho) = 1, \text{ and } \{\eta_k(\rho)\}_{k\in\mathbf{P}} \in bv_{j+1}$$

for each $j \in \mathbf{P}$ (cf. [5; II]).

By theorem (3.20) it therefore follows that

$$F[X; W_\sigma(\rho)] = (X^\psi)^{\sim X} \quad \text{with} \quad \psi = \{-k^\sigma\}_{k\in\mathbf{P}}$$

For further examples of processes, also in connection with theorem (3.7) see [4], [5], [6], [9], [17] and the literature cited there. In this direction one has the following Bernstein-type inequality

THEOREM (4.11). *Let* $\nu > 0$. *If* $(X,\{p_r\})$, $\{P_k\}$ *satisfy the* (C^κ)*-condition* (4.3) *for some* $\kappa > 0$, *then to each* $r \in J$ *there exists* $t \in J$ *and a constant* $B(r,t;\kappa) > 0$ *such that for all polynomials* $f = \sum_{k=0}^{n} P_k f \in \Pi$ *it follows that*

$$p_r(\sum_{k=0}^{n} k^\nu P_k f) \leqslant B(r,t;\kappa) C\, n^\nu\, p_t(\sum_{k=0}^{n} P_k f)$$

Proof. We reduce the proof to (3.10) with $\rho\to\infty$ replaced by $n\to\infty$. Let $\nu > 0$ and $e(x) \in C_{oo}^{\infty}([0,\infty))$, the class of infinitely differentiable functions with compact support on $[0,\infty)$, such that $e(x) = x^\nu$ if $0 \leqslant x \leqslant 1$ and $e(x) = 0$ if $x \geqslant 2$. Evidently the sequence $\lambda(n)$ with $\lambda_k(n) = e(k/n)$ belongs to bv_{j+1} for each $j \in \mathbf{P}$ uniformly in $n \in \mathbf{P}$ (cf. [5; I, II]; the dependence of the parameter $n \in \mathbf{P}$ being of Fejér-type). Thus we choose $j = [\kappa] + 1$ with $[\kappa]$ the greatest integer less than or equal to κ. Now we identify $\psi_k = k^\nu$, $\phi^{-1}(n) = n^\nu$; finally, $\tau \in \omega$ with $\tau_k(n) = (n/k)^\nu \lambda_k(n)$ is a multiplier on X. Particularly, for $f = \sum_{k=0}^{n} P_k f$ we have by (3.11) as $\tau_k(n) = 1$ if $k \leqslant n$

$$p_r(\sum_{k=0}^{n} (k/n)^\nu P_k f) = p_r(\sum_{k=0}^{\infty} \lambda_k(n) P_k f) \leqslant B(r,t;\kappa) \|\lambda\|_{bv_{j+1}}\, p_t(f) ,$$

and the theorem is proved with $C = \|\lambda\|_{bv_{j+1}}$.

V. APPLICATIONS TO CESÀRO BOUNDED ORTHOGONAL SYSTEMS.

V.1. TRIGONOMETRIC SERIES.

As a first example we treat trigonometric expansions in weighted locally convex spaces of 2π-periodic functions. Here theorem (5.2) (cf. [14]) gives necessary and sufficient conditions upon the weight functions $U_r(x)$ such that the (C,1)-means of the Fourier series expansion satisfy condition (4.3). This in turn determines examples of locally convex spaces $X^p_{D,J}$ and $X^p_{V,J}$ for which the (C^κ)-condition is satisfied for $\kappa = 1$ but not for $\kappa = 0$.

Let the system $\{P_k\}_{k\varepsilon\mathbf{P}}$ be defined by

(5.1) $P_0f(x) = f^\wedge(0)$, $P_kf(x) = f^\wedge(k)e^{ikx} + f^\wedge(-k)e^{-ikx}$ $(k \in \mathbf{N})$

$f^\wedge(k)$ denoting the usual Fourier coefficients

$$f^\wedge(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iku}du \qquad (k \in \mathbf{Z})$$

THEOREM (5.2). [14; p. 223/224]: *Assume that* $1 \leqslant p < \infty$, $f(x)$ *is integrable on* $[0,2\pi]$, $U_r(x) \geqslant 0$, $f(x)$ *and* $U_r(x)$ *have period* 2π. *Then the following are equivalent:*

(5.3) $$\lim_{n\to\infty}\int_0^{2\pi} |(C,1)_nf(x) - f(x)|^pU_r(x)dx = 0$$

for every function f *satisfying* $a_r^p(f) := (\int_0^{2\pi} |f(x)|^pU_r(x)dx)^{1/p} < \infty$.

(5.4) $$\int_0^{2\pi} |(C,1)_nf(x)|^pU_r(x)dx \leqslant C_p\int_0^{2\pi} |f(x)|^pU_r(x)dx ,$$

the constant C_p *being independent of* f *and* n.

(5.5) *For every interval* I *with* $|I| \leqslant 2\pi$ ($|I|$ *the length of* I) *one has with a constant* K, *independent of* I,

$$A_p := \int_I U_r(x)dx \ (\int_I [U_r(y)]^{-1/(p-1)}dy)^{p-1} \leqslant K|I|^p \qquad (1 < p < \infty)$$

$$A_1 := \int_I U_r(x)dx \ \operatorname*{ess\,sup}_{y\in I}[U_r(y)]^{-1} \leqslant K|I| \qquad (p=1)$$

REMARK. *In the case* $1 < p < \infty$ *one may replace the* (C,1)*-means in theorem* (5.2) *by the usual partial sums* $S_n(f;x)$ (cf. [10]).

LEMMA (5.6). *For every subinterval* $I \subset \mathbf{R}$ *the weights* $U_r(x) = |x|^r$ *satisfy the condition*

$$A_p \leqslant K|I|^p \quad \textit{if} \quad -1 < r < p-1 \qquad (1 < p < \infty)$$

$$A_1 \leqslant K|I| \quad \textit{if} \quad -1 < r \leqslant 0 \qquad (p=1)$$

with a constant $K = K(r,p)$ *independent of* I.

In view of the estimate

(5.7) $$\frac{2}{\pi} \leqslant \frac{\sin x/2}{x/2} \leqslant 1 \qquad x \in [0,\pi]$$

Lemma (5.6) immediately implies

LEMMA (5.8). *The weight functions*

$$U_r(x) = |2 \sin x/2|^r \qquad (x \in \mathbb{R})$$

satisfy (5.5) *for*

$$-1 < r < p-1 \quad \textit{if} \quad 1 < p < \infty ,$$
$$-1 < r \leqslant 0 \quad \textit{if} \quad p = 1 .$$

Let us observe that on the fundamental interval $[-\pi,\pi]$ the functions $U_r(x)$ have to be defined here such that they are symmetric in some neighborhood of its singularities and zeros.

Via the weights $U_r(x)$ of Lemma (5.8) we define the Banach spaces

$$(5.9) \qquad X_{2\pi}^{r,p} := \{f \in L_{2\pi}^1;\ a_r^p(f) := $$
$$:= (\int_{-\pi}^{\pi} |f(x)|^p U_r(x)dx)^{1/p} < \infty\} \qquad (1 \leqslant p < \infty)$$

as subspaces of $L_{2\pi}^1$. We have in the sense of continuous embedding

$$(5.10) \qquad L_{2\pi}^\infty \subset X_{2\pi}^{r,p} \subset L_{2\pi}^1 \qquad r \in (-1,p-1),\ 1 \leqslant p < \infty$$

and $L_{2\pi}^2$ is dense in $X_{2\pi}^{r,p}$. Therefore the projections P_k, $k \in \mathbb{P}$, defined in (5.1), are continuous, total and fundamental on $X_{2\pi}^{r,p}$ for all $r \in (-1,p-1)$.

Given some open interval $J \subset (-1,p-1)$, by

$$(5.11) \qquad X_{D,J}^p := \bigcap_{r\varepsilon J} X_{2\pi}^{r,p} \quad \text{and} \quad X_{V,J}^p := \bigcup_{r\varepsilon J} X_{2\pi}^{r,p} \qquad (1 \leqslant p < \infty)$$

there are defined locally convex spaces in which the (C^κ)-condition (4.3) holds with $\kappa=1$ for $p=1$ and by [10] even with $\kappa=0$ for $1 < p < \infty$.

Evidently, for $p=1$ the $(C,0)$-means are not equicontinuous on $X_{D,J}^1$ or $X_{V,J}^1$ as the example

$$f_n(x) = \operatorname{sgn} D_n(x), \quad D_n(x) = 1/2 + \sum_{k=1}^n \cos kx$$

shows. Indeed, one has $(n\to\infty)$

$$a_r^1(S_n(f_n;x)) \geqslant (1/2)\|S_n(\operatorname{sgn} D_n;x)\|_{L_{2\pi}^1} = O(\log n) .$$

Under the system of norms $\{a_r^p\}_{r\in J}$ which are in concordance, $X_{D,J}^p$ $1 \leq p < \infty$, is a countably normed, barreled and B-complete Hausdorff space, metrisable but not normable. As the closed graph theorem holds we have $M_C(X_{D,J}^p\ ;\ \{P_k\}) = M(X_{D,J}^p\ ;\ \{P_k\})$. On each $X_{2\pi}^{r,p} \subset L_{2\pi}^1$, $r \in (-1,p-1)$, and therefore on $X_{D,J}^p$ and $X_{V,J}^p$ the system $\{P_k\}_{k\in P}$ is continuous, fundamental, total and mutually orthogonal.

$X_{V,J}^p$, $1 \leq p < \infty$, is a countable strict inductive limit, complete, barreled and Hausdorff but not metrisable and not necessary B-complete; however by the closed graph theorem we have $M_C = M$.

Let us conclude with an application of Theorem (4.11) in connection with the spaces (5.11) and weights (5.8).

COROLLARY (5.12). *Let* $\nu > 0$. *Then to each* $r \in (-1,p-1)$ *there exists a* $t \in (-1,p-1)$ *such that for each trigonometric polynomial* $\sum_{k=-n}^{n} c_k e^{ikx}$ *with some constant* $D(r,t;\kappa,\nu) > 0$

$$\left(\int_{-\pi}^{\pi} \left|\sum_{k=-n}^{n} |k|^{\nu} c_k e^{ikx}\right|^p |2 \sin x/2|^r dx\right)^{1/p} \leq$$

$$\leq D(r,t;\kappa,\nu) n^{\nu} \left(\int_{-\pi}^{\pi} \left|\sum_{k=-n}^{n} c_k e^{ikx}\right|^p |2 \sin x/2|^t dx\right)^{1/p}$$

V.2. LAGUERRE SERIES.

Let $L^p(0,\infty)$, $1 \leq p < \infty$, denote the usual Lebesgue spaces (with respect to ordinary Lebesgue measure) of functions for which the norms

$$\|f\|_p := \left(\int_0^{\infty} |f(x)|^p dx\right)^{1/p} \qquad (1 \leq p < \infty)$$

are finite. $L_{loc}^p(0,\infty)$ denotes the set of functions which belong locally to $L^p(0,\infty)$, i.e. on every compact subset of $(0,\infty)$.

With the weight functions, defined on $(0,\infty)$ for some (fixed) $\alpha > -1$

$$(5.13) \qquad U_{b,r}(x) := x^{\alpha/2} x^b (1+x)^{r-b} \exp(-x/2) \qquad (b,r \in \mathbb{R})$$

let us introduce the Banach spaces

(5.14) $X^p_{b,r} := \{f \in L^p_{loc}(0,\infty);\ \ \pi^p_{b,r}(f) := \|f(x)U_{b,r}(x)\|_p < \infty\}$

Then for every open subset $J \subset \mathbf{R}$, by

(5.15) $X^p_{D,J} := \bigcap_{r\in J} X^p_{b,r}$ and $X^p_{V,J} := \bigcup_{r\in J} X^p_{b,r}$ $\qquad (1 \leq p < \infty)$

locally convex spaces are defined in which considerations analogous to V.1 are valid. Therefore the closed graph theorem holds so that each multiplier τ is continuous. Obviously these and the following considerations are also true for a variation of the parameter b in an open set $J \subset \mathbf{R}$ or of the pair (b,r) in an open $J \subset \mathbf{R}_2$.

Let $L^\alpha_k(x)$, $\alpha > -1$, denote the kth Laguerre polynomial given via

$$\sum_{k=0}^{\infty} L^\alpha_k(x)s^k = (1-s)^{-\alpha-1} \exp(-\tfrac{sx}{1-s})$$

Then to each $f \in X^p_{D,J}$ or $f \in X^p_{V,J}$, respectively, one may associate its (well-defined) (cf. [15; p.17]) Laguerre series expansion

(5.16) $$f \sim \sum_{k=0}^{\infty} P^\alpha_k f$$

where

(5.17) $$(P^\alpha_k f)(x) := (\frac{k!}{\Gamma(k+\alpha+1)} \int_0^\infty f(y)e^{-y}y^\alpha L^\alpha_k(y)dy)L^\alpha_k(x)$$

These projections $\{P^\alpha_k\}_{k\in P}$ form a total, fundamental and mutually orthogonal system on $X^p_{D,J}$ and $X^p_{V,J}$. With the results in [13] and [15] we determine now open intervals $J = J(\kappa,p) \subset \mathbf{R}$ such that on $X^p_{D,J}$ and $X^p_{V,J}$ the (C^κ)-condition (4.3) holds with $r \leq t$, $r,t \in J$ for $\kappa=1$; but (4.3) isn't valid with $\kappa=0$ for all $r \in J$ and any choice of $t \in J$. This is an immediate consequence of the fact that the conditions about the parameters b and r in [13] concerning (C,0)-summability are sharp. However, it was not Muckenhoupt's aim to study summability conditions in a locally convex frame. In particular for $\alpha > -1/2$ and

(5.18) $$1/4 - 1/p < b < 3/4 - 1/p \qquad (1 \leq p < \infty)$$

one has the following intervals $J(\kappa,p)$ for the parameter r:

$$(5.19)\quad \text{for } \kappa=0 \quad J(0,p) = \begin{cases} (1/4 - 1/p,\ 3/4 - 1/p), & p \in [4/3,4] \\ \emptyset & , p \notin [4/3,4] \end{cases}$$

In case $\kappa=0$ the restriction $r < t$, $r,t \in J$, is necessary for $p=4/3$ and $p=4$. Thus, if $p \notin [4/3,4]$, then the (C,0)-means are not equicontinuous on $X^p_{D,J}$ or $X^p_{V,J}$ respectively. On the other hand,

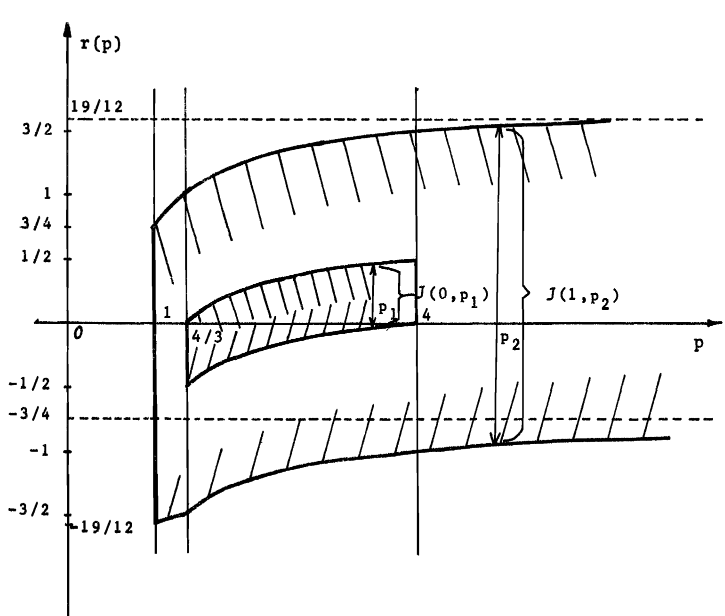
$$(5.20)\quad \text{for } \kappa=1 \ J(1,p) = \begin{cases} (-1/(3p) - 5/4,\ 7/4 - 1/p), & 1 \leqslant p < 4/3 \\ (-1/p - 3/4\ ,\ 7/4 - 1/p), & 4/3 \leqslant p \leqslant 4 \\ (-1/p - 3/4\ ,\ 19/12 - 1/(3p)), & 4 < p < \infty \end{cases}$$

Furthermore, for $b=r$, $\alpha > -1$, in (5.13), i.e. $U_{r,r}(x) = x^{\alpha/2}x^r\exp(-x/2)$ there are other open intervals $J(1,p)$ (cf. [15, p. 11]) such that the (C,1)-means, but not the (C,0)-means are equicontinuous on $X^p_{D,J}$ or $X^p_{V,J}$, $1 \leqslant p < \infty$, respectively, namely

$$(5.21)\quad \left\{ \begin{array}{l} -1/p-\min(\alpha/2,1/4) < r < 1 - 1/p + \min(\alpha/2,1/4) \\ -2/(3p) - 1/2 < r < 7/6 - 2/(3p) \end{array} \right\} \ (1 \leqslant p < \infty)$$

So (5.20) means, in the case $1 \leqslant p < 4/3$, for example, that to each $r \in (-1/(3p) - 5/4,\ 7/4 - 1/p)$ and each $t \in [r,\ 7/4 - 1/p)$ there exists a constant $C(r,t) > 0$ such that

$$\pi^p_{b,r}((C,1)_n f) \leqslant C(r,t)\pi^p_{b,t}(f) \qquad (f \in X^p_{V,J} \text{ or } f \in X^p_{D,J})$$



The figure gives the upper and lower bounds of the parameter $r = r(p)$ of conditions (5.19), (5.20) which determine the allowable intervals $J(0,p)$ and $J(1,p)$.

Obviously $J(0,p) \subset J(1,p)$, and the restriction $r \leqslant t$ in (5.18) - (5.20) may not be omitted. In the case $\kappa=1$ the parameter r generally still depends on the parameter $b = b(\alpha,p)$.

Further examples of weighted locally convex spaces with the more general weight functions $U_{b,r}(x)$ of (5.13) may be derived from the results in [13] and [15] by variation of the parameters b and r under the given restrictions.

Correspondingly one may treat Hermite expansions in suitable weighted functions spaces (cf. [11], [12]) using results of [13] and [15] or ultraspherical expansions by taking inductive and projective limits with respect to the parameter $p \in [1,\infty)$ (cf. [2], [12]). Some examples of domains J valid for Hermite expansions are given in [11; cf. (3.5) and (3.6)] in the order-preserving case, but they are valid on spaces analogous to (5.15) also in the general case. One can obtain further examples of locally convex spaces satisfying the condition (4.3) from the examples in V.1 and V.2 by forming countably normed spaces and inductive limits with respect to the free parameters in an open set A, for example in V.2 with $p \in A \subset [1,\infty)$

$$X_{D,A} := \bigcap_{p\in A} X^p_{D,J} \quad \text{and} \quad X_{V,A} := \bigcup_{p\in A} X^p_{D,J} \quad \text{or}$$

$$X^I_{V,A} := \bigcup_{p\in A} X^p_{V,J}$$

In the last case the closed graph theorem may fail on $X^I_{V,A}$ as this inductive limit is not countable, and hence one may then get $M_C \subset M$ with proper inclusion.

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SOBRE UN TEOREMA DE R. PALAIS Y LOS METODOS DE PROYECCION $\Pi_\Phi\{P_n\}$

Carmen Casas

En esta nota examinamos la relación entre un teorema de Palais [3], reformulado adecuadamente, y un teorema de la teoría de los métodos de proyección Π de Galerkin [2]. Mostramos que en el caso considerado por Palais, de un álgebra A_Φ "approximately tame", a. t., ambos teoremas son equivalentes. En caso de álgebras A generales el teorema de Palais sugiere la introducción de un método de proyección que llamamos Π_Φ, y damos para el método Π_Φ dos teoremas de estabilidad así como un criterio general para que se verifique $T \in \Pi_\Phi$. Previamente examinamos la relación entre la noción de álgebra a. t. y la de ideal simétrico mono-normante, y de paso extendemos el teorema de Palais al caso de bases equivalentes a ortonormales y sustituímos la condición "tame" por otra más débil vinculada a la noción de operador casi triangular.

Cuestiones más finas, tales como caracterización de la clase Π_Φ, caso de bases generales, etc., serán consideradas en un trabajo próximo.

El tema me fue sugerido por el Prof. M. Cotlar a quien agradezco su generosa ayuda.

1. PRELIMINARES.

1.1. NOTACIONES. En todo lo que sigue H designará un espacio fijo de Hilbert, $\{e_n\}$ una base ortonormal fija del mismo, H_n el subespacio generado por $\{e_1,\dots,e_n\}$, P_n la proyección ortogonal de H sobre H_n, B(H) el espacio de los operadores lineales acotados $A: H \longrightarrow H$ con la norma usual $\|A\|$, $K_n = \{K \in B(H);\ \dim. K(H) = n\}$, $K = \bigcup_{n=1}^{\infty} K_n$, y Ω_∞ el espacio de los operadores $X \in B(H)$ compactos con la misma norma $\|X\|$. Si A es invertible y $\{\varphi_j\} = \{A e_j\}$, entonces $\{\varphi_j\}$ se llama base equivalente a ortonormal; poniendo

$$P'_n\, x = \sum_{j=1}^{n} c_j\, \varphi_j, \text{ si } x = \sum_{j=1}^{\infty} c_j\, \varphi_j, \text{ vale } P'_n = A\, P_n A^{-1} \tag{1}$$

En algunas situaciones más generales consideramos en vez de $\{e_n\}$ una base $\{\varphi_n\}$ equivalente a ortonormal con $\{P'_n\}$ en vez de $\{P_n\}$. Observemos que $P_n x \longrightarrow x$, $\forall x \in H$, o sea $P_n \longrightarrow 1$ fuertemente pero no en norma. Sin embargo se tiene (cfr. [3] pág. 274):

1.1.1. Si $P_n x \longrightarrow x$, $\forall x \in H$, $\|P_n\| \leqslant 1$, entonces $\|P_n A - A\| \longrightarrow 0$ y $\|AP_n - A\| \longrightarrow 0$ para todo $A \in \Omega_\infty$.

$A \in B(H)$ se dice triangular (respectivamente casitriangular) respecto de la sucesión $\{P_n\}$ (cfr. [4],[5]) si $P_n A P_n - P_n A = 0$ $\forall n$. (respect. $\|P_n A P_n - P_n\| \longrightarrow 0$).

Si $A \in \Omega_\Phi$ entonces, por 1.1.1, la última condición $\|P_n A P_n - P_n A\| \to 0$ equivale a $\|P_n A P_n - A P_n\| \longrightarrow 0$.

Para todo $A \in \Omega_\infty$ indicaremos con $\{s_j(A)\}$ a la sucesión de los números de Schmidt de A (cfr. [1]) de modo que

$$A\,x = \sum_{j=1}^{\infty} s_j(x,\, \varphi_j)\, \psi_j \;, \quad s_j = s_j(A) \tag{2}$$

donde $\{\varphi_j\}$ (resp: $\{\psi_j\}$) es una base (sistema) ortonormal.

$$\text{Si } \hat{c} = \{\xi = (\xi_j) \in c_0 \colon \xi_j = 0 \text{ desde un } j\}\;, \tag{3}$$

$\Phi(\xi) = \Phi(\xi_1, \xi_2, \ldots)$ es una función normante simétrica definida en $\hat{c}$ (cfr. [1], cap. III), y si

$$c_\Phi = \{\xi = (\xi_j) \in c \colon \|\xi\|_\Phi \equiv \sup_n \Phi(\xi_1 \ldots \xi_n, 0 \ldots 0) < \infty\} \tag{4}$$

indicaremos con $\Omega_\Phi = \{X \in \Omega_\infty \colon \{s_n(X)\} \in c_\Phi\}$ al ideal normado simétrico asociado a Φ, con

$$\|X\|_\Phi = \|\{s_n(X)\}\|_\Phi \tag{5}$$

Si $A \in \Omega_\infty$ es dado por (2) y si Q_n es el proyector ortogonal sobre el subespacio $\{\varphi_1, \ldots, \varphi_n\}$ designemos con

$$A_n = Q_n A = \sum_{j=1}^{n} s_j(x,\, \varphi_j)\, \psi_j \tag{2a}$$

Se tiene entonces [1] que

$$\min_{K \in \mathcal{K}_n} \|A - K\|_\Phi = \|A - A_n\|_\Phi = \Phi(s_{n+1}(A), s_{n+2}(A), \ldots) \tag{6}$$

$$\inf_{K\in\mathcal{K}} \|A - K\|_{\Phi} = \lim_{n\to\infty} \Phi(s_{n+1}(A), s_{n+2}(A), \ldots) \qquad (6a)$$

1.2. LA CONDICION TAME. $\mathcal{A} \subset B(H)$ designará un álgebra de Banach de operadores, cuya topología es más fina que la inducida por la de $B(H)$. $B(H_n)$ designará la subálgebra de $B(H)$ formada por los operadores A tales que $A(H_n)=H_n$ y $A(H_n^{\perp}) = 0$, de modo que $P_nAP_n \in B(H_n)$, $\forall\, A \in B(H)$.
Siguiendo a Palais [3] diremos que $\mathcal{A}$ cumple la condición "tame", o que $\mathcal{A}$ es un álgebra "approximately tame" respecto de la base $\{e_n\}$ fijada si:

1) $B(H_n) \subset \mathcal{A}$ $\forall n=1,2,\ldots$; 2) $P_nAP_n \to A$ en la norma de $\mathcal{A}$, $\forall\, A \in \mathcal{A}$

En particular si $\mathcal{A} = \Omega_{\Phi}$ entonces 2) equivale a

$$\text{2a)} \quad \|P_nAP_n - A\|_{\Phi} \longrightarrow 0 \qquad (7)$$

y 2) implica que $\mathcal{A} \subset \Omega_{\Phi}$.
La condición 2a) implica estas otras dos

3) $\|P_nAP_n - P_nA\|_{\Phi} \to 0$ y 3a) $\|P_nAP_n - AP_n\|_{\Phi} \to 0$ puesto que

$\|P_nAP_n - P_nA\|_{\Phi} = \|P_n(P_nAP_n) - P_nA\|_{\Phi} \leqslant \|P_n\|\ \|P_nAP_n - A\|_{\Phi}$

A su vez la 3) implica obviamente la

4) $\|P_nAP_n - P_nA\| \to 0$ o sea toda $A \in \mathcal{A}$ es casitriangular.

Diremos que un operador $A \in \Omega_{\Phi}$ es *casitriangular-Φ a derecha* (respecto de $\{P_n\}$) si verifica 3), y *casitriangular-Φ a izquierda* si verifica 3a).
La casi triangularidad-Φ es pues una condición más débil que la "tame" de Palais.

1.3. EL TEOREMA DE PALAIS.

Pongamos:

$$
\begin{aligned}
&GL(H) = \{T \in B(H) : T \text{ invertible en } B(H)\}\\
&GL(n,\{P_n\}) = \{T \in GL(H) : T = I + A,\ A \in B(H_n)\}\\
&GL(\infty,\{P_n\}) = \varinjlim_{n\to\infty} GL(n) = \text{límite inductivo de las } GL(n),\\
&G(\mathcal{A}) = \{T \in GL(H) : T = I + A,\ A \in \mathcal{A}\}\\
&\mathcal{O} = \{A \in \mathcal{A} : 1 + A \in GL(H)\}
\end{aligned}
\qquad (8)
$$

Entonces se tiene:

1.3.1. TEOREMA DE PALAIS. *Si $\mathcal{A}$ es un álgebra "approximately tame"* *de operadores en* H*, entonces la inyección* j: G L (∞) $\longrightarrow$ G (A) *es una equivalencia homotópica. O sea, existe una aplicación continua* q_1 ; G (A) $\longrightarrow$ G L (∞), q_1(G(A)) = G L (∞) $\subset$ G(A), *tal que* jq_1 *así como* $q_1 j$ *son homotópicas a la aplicación identidad.*

Para lo que sigue conviene recordar la idea de la demostración de este teorema. Primeramente se define P_t para todo $t > 0$, por la fórmula

$$P_t = P_n + (t - n) (P_{n+1} - P_n), \quad n \leqslant t \leqslant n+1 \tag{8a}$$

Luego como $\mathcal{O}$ es un abierto de $\mathcal{A}$, se define en $\mathcal{O}$ la función:

$$t'(A) = \inf\{t \geqslant 0,\ P_t\, A\, P_t + 1 = Q_t + 1 \text{ es inversible en } \mathcal{A}\} \tag{8b}$$

Se prueba que t'(A) es contínua superiormente, y que por tanto existe una función contínua t(A) en $\mathcal{O}$, tal que

$$t(A) \geqslant t'(A), \quad A \in \mathcal{O} \tag{8c}$$

Se define q_τ: G (A) $\longrightarrow$ G L (∞) por la fórmula

$$\begin{aligned} q_\tau(1+A) &= I + P_{\frac{1}{\tau}t(A)}\, A\, P_{\frac{1}{\tau}t(A)} = \\ &= P_{\frac{1}{\tau}t(A)} (1+A)\, P_{\frac{1}{\tau}t(A)} + (1 - P_{\frac{1}{\tau}t(A)}) \end{aligned} \tag{8d}$$

Se prueba entonces que

1.3.2. Si $\mathcal{A}$ es "approximately tame" entonces $\forall$ A $\in$ $\mathcal{A}$,
$q_\tau(1 + A) \longrightarrow 1 + A$ en $\mathcal{A}$, cuando $\tau \longrightarrow 0$,
$q_n(1 + A) \longrightarrow 1 + A$ en $\mathcal{A}$ si $n \longrightarrow \infty$.

De 1.3.2, resulta enseguida que $q_1 j$ es homotópica a la identidad de G L (∞), así como j q_1 homotópica a la identidad de G (A), lo que prueba el teorema 1.3.1.

1.4. LA CLASE Π $\{P_n\}$.

Sea $\{P_n\}$ una sucesión de proyectores de H que convergen fuertemente a 1, $P_n x \longrightarrow x$, $x \in H$. Diremos (cf [2]) que el operador $A \in B(H)$ pertenece a la clase $\Pi(P_n)$ si A es inversible en B(H) y si $\forall n > n_0$ es $P_n\, A\, P_n$ invertible en $B(H_n)$ (o sea considerando $P_n\, A\, P_n$ como operador de H_n en H_n), y además

$$(P_n\ A\ P_n)^{-1}\ P_n\ x \longrightarrow A^{-1}x, \quad \forall\ x$$

Son de especial interés los dos casos siguientes:

a) $\{P_n\}$ es la sucesión de proyectores ortogonales correspondientes a una base ortogonal $\{e_n\}$.

b) $\{P_n\}$ es la sucesión de proyectores correspondientes a una base $\{\varphi_j\}$no necesariamente ortonormal, pero equivalente a la base ortonormal $\{e_j\}$.

Vale la siguiente propiedad (cf. [2]) de frecuente aplicación:

1.4.1. $A \in \Pi\{P_n\}$ *si y sólo si* A *es invertible y existe un* $c > 0$ *tal que desde un* $n > n_0$ *se verifica*

$$\|(P_n\ A\ P_n)x\| \geqslant c\ \|P_n\ x\|\ ,\ x \in H \tag{9}$$

$$P_n\ A\ P_n\ (H_n) = H_n \tag{9a}$$

Mencionaremos aún los siguientes teoremas. (cf. [2])

1.4.2. *Si* $A = 1 + S + T$, $\|S\| < 1$ *y* T *un operador compacto entonces* $A \in \Pi\{P_n\}$ *, siendo* $\{P_n\}$ *cualquier sucesión de proyectores ortogonales.*

1.4.3. (Teorema de estabilidad I). *Si* $A \in \Pi\{P_n\}$*, entonces existe* $\gamma > 0$ *tal que para todo operador* B *de* H *en* H*, tal que* $\|B\| < \gamma$ *, entonces* $A + B \in \Pi\{P_n\}$*, o sea* $\Pi\{P_n\}$ *es un conjunto abierto.*

1.4.4. (Teorema de estabilidad II). *Si* $A \in \Pi\{P_n\}$ *y* T *es un operador compacto de* H *en* H *entonces* $A + T \in \Pi\{P_n\}$.

2. RELACION ENTRE IDEALES MONONORMANTES Y LA CONDICION DE PALAIS.

Siguiendo a [1] diremos que la función normante simétrica $\Phi(\xi)$ es mononormante, si para todo $\xi \in c_\Phi$ es

$$\lim_{n\to\infty} \Phi(\xi_{n+1}, \xi_{n+2}, \dots) = \lim_{n,p\to\infty} \Phi(\xi_{n+1}, \dots, \xi_{n+p}, 0, 0, \dots) = 0$$

De (6) y (6a) del §1, se deduce enseguida que para toda Φ normante simétrica son equivalentes las condiciones siguientes:

a) Φ es mononormante.

b) Para todo $A \in \Omega_\Phi$ existe una sucesión $\{B_n\} \subset K_n$ con $\|B_n - A\|_\Phi \longrightarrow 0$.

c) Para todo $A \in \Omega_\Phi$ es $\|A - A_n\|_\Phi \to 0$, donde $A_n = Q_n A$ es dado por (2a), §1.

2.1. PROPOSICION. *Para toda función normante Φ, son equivalentes las condiciones siguientes:*

a) *Φ es mononormante.*

b) *la condición* 2a), *osea* $\|P_n A P_n - A\|_\Phi \to 0$ *si* $n \to \infty$, *se verifica para todo* $A \in \Omega_\Phi$ *y para toda sucesión de proyectores ortonormales* $P_n \in K_n$ *asociados a una base ortonormal* $\{e_n\}$.

c) *la condición* 2a) *se verifica por lo menos para una sucesión* $\{P_n\}$ *ortogonal asociada a una base ortonormal* $\{e_n\}$.

d) *la condición* 2a) *se verifica para toda* $A \in \Omega_\Phi$ *y para toda sucesión de proyectores* $\{P_n\}$ *asociados a una base equivalente a una ortonormal.*

2.2. LEMA. *Sea* $\{P_n\}$ *una sucesión de proyectores ortogonales asociados a una base ortonormal* $\{e_n\}$, *Φ una función mononormante simétrica, entonces para todo* $A \in \Omega_\Phi$ *vale*

$$\|A(1 - P_n)\|_\Phi \to 0, \; y \; \|(1 - P_n)A\|_\Phi \to 0$$

Demostración. Consideraremos primero el caso A de rango 1, entonces existen dos vectores φ, ψ, tales que $\forall x \in H$

$$Ax = c(x, \psi)\varphi \;, \; \|\varphi\| = 1, \; c > 0$$

Además $\|A(1 - P_n)\| = \sup_{\|x\|=1} \|A(1 - P_n)x\| = \sup_{\|x\|=1} \|c((1 - P_n)x,\psi)\varphi\| =$

$$= \sup_{\|x\|=1} c\|(x,(1 - P_n)\psi)\varphi\| = c \sup_{\|x\|=1} \|(x,(1 - P_n)\psi)\varphi\| \leq c\|(1-P_n)\psi\|$$

O sea $\|A(1 - P_n)\| \leq c\|(1 - P_n)\psi\|$ y como $(P_n - 1)\psi \to 0$ resulta

$$\lim_{n\to\infty} \|A(1 - P_n)\| = 0 \tag{10}$$

Como $A(1 - P_n)$ es de rango 1, y $A \in \Omega_\Phi$ es $\|A(1 - P_n)\|_\Phi = \|A(1 - P_n)\|$ y de (10) resulta

$$\|A(1 - P_n)\|_\Phi \to 0 \tag{11}$$

Consideremos ahora A un operador de rango finito m, entonces $A = \sum_{0\leq k\leq m} \lambda_k B_k$, donde los B_k son operadores de rango 1.

Como $\|A(1 - P_n)\|_\Phi \leq \sum_{0\leq k\leq m} |\lambda_k| \|B_k(1 - P_n)\|_\Phi$, y por (11) cada término tiende a 0, resulta $\|A(1 - P_n)\|_\Phi \to 0$.

Finalmente sea $A \in \Omega_\Phi$, por §2.b existe un $B \in K_N$ tal que

$\|B - A\|_\Phi < \frac{\varepsilon}{3}$.

Además por (11) $\|B(1 - P_n)\|_\Phi < \frac{\varepsilon}{3}$ $\forall$ $n > n_0$ luego

$\|A(1 - P_n)\|_\Phi \leqslant \|A - B\|_\Phi + \|B(1 - P_n)\|_\Phi + \|(B - A)P_n\|_\Phi < \varepsilon$, $\forall$ $n > n_0$

o sea

$$\|A(1 - P_n)\|_\Phi \longrightarrow 0 \qquad \text{c.d.d.}$$

Demostración de la proposición 2.1. a) $\Rightarrow$ b)

$\|A - P_n A P_n\|_\Phi \leqslant \|A(1 - P_n)\|_\Phi + \|(1 - P_n)A P_n\|_\Phi \leqslant \|A(1 - P_n)\|_\Phi +$

$+ \|(1 - P_n)A\|_\Phi \|P_n\|$

y por el lema 2.2 cada término tiende a 0, luego $\|A - P_n A P_n\|_\Phi \to 0$.

b) $\Rightarrow$ c) trivial .

a) $\Rightarrow$ d) Sea $\{e_j\}$ una base ortonormal de H y $\{\varphi_j\}$ su base equivalente, y $\{P_n\}$ y $\{P'_n\}$ los proyectores asociados a $\{e_j\}$ y $\{\varphi_j\}$ respectivamente; $\varphi_j = A e_j$ y $P'_n = A P_n A^{-1}$ por 1.1 (1). Sea $B \in \Omega_\Phi$; por b) vale $\|P_n B P_n - B\|_\Phi \longrightarrow 0$. Entonces $\|P'_n B P'_n - B\|_\Phi =$
$= \|A P_n A^{-1} B A P_n A^{-1} - A A^{-1} B A A^{-1}\|_\Phi =$
$= \|A(P_n A^{-1} B A P_n - A^{-1} B A) A^{-1}\|_\Phi \leqslant \|A\| \|A^{-1}\| \|P_n A^{-1} B A P_n - A^{-1} B A\|_\Phi$
y la última expresión tiende a 0 pues $A^{-1} B A \in \Omega_\Phi$ es decir
$\|P'_n B P'_n - B\|_\Phi \longrightarrow 0$.

d) $\Rightarrow$ c) trivial .

c) $\Rightarrow$ a) Si se verifica 2a) para una sucesión de proyectores, como $Q_n = P_n A P_n \in K_n$, 2a) significa que existe $\{Q_n\} \in K$ con $\|A - Q_n\|_\Phi \longrightarrow 0$ y por §2.b) Φ es mononormante.

2.4. COROLARIO. *Si la propiedad* 2a) *se verifica para una sucesión* $\{P_n\}$ *de proyectores ortogonales asociados a una base* $\{e_n\}$ *ortonormal se cumple para cualquier otra sucesión* $\{P_n\}$ *asociados a otra base ortonormal.*

Observemos que vale el siguiente lema.

2.5. LEMA. *La función normante simétrica* Φ *es mononormante, si la condición* $\|A - A_n\|_\Phi \longrightarrow 0$ *se verifica para todo* $A \in \Omega_\Phi$, $A > 0$. (*donde* A_n *es dado por* 2a §1).

Sea $\xi = \{\xi_n\} \in C_\Phi$, y $\{\varphi_j\}$ una base ortonormal de H. Consideremos el operador $Ax = \sum_{i=1}^{\infty} |\xi_i| (x,\varphi_i)\varphi_i$; entonces $A \in \Omega_\Phi$ y $A > 0$.

$$A_n x = \sum_{i=1}^{n} |\xi_i| (x,\varphi_i)\varphi_i \text{ y } \|A-A_n\|_\Phi = \sup_k \Phi(\xi_{n+1},\xi_{n+2},\ldots,\xi_{n+k},0,\ldots,0).$$

Ya que por hipótesis $\|A - A_n\|_\Phi \longrightarrow 0$, resulta

$$\lim_{n,k\to\infty} \Phi(\xi_{n+1},\xi_{n+2},\ldots,\xi_{n+k},0,\ldots,0) = 0$$

es decir Φ es mononormante.

Como Φ_Π no es mononormante resulta:

2.6. COROLARIO. *Existe un* $A \in \Phi_\Pi$, $A > 0$ *tal que* $\|A - A_n\|_\Phi$ *no converge para* $n \longrightarrow \infty$.

Como Φ_Π y los Φ_p son mononormantes obtenemos el siguiente corolario que contiene el teorema E de Palais (cf. [3]):

2.7. COROLARIO. *Si* Φ *es mononormante, entonces* $A = \Omega_\Phi$ *es un álgebra 'approximately tame' en sentido de Palais.*

En particular, las álgebras Ω_p y Ω_Π son 'approximately tame' respecto de toda base ortonormal así como de toda base equivalente a ortonormal.

3. EL TEOREMA DE PALAIS Y LA CLASE $\Pi\{P_n\}$.

Sea Φ mononormante y $A = \Omega_\Phi$. Por 2.7 sabemos que A es 'approximately tame' respecto de $\{P_n\}$ generales asociados a bases equivalentes a ortonormales, luego a $\Omega_\Phi = A$ se aplica el teorema de Palais 1.3.1 y 1.3.2.

Ahora relacionaremos este teorema (en caso de $A = \Omega_\Phi$) con el método de proyección $\Pi\{P_n\}$ (ver 1.4). Observemos antes que si $G(A)$ es la clase definida en (8) del 1.3, se tiene

3.1. LEMA. *Sea* Φ *normante simétrica* $A = \Omega_\Phi$. *Entonces para todo* $1 + A \in G(A)$ *es* $(1+A)^{-1} \in G(A)$. *Luego poniendo* $V(1+A) = (1+A)^{-1}$, *tenemos que* $V: G(A) \to G(A)$ *es un operador contínuo en* $G(A)$.

Demostración. Si $C = (1+A)^{-1}$, $(1+A)C = 1$, entonces $C = 1-AC$, y de $A \in A$ es $AC \in A$ y $1-AC = C$ es invertible, luego $(1+A)^{-1} = C \in G(A)$.

Además: $A_n \longrightarrow A$ en A implica $(1+A_n)^{-1} \longrightarrow (1+A)^{-1}$ en A, pues $1 + A_n = 1 + A + \varepsilon_n$ con $\|\varepsilon_n\|_\Phi \longrightarrow 0$,

$1 + A_n = (1+A)[1 + (1+A)^{-1}\, \varepsilon_n] = (1+A)(1+\eta_n)$ donde $\|\eta_n\|_\Phi \longrightarrow 0$.

Luego para $n > n_0$ $1 + \eta_n$ es invertible y $(1+\eta_n)^{-1} \longrightarrow 1$,

entonces $(1+A_n)^{-1} \longrightarrow (1+A)^{-1}$. c.d.d.

En todo lo que sigue Φ es mononormante y $A = \Omega_\Phi$.

Por lo visto en 1.3.1, si q_τ: $G(A) \longrightarrow G(A)$ es definido por $q_\tau\ (1+A) = 1 + P_{\frac{1}{\tau}t(A)}\ A\ P_{\frac{1}{\tau}t(A)}$ entonces $q_\tau\ (1+A) \longrightarrow 1+A$ si $\tau \to 0$ en A, en particular $q_n\ (1+A) \to 1+A$ si $n \longrightarrow \infty$ de modo que q_τ da una homotopía de q_1 en 1.

Como $\mathcal{Y}$ es continuo, obtenemos que: $\mathcal{Y}q_\tau \longrightarrow \mathcal{Y} \therefore \mathcal{Y}q_1 \cong \mathcal{Y}$ y $q_\tau\mathcal{Y} \longrightarrow \mathcal{Y}$ $\therefore q_1\mathcal{Y} \cong \mathcal{Y}$, luego

$$(\mathcal{Y}q_\tau - q_\tau\mathcal{Y})\ (1+A) \longrightarrow 0 \quad \text{en} \quad A \tag{12}$$

$$\mathcal{Y}q_1 \cong q_1\mathcal{Y} \tag{12a}$$

Ahora como $A \in \Omega_\Phi$, por 1.4.2 $1+A \in \Pi\{P_n\}$ de modo que desde un $\tau > \tau_0$, es $P_{\frac{1}{\tau}t(A)}\ (1+A)\ P_{\frac{1}{\tau}t(A)} : H_n \longrightarrow H_n$ invertible en $B(H_n)$; y existe $[P_{\frac{1}{\tau}t(A)}\ (1+A)\ P_{\frac{1}{\tau}t(A)}]^{-1}: H_n \longrightarrow H_n$, donde $t(A)$ fue dado en (8c) de 1.3.1.

3.2. PROPOSICION. *Se puede elegir la función continua* $t(A)$ *de modo que desde un* $\tau > \tau_0$ *se verifique*

$$\mathcal{Y}q_\tau(1+A) = P_{\frac{1}{\tau}t(A)}\,[P_{\frac{1}{\tau}t(A)}\ (1+A)\ P_{\frac{1}{\tau}t(A)}]^{-1}\ P_{\frac{1}{\tau}t(A)} + (1-P_{\frac{1}{\tau}t(A)}) \tag{13}$$

$$q_\tau\mathcal{Y}\ (1+A) = P_{\frac{1}{\tau}t(A)}\ (1+A)^{-1}\ P_{\frac{1}{\tau}t(A)} + (1 - P_{\frac{1}{\tau}t(A)}) \tag{14}$$

de modo que por (12a) es:

$$P_{\frac{1}{\tau}t(A)}\,[P_{\frac{1}{\tau}t(A)}\,(1+A)P_{\frac{1}{\tau}t(A)}]^{-1}P_{\frac{1}{\tau}t(A)} - P_{\frac{1}{\tau}t(A)}\,(1+A)^{-1}P_{\frac{1}{\tau}t(A)} \to 0 \text{ en } A \tag{15}$$

Demostración. Observemos que, para $\tau > \tau_0$ es

$$\mathcal{Y}q_\tau\ (1+A) = [P_{\frac{1}{\tau}t(A)}\ (1+A)\ P_{\frac{1}{\tau}t(A)} + (1 - P_{\frac{1}{\tau}t(A)})]^{-1} =$$

$$= P_{\frac{1}{\tau}t(A)} [P_{\frac{1}{\tau}t(A)} (1+A) P_{\frac{1}{\tau}t(A)}]^{-1} P_{\frac{1}{\tau}t(A)} + (1 - P_{\frac{1}{\tau}t(A)})$$

Para verificar esta igualdad, basta multiplicar a derecha y a izquierda por $P_{\frac{1}{\tau}t(A)} (1+A) P_{\frac{1}{\tau}t(A)} + (1-P_{\frac{1}{\tau}t(A)})$ y comprobar que da 1.

Esto prueba (13).

Como $\mathcal{V}(1 + A) = (1 + A)^{-1} \in G(\mathcal{A})$, $(1 + A)^{-1} = 1 + D$, $D \in \mathcal{A}$;

$q_\tau \mathcal{V}(1+A) = q_\tau (1+A)^{-1} = P \frac{1}{\tau} t(D) (1+A)^{-1} P \frac{1}{\tau} t(D) + 1 - P \frac{1}{\tau} t(D)$.

Eligiendo $t(D) = t(A)$ una función contínua mayor que $t'(A)$ y $t'(D)$ se puede reemplazar $t(D)$ por $t(A)$ y obtenemos (14). c.d.d.

En particular (15) da

$$P_n(P_n(1 + A) P_n)^{-1} P_n - P_n (1 + A)^{-1} P_n \longrightarrow 0 \text{ en } \mathcal{A} \qquad (15a)$$

Interpretaremos (15a) en términos de la teoría de la proyección. Si $\mathcal{A}$ es un álgebra que contiene los operadores de rango finito y $\mathcal{A} \subset \Omega_\infty$, entonces $1 + A \in G(\mathcal{A})$ implica $1 + A \in \Pi\{P_n\}$ (ver 1.4.2) ($\{P_n\}$ cualquier sucesión de proyectores correspondiente a una base equivalente a una ortonormal). O sea, mientras que el teorema 1.4.2. dice tan sólo que (15a) vale en la topología fuerte, en cambio si $\mathcal{A} = \Omega_\Phi$, Φ mononormante, 3.2 nos dice que (15a) vale en la topología de la norma $\| \ \|_\Phi$, o sea el teorema (I) de Palais expresa un hecho formalmente más fuerte que $1 + A \in \Pi\{P_n\}$.

Esto sugiere estudiar una nueva clase de operadores que llamaremos $\Pi_\Phi\{P_n\}$ que serán los que verifican (15a) en la topología de $\mathcal{A}$, que pasamos a estudiar en las secciones siguientes.

4. LA CLASE Π_Φ.

4.1. DEFINICION. *Sean Φ una función normante simétrica, Ω_Φ el ideal normado correspondiente, y $\{P_n\}$ una sucesión de proyectores ortogonales que convergen fuertemente a la unidad en* $H(P_n x \longrightarrow x)$. *Sea* $H_n = P_n(H)$. *Diremos que un operador* $A \in \Pi_\Phi\{P_n\}$ *si:* i) A *es invertible,* ii) *los operadores* $P_nAP_n : H_n \longrightarrow H_n$ *son invertibles en* $B(H_n)$, *o sea existe* $(P_nAP_n)^{-1} : H_n \longrightarrow H_n$ *y* iii) *vale* $\|(P_nAP_n)^{-1} P_n - P_nA^{-1}P_n\|_\Phi \longrightarrow 0$.

Observemos que iii) implica que $(P_nAP_n)^{-1} P_n - P_nA^{-1}P_n \in \Omega_\Phi$, mientras que $A \notin \Omega_\Phi$.

4.2. PROPOSICION. *Una condición necesaria y suficiente para* $A \in \Pi_\Phi\{P_n\}$, *es que se verifiquen las condiciones siguientes:*

(a) $(P_nAP_n)^{-1}$ *existe como operador de* H_n *en* H_n, *y* $\forall\ n > n_0$

$\|(P_nAP_n)^{-1} P_n\| \leq C,\ C > 0$

(b) $\|P_nA(1-P_n)A^{-1}P_n\|_\Phi \longrightarrow 0$

Demostración. Supongamos $A \in \Pi_\Phi\{P_n\}$, entonces se verifican i) ii) y iii) de la definición, es decir $\forall n > n_0$

$\|(P_nAP_n)^{-1}P_n-P_nA^{-1}P_n\|_\Phi < \varepsilon$, o sea

$\|(P_nAP_n)^{-1}P_n-P_nA^{-1}P_n\| < \varepsilon$ y $\|(P_nAP_n)^{-1}P_n\| < \|P_nA^{-1}P_n\| + \varepsilon \leq$

$\leq \|A^{-1}\| + \varepsilon = C$, lo que prueba (a).

Por otra parte:

$\|P_nA(1-P_n)A^{-1}P_n\|_\Phi = \|P_n-P_nAP_nA^{-1}P_n\|_\Phi = \|P_nAP_n[(P_nAP_n)^{-1}P_n-P_nA^{-1}P_n]\|_\Phi \leq$

$\leq \|P_nAP_n\|\ \|(P_nAP_n)^{-1}P_n - P_nA^{-1}P_n\|_\Phi \leq \|A\|\ \|(P_nAP_n)^{-1}P_n-P_nA^{-1}P_n\|_\Phi$ y el último miembro tiende a 0 pues $A \in \Pi_\Phi\{P_n\}$, lo que prueba (b).

Supongamos ahora que se verifican (a) y (b).
Las propiedades i) y ii) se verifican obviamente. Para verificar iii), observemos primero la siguiente identidad:

$(P_nAP_n)(P_nA^{-1}P_n) = P_n - P_nA(1-P_n)\ A^{-1}P_n$. Luego

$P_nA^{-1}P_n = (P_nAP_n)^{-1}\ [P_n-P_n\ A(1-P_n)A^{-1}\ P_n] =$

$= (P_nAP_n)^{-1}P_n - (P_nAP_n)^{-1}\ P_nA(1-P_n)\ A^{-1}P_n$, de donde

$(P_nAP_n)^{-1}P_n - P_nA^{-1}P_n = (P_nAP_n)^{-1}\ P_nA(1-P_n)A^{-1}\ P_n$ y

$\|(P_nAP_n)^{-1}\ P_n - P_nA^{-1}P_n\|_\Phi \leq \|(P_nAP_n)^{-1}P_n\|\ \|P_nA(1-P_n)A^{-1}P_n\|_\Phi$

Por las condiciones (a) y (b) el último miembro tiende a 0 lo que prueba iii), c.d.d.

OBSERVACION. La condición (a) de 4.2 es cierta para un operador A si y sólo si $A \in \Pi\{P_n\}$ ya que (a) es equivalente a las condiciones (9) y (9a) de 1.4.1.
Luego 4.2. puede enunciarse en la siguiente forma

4.2.(bis) PROPOSICION. $A \in \Pi_\Phi\{P_n\}$ *si y sólo si:*

a) $A \in \Pi\{P_n\}$ *y*

b) *si* $\|P_nA(1-P_n)A^{-1}P_n\|_\Phi \longrightarrow 0$

4.3. COROLARIO. *Si* $A \in \Pi\{P_n\}$ *y si* A *es casi triangular* Φ *a derecha o* A^{-1} *es casi triangular* Φ *a izquierda respecto de* $\{P_n\}$ *(ver 1.2) entonces* $A \in \Pi_\Phi(P_n)$.

Demostración. Supongamos por ejemplo que $A \in \Pi\ \{P_n\}$ y que A es casi triangular Φ a derecha respecto a $\{P_n\}$. Para probar que $A \in \Pi_\Phi\{P_n\}$ basta verificar b) del teorema 4.2 bis. Pero $\|P_nA(1-P_n)A^{-1}P_n\|_\Phi \leqslant \|P_nA-P_nAP_n\|_\Phi \quad \|A^{-1}\|$ y por la casi triangularidad Φ a derecha, la última expresión tiende a 0. c.d.d.

4.4. COROLARIO. *Si* $A = 1+S+T$, *donde* S *es casi triangular* Φ *a derecha respecto a* $\{P_n\}$, $\|S\| < 1$, *y* T *un operador de rango finito, entonces* $A \in \Pi_\Phi\{P_n\}$.

En efecto: Por 1.4.2, $A \in \Pi\{P_n\}$ y $A = 1+S+T$ es casi triangular Φ a derecha respecto a $\{P_n\}$, luego por 4.3 $A \in \Pi_\Phi\{P_n\}$.

4.5. DEFINICION. *Diremos que* Φ *es casi mononormante si* Φ *es normante simétrica y para cualquier* $T \in \Omega_\Phi$, T *es casi triangular* Φ *a derecha respecto a cualquier sucesión de proyectores ortonormales que convergen fuertemente a* 1. $(P_nx \to x)$.

Observemos que si Φ es mononormante, entonces Φ es casi mononormante, pues por 2.7 Ω_Φ es 'approximately tame' respecto a cualquier sucesión de proyectores ortogonales $\{P_n\}$ con $P_nx \to x$, y por 1.1.1, si $A \in \Omega_\Phi$ A es casi triangular a derecha respecto a $\{P_n\}$.

4.6. COROLARIO. *Si* Φ *es casi mononormante, si* A *es casi triangular* Φ *a derecha respecto a* $\{P_n\}$ *y* $A \in \Pi\{P_n\}$, *y si* $T \in \Omega_\Phi$ *entonces* $A+T \in \Pi_\Phi$.

En efecto por 1.4.4 $A+T \in \Pi\{P_n\}$ y A+T es casi triangular Φ a derecha respecto a $\{P_n\}$, luego por 4.3, $A+T \in \Pi_\Phi$.

4.7. COROLARIO. *Si* Φ *es mononormante las clases* $\Pi\{P_n\}$ *y* $\Pi_\Phi\{P_n\}$ *coinciden.*

Basta aplicar el corolario 4.3. Luego en caso de un Ω_Φ mononormante el teorema de Palais expresa en forma geométrica el mismo hecho que el teorema 1.4.2.

Un teorema de V. Neumann (cf.[6]) dice que todo operador simétrico A puede escribirse como una suma de un operador de Ω_2 de norma arbitrariamente pequeña y un operador de la forma $\sum_j \lambda_j (x,\varphi_j)\varphi_j$ donde $\{\varphi_j\}$ es un sistema ortonormal, pero $\{\lambda_j\}$ no tiende necesariamente a cero, de modo que este operador no es necesariamente compacto. Luego para todo A y para todo n finito existe un sistema ortonormal $\{\varphi_1^{(n)}, \varphi_2^{(n)}, \ldots, \varphi_n^{(n)}\}$ y una n-upla numérica $\{\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{n,n}\}$ tales que:

1) Para todo $x \in H$, vale $Ax = \sum_{j=1}^{n} \lambda_{jn}(x,\varphi_j^{(n)})\varphi_j^{(n)} + R_n(x)$ (16)

   con $\sum (\lambda_{jn})^2 \geqslant C > 0$ (16a)

2) El proyector $P_n x = \sum_{j=1}^{n} (x,\varphi_j^{(n)})\, \varphi_j^{(n)}$ (17)

   verifica $P_n x \to x$, $\forall\, x$ (17a)

3) $\|P_n R_n\| \longrightarrow 0$ y $\|R_n P_n\| \longrightarrow 0$ para $n \longrightarrow \infty$

Podemos entonces dar el siguiente criterio general para la clase $\Pi_\Phi\{P_n\}$.

4.8. TEOREMA. *Sea* Φ *normante simétrica,* A *un operador invertible tal que para todo* n *existen* $\{\varphi_j^{(n)}\}$ *y* $\{\lambda_{jn}\}$ *que verifican las condiciones* 1), 2), 3) *y la* 3a), *más fuerte que* 3), *siguiente:*

3a) $\|R_n P_n\|_\Phi \longrightarrow 0$ *y* $\|P_n R_n\|_\Phi \longrightarrow 0$ (17b)

entonces $A \in \Pi_\Phi\{P_n\}$.

Demostración. Basta verificar las condiciones a) y b) de 4.2. Veamos primero que $(P_n A P_n)^{-1}$ existe como operador de H_n en H_n. Observemos que de (16) y (17b) es claro que para n grande es $P_n A = AP_n + S_n$, donde $\|S_n\| \leqslant \|S_n\|_\Phi < \varepsilon_n$ y $\varepsilon_n \longrightarrow 0$. (18)

Luego: $$P_n A^{-1} P_n \cdot P_n A P_n = P_n A^{-1} A P_n + P_n A^{-1} S_n P_n = $$
$$= P_n + S_n' \quad , \quad \|S_n'\| < \varepsilon_n^1 \longrightarrow 0.$$

Como P_n es igual a 1 en H_n, es $P_n + S_n'$ invertible en H_n, y $[(P_n+S_n')^{-1} P_n A^{-1} P_n]\, P_n A P_n = P_n = 1$ en H_n, luego $(P_n+S_n')^{-1} P_n A^{-1} P_n$ es el inverso a izquierda de $P_n A P_n$. Análogamente se verá que $P_n A P_n$ tiene inversa a derecha, luego $(P_n+S_n')^{-1} P_n A^{-1} P_n = (P_n A\, P_n)^{-1}$.

Además para $n \longrightarrow \infty$ es $\|(P_n+S_n')^{-1}\| \longrightarrow \|P_n\| = 1$ y como $P_n A^{-1} P_n x = \sum_{j=1}^{n} \frac{1}{\lambda_j}(x,\varphi_j)\varphi_j$ es (en H_n) $\|P_n A^{-1} P_n\| \leqslant \sup \frac{1}{\lambda_j} \leqslant C$,

resulta que: $\|(P_nAP_n)^{-1}\| = \|(P_n + S'_n)^{-1} P_nA^{-1}P_n\| \leq$
$\leq \|(P_n + S'_n)^{-1}\| \ \|P_nA^{-1}P_n\| \leq (1 + \varepsilon)\ C = C_1$. Esto prueba a).

Para probar b), observemos que por (18) es
$\|P_nA(1-P_n)A^{-1}P_n\|_\Phi = \|(AP_n+S_n)(1-P_n)A^{-1}P_n\|_\Phi = \|S_n(1-P_n)A^{-1}P_n\|_\Phi \to 0.$

4.9. TEOREMA DE ESTABILIDAD I. *Si* $A \in \Pi_\Phi\{P_n\}$, *A casi triangular a derecha y* A^{-1} *casi triangular a izquierda respecto a* $\{P_n\}$, *sucesión de proyectores ortogonales,* $P_nx \to x$, *entonces existe un* $\delta > 0$ *tal que* $A+T \in \Pi_\Phi\{P_n\}$, *para todo* $T \in \Omega_\Phi$ *con* $\|T\| < \delta$.

Demostración. Por a) de 4.2 bis es $A \in \Pi\{P_n\}$ y por 1.4.3, si $\|T\| < \delta_1$, $A + T \in \Pi\{P_n\}$ luego por 4.2 bis basta probar que $A + T$ verifica la condición b). Observamos primero que:
$(A + T)^{-1} = [A(1 + A^{-1}\ T)]^{-1} = [1 + A^{-1}\ T + (A^{-1}\ T)(A^{-1}\ T)\ldots]A^{-1}$
si hemos elegido $\|T\| < \delta_1$ ($\delta < \delta_1$) de modo que $\|A^{-1}T\| < 1$. Si llamamos $B_1 = A^{-1}T + A^{-1}T\ A^{-1}T + \ldots$, entonces $B_1 \in \Omega_\Phi$ y vale
$(A + T)^{-1} = (1 + B_1)\ A^{-1} = A^{-1} + B_1\ A^{-1}$, luego
$\|P_n(A + T)(1 - P_n)(A + T)^{-1}\ P_n\|_\Phi = \|P_nA(1-P_n)\ A^{-1}P_n + P_nA(1-P_n)$
$B_1A^{-1}P_n + P_nT(1-P_n)A^{-1}\ P_n + P_nT(1-P_n)B_1A^{-1}\|_\Phi \leq$
$\leq \|P_nA(1-P_n)A^{-1}P_n\|_\Phi + \|P_nA(1-P_n)\ B_1A^{-1}P_n\|_\Phi + \|P_nT(1-P_n)\ A^{-1}\ P_n\|_\Phi +$
$+ \|P_nT(1-P_n)\ B_1A^{-1}P_n\|_\Phi$. Veamos que cada término de esta última expresión tiende a 0.
$\|P_nA(1-P_n)A^{-1}P_n\|_\Phi \longrightarrow 0$ pues $A \in \Pi_\Phi\{P_n\}$
$\|P_nA(1-P_n)B_1A^{-1}P_n\|_\Phi \leq \|P_nA-P_nAP_n\|\ \|B_1A^{-1}P_n\|_\Phi \longrightarrow 0$
porque A es triangular a derecha y $B_1 \in \Omega_\Phi$;
$\|P_nT(1-P_n)A^{-1}P_n\|_\Phi \leq \|P_nT\|_\Phi\ \|(1-P_n)A^{-1}P_n\| \longrightarrow 0$,
ya que A^{-1} es triangular a izquierda, y $T \in \Omega_\Phi$;
$\|P_nT(1-P_n)B_1A^{-1}\|_\Phi \leq \|T(1-P_n)\|\ \ \|B_1A^{-1}\|_\Phi$ y $\|T(1-P_n)\| \longrightarrow 0$ por 1.1.1.
Luego $\|P_n(A+T)(1-P_n)(1+T)^{-1}\ P_n\|_\Phi \longrightarrow 0$, c.d.d.

4.10. TEOREMA DE ESTABILIDAD II. *Si* $A \in \Pi_\Phi$, *con* $A \in \Pi\{P_n\}$ *y A casi triangular a derecha respecto de* $\{P_n\}$, *sucesión de proyectores ortogonales,* $P_nx \to x$, *entonces* $A + T \in \Pi_\Phi$ *para todo* T *de rango finito.*

Demostración. Observemos primero que por ser T de rango finito, podemos suponer $T = Q\ T$, donde Q es el proyector sobre el rango de T. Q y $T \in \Omega_\Phi$.

Por 1.4.4 $A + T \in \Pi\{P_n\}$. Para ver que $A + T \in \Pi_\Phi\{P_n\}$ basta probar b) de 4.2 bis.

$$P_n(A+T)(1-P_n)(A+T)^{-1}P_n = P_nA(1-P_n)(A+T)^{-1}P_n + P_nT(1-P_n)(A+T)^{-1}P_n =$$
$$= P_nA(1-P_n)(1+A^{-1}T)^{-1}A^{-1}P_n + P_nT(1-P_n)(A+T)^{-1}P_n$$

Si llamamos C a $(1+A^{-1}T)^{-1}$, $C(1+A^{-1}T) = 1$, $C+CA^{-1}T = 1$ y $C = 1-CA^{-1}T$; entonces:

$$\|P_n(A+T)(1-P_n)(A+T)^{-1}P_n\|_\Phi = \|P_nA(1-P_n)(1-CA^{-1}T)A^{-1}P_n\|_\Phi +$$
$$+ \|P_nQT(1-P_n)(A+T)^{-1}P_n\|_\Phi \leq \|P_nA(1-P_n)A^{-1}P_n\|_\Phi +$$
$$+ \|P_nA(1-P_n)CA^{-1}TA^{-1}P_n\|_\Phi + \|P_nQT(1-P_n)(A+T)^{-1}P_n\|_\Phi$$
y cada uno de los últimos términos tiende a 0. En efecto:

$\|P_nA(1-P_n)A^{-1}P_n\|_\Phi \to 0$ porque $A \in \Pi_\Phi\{P_n\}$

$\|P_nA(1-P_n)CA^{-1}TA^{-1} P_n\|_\Phi \leq \|P_nA(1-P_n)\| \; \|A^{-1}\|^2 \; \|C\| \; \|T\|_\Phi \to 0$

por ser A triangular y

$\|P_nQT(1-P_n)(A+T)^{-1}P_n\|_\Phi \leq \|Q\|_\Phi \; \|T(1-P_n)\| \; \|(A+T)^{-1}\| \to 0$

porque $\|T(1-P_n)\| \to 0$ por 1.1.1, c.d.d.

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UN EJEMPLO DE GEOMETRIAS METRICAS EUCLIDIANAS EN CUALQUIER DIMENSION EN LAS QUE NO RIGE EL AXIOMA DE PARALELISMO DE EUCLIDES

Heinz-Reiner Friedlein

Por los axiomas en [2] ó [4] es posible construir geometrías en cualquier dimensión. Adjuntamos a los axiomas de [2] el siguiente:

EUC. Para g ⊥ h existe un punto S I g, h y dos rectas que pasan por S con s ⊥ t tales que g ⊥ s ⊥ t ⊥ h. Se dice que las rectas s, h ó g, t están en una escalera.

Decimos que la geometría así descrita es una geometría métrica euclidiana. Es conocido que ciertas geometrías afines son geometrías métricas euclidianas.

Ahora queremos construir un ejemplo de una geometría métrica euclidiana que no es geometría afín. El ejemplo que vamos a construir es una generalización de un ejemplo para el plano métrico euclidiano que figura en [1].

Designaremos los puntos y las rectas de las geometrías por letras mayúsculas y minúsculas, respectivamente. Un haz propio con soporte P es el conjunto de todas las rectas que pasan por P.

El teorema principal en [2] nos muestra que es posible extender cada geometría métrica euclidiana a una geometría proyectiva o bien que la geometría métrica euclidiana es sumergible en una geometría proyectiva, que se dice geometría proyectiva euclidiana.

TEOREMA 1. a) *En un espacio métrico euclidiano todos los haces propios tienen la misma cardinalidad.*

b) *Además, cada haz de la geometría proyectiva euclidiana tiene la misma cardinalidad que un haz propio en la geometría métrica euclidiana.*

Un haz de la geometría proyectiva euclidiana es definido en forma análoga a un haz propio de una geometría euclidiana.

Demostración. a) Sean dos haces diferentes que tienen como soportes los puntos diferentes O y P respectivamente. Sea g I O; según

[2] existe exactamente un h I P y perpendicular a g. Esto implica un mapeo biyectivo del conjunto de las rectas del haz determinado por O sobre el conjunto de las rectas que pasan por P.

b) Sea G un haz que contiene rectas proyectivas, que no son rectas métricas. Elegimos un punto métrico O como centro de un "snail map" (comparar [3]). Existen en G dos rectas métricas euclidianas diferentes a, b, tal que a, b no tienen ni un punto métrico ni una recta métrica euclidiana en común que sea ortogonal a las dos; entonces existe un "snail map" con centro O tal que la imagen G' de G bajo este mapeo es un haz propio. Las preimágenes de las rectas de G' son - por construcción (comparar [3]) - rectas proyectivas y están en correspondencia 1-1 con el snail map.

Queda por demostrar que los haces G con la propiedad que todas las rectas de G son perpendiculares a un hiperplano de la geometría métrica euclidiana que pasa por O tienen la misma cardinalidad. Pero como estamos en una geometría proyectiva, se sabe que los haces de esta geometría tienen la misma cardinalidad.

DEFINICION. *Una geometría afín que es una geometría métrica euclidiana se llama geometría euclidiana.*

Desde luego,no toda geometría afín es una geometría euclidiana. Por ejemplo, las geometrías afines con coordenadas en un cuerpo no conmutativo no son geometrías euclidianas (comparar [1]).

TEOREMA 2. *Cada geometría métrica euclidiana es sumergible en una geometría euclidiana.*

Demostración. Según [2] ó [3] cada geometría métrica euclidiana es sumergible en una geometría métrica proyectiva E_p. Elegimos en E_p subconjuntos Π_o, Γ_o de puntos y rectas respectivamente.

$\Pi_o = \{P \in E_p \mid \nexists\ g \neq h, g, h \ I\ P$ tal que $g, h \perp H$, H hiperplano y g, h rectas en la geometría métrica euclidiana$\}$

$\Gamma_o := \{r \in E_p \mid Q, R \in \Pi_o, Q \neq R$ y $Q, R \ I\ r\}$

Entonces Π_o y Γ_o determinan una geometría afín y todos los puntos métricos euclidianos se encuentran en Π_o y todas las rectas métricas euclidianas pertenecen a Γ_o. Según [3] cada recta de Γ_o es eje de una reflexión. Esto implica que Π_o, Γ_o determinan una geometría euclidiana [3].

Con este resultado podemos construir nuestro ejemplo.

Consideremos el anillo $\overline{C}_o = \langle \{1/p \mid p = 2, 4\ell + 1, \ell \in N\} \rangle$

Sea $\Pi := \{(\alpha_1,\ldots,\alpha_n,\ldots) \mid \alpha_1,\ldots,\alpha_n,\ldots \in \bar{C}_o\}$ un conjunto de puntos cuyas coordenadas pertenecen a $\bar{C}_o$. Es claro que Π es un subconjunto de los puntos de una cierta geometría euclidiana A (R) con coordenadas en R. Además A (R) es una geometría afín y el espacio vectorial con producto interno correspondiente es un espacio de Hilbert.

NOTA. Es posible demostrar que la geometría métrica euclidiana determina este producto interior (comparar [2]).

Como conjunto Γ de rectas definimos: Si P es un punto de Π entonces las rectas de Γ son las rectas g de A (R) con la siguiente propiedad: g es la unión de P con un punto $Q \in \Pi$ y $Q \neq P$.

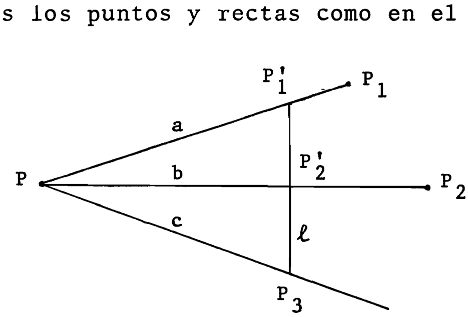
Ahora queremos demostrar que (Π, Γ, I) es una geometría métrica euclidiana pero no una geometría euclidiana.

TEOREMA 3. (Π, Γ, I) *determina una geometría métrica euclidiana en la que no vale el axioma de paralelismo de Euclides.*

Demostración. Usamos los axiomas de [2]. Estos axiomas exigen también para la geometría métrica euclidiana una relación $\perp$ y reflexiones en puntos y rectas. Pero A (R) tiene estas propiedades. Axioma 1 de [2] se cumple pues Π, Γ son subconjuntos de los puntos y rectas en A (R).

Axioma 2. Sean $\bar{g}_o, \bar{h}_o, \bar{j}_o, \bar{a}_o, \bar{g}_1, \bar{j}_1$ reflexiones en A (R) con ejes $g_o, h_o, j_o, a_o, g_1, j_1$ respectivamente y g_o, h_o, j_o I A; h_o, g_1, j_1 I B; $A \neq B$. Entonces tenemos $\bar{g}_o \cdot \bar{h}_o \cdot \bar{j}_o = \bar{a}_o$ con a_o I A y $\bar{j}_1 \cdot \bar{h}_o \cdot \bar{y}_o = \bar{a}_1$ con a_1 I B.

Axioma 3. Dados los puntos y rectas como en el dibujo



$\bar{a} \cdot \bar{b} \cdot \bar{c} = \bar{d}$ con a, b I P implica según [2] que a, b, c se encuentran en un plano métrico euclidiano. Sea ℓI P_3 la recta perpendicular a b por P'_2 tal que corta la recta a en P'_1. Falta demostrar que P'_1, $P'_2 \in \Pi$. Pero esto sale de [1],pág. 288/299,fórmula (1) y (3).

Axioma 4. Sean $g = (P, P')$, $h = (P,Q')$ diferentes y $(P,P') \not\perp (P,Q')$. Entonces (P,P'), (P,Q') determinan un plano euclidiano A_2 (R) de A (R). Se sabe que en A_2 (R) la reflexión $\overline{P}$ en el punto P es el producto $\overline{g}.\overline{r}$ con $r \perp g$, $r \in A_2$ (R). Análogamente como en la demostración para axioma 3, existe un punto A I r y $A \in \Pi$ tal que $(A, Q') \not\perp r$. El teorema de tres reflexiones [1] implica que $\overline{(P,P')}.\overline{(P,Q')}.\overline{r}$ es una reflexión en una cierta recta d I P.

El axioma 5 sigue inmediatamente de la demostración de los axiomas 3 y 4.

Axiomas 6 y 7 se satisfacen, pues estamos en A (R).

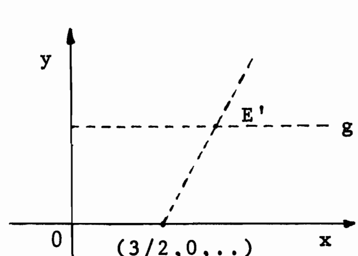
Axioma 8 es trivial pues en A (R) no existen puntos P,Q distintos con $\overline{P}.\overline{Q} = \overline{Q}.\overline{P}$.

Axioma 9. La existencia de tres puntos que no se encuentran en una sola recta es trivial (ver la demostración del axioma 4) y desde luego dos puntos diferentes se encuentran sobre cada recta.

Además la geometría construída cumple trivialmente Euc. Por tanto Π,Γ son los conjuntos de puntos y rectas respectivamente de una geometría métrica euclidiana.

Ahora vamos a demostrar que esta geometría no satisface el teorema de las paralelas de Euclides.

Como cada geometría afín satisface este teorema, estamos listos con la demostración si hemos encontrado dos rectas distintas g,h con g, h I E, $x \cap g = \emptyset$, $x \cap h = \emptyset$; $g, h \in \Gamma$, $P \in \Pi$. Como $1/3 \notin \overline{C}_0$ entonces $(1/3,0,...) \notin \Pi$ pero 0, $E' = (1,1,0,...) \in \Pi$. Además sabemos que los ejes coordenados x e y pertenecen a Γ. Pero en el plano euclidiano que determinan x e y se tienen dos rectas diferentes g, h que pasan por E' y que no cortan la recta x. Elegimos g como la recta ortogonal a y con g I E' y como h la recta



en el plano euclidiano de ecuación $y = 3/2\ x - 1/2$, h pertenece a

Γ pues $(0,-1/2,0,\ldots)$ y E' se encuentran sobre h y E'. $(0,-1/2,0,\ldots,0,\ldots) \in \Pi$ pero la intersección con x es $(3/2,0,0,\ldots) \notin \Pi$.

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A SURVEY OF MODERN APPLICATIONS OF THE METHOD OF CONFORMAL MAPPING

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INTRODUCTION. Classical applications of conformal mapping to many stationary problems of mathematical physics go back over a century and continue to the present. These applications deal, in general, with solutions of Laplace's equation which remains invariant if the real plane is subjected to a conformal transformation. Consequently, any complicated configuration can be transformed into a more convenient one without modification of the governing partial differential equation.

Applications of conformal mapping techniques to the mathematical theory of elasticity are considerable more complex. The method is due to the great Russian mathematician Mushkelishvili. An excellent survey of applications of this method is available in the book *Elasticity and Plasticity* by J.N. Goodier and P.G. Hodge, Jr. (John Wiley and Sons, 1958).

It is the purpose of this paper to present a review of non-classical applications of conformal mapping in several fields of technology and applied sciences: acoustics, electromagnetic theory, vibrations, viscous flow problems, etc.

The present review should not be considered exhaustive but rather informative.

ELECTROMAGNETIC THEORY.

In general, exact analytical calculations of wavefields between conducting surfaces are only possible for configurations of simple shape, for example planes or cylinders having a circular cross-section. However, more complicated boundary configurations are needed in many technological applications.

It must be pointed out that the standard circular an rectangular waveguides do not satisfy all present and future requirements. Hence the need of investigating waveguides of very general cross section.

Since many curved surfaces can be transformed into simpler surfaces, the boundary configurations can be simplified using the method of conformal transformation.

When such a transformation is used the space in which the wave propagates becomes considerable more complex since the dielectric constant and the permeability become functions of position. In some cases the space becomes anisotropic ([1]).

Some authors feel that the complex transformed equations with their simplified boundary configurations are more suitable for numerical evaluations than the original differential system ([1]).

The most important contributions in this field are due to Meinke and his coworkers ([1],[2]). The curved boundary surfaces are transformed into parallel planes and rectangular coordinates are then used. The transformed governing differential system has the same structure for all transformed systems and they only differ in some position dependent factors which describe the non-uniformily of the field. The solution of the differential system is then expressed in terms of an infinite sum of orthogonal functions and a numerical evaluation of the equations is then possible.

The transformation expressions can be obtained by mathematical formulations, graphical methods or by utilizing the electrolytic tank technique. Other contributions in this field are due to Tischer ([3],[4]), Wohlleben ([5]), Chi and Laura ([6],[7]), Bava and Perona ([8]), Baier ([9]), etc.

It has been shown by Richter ([10]) that wave propagation around a cylinder represents an approximation to wave propagation around a sphere (e.g. the earth) when the influence of the curvature of the sphere perpendicular to the direction of propagation can be neglected. He proved that the conformal mapping applied to the cylinder yields an earth - flattening technique which agrees with first - order approximations obtained in spherical coordinates. It is interesting to point out that Pryce ([10]) treats wave propagation around the earth in spherical coordinates using a range transformation suggested by Pekeris and a height transformation suggested by Copson and that Richter proved that both independently proposed transformations follow directly from the application of conformal mapping ([10]).

Richter's paper is the only contribution which makes use of conformal mapping in a problem of radio propagation in the atmosphera.

FLOW AND HEAT TRANSFER IN DUCTS OF ARBITRARY SHAPE.

The analysis of the flow and heat transfer in ducts of arbitrary shape has been the subject of investigation for many years. A common application of such conduits is in space vehicles ([11]). A very general approach has been developed by Sparrow and Haji - Sheikh and several practical cases have been studied in an excellent paper ([11]).

A different approach has been followed in Refs. [12] and [13]. The technique introduced therein is not as general as that developed in Ref. [11] but it is more convenient for the specific problem tackled by the authors.

HEAT CONDUCTION PROBLEMS.

Laura and his coworkers have made use of conformal mapping techniques in the analysis of unsteady heat conduction problems in bars of arbitrary cross section ([14]-[16]).

Yu ([17]) has extended the method to deal with temperature dependent conductivity materials.

ANALYTICAL PREDICTION OF DRYING PERFOMANCE IN NON-CONVENTIONAL SHAPES. APLICATION TO THE DRYING OF APPLES.

The diffusional flow of water is an important part of many food drying processes. In general, several mechanisms are expected when considering a drying process. They are sometimes divided into two broad all-including cathegories: one in which drying occurs as if the system were pure water being evaporated, one with internal control. The first type may or may not occur but the second type is always present.

The internal control period of drying is described as a diffusional process, which follows Fick's second law in Reference [18]. Experimental data was obtained for a rectangular paralellepiped. This shape having natural coordinates, has a simple mathematical solution. Non-conventional shapes, which do not have natural coordinates on the x, y plane, are solved analitically by using conformal mapping. The initial and boundary conditions are those classical for a drying problem: uniform initial concentration, zero surface concentration. The transformed differential system is solved by the collocation along arcs procedure, obtaining moisture concentration distribution as a function of time and position.

For practical reasons, the total amount of moisture diffusing from the solid body is more useful than the concentration distribution. The solution is therefore integrated over the volume of the solid bodies.

The non-conventional cross sections considered are: cardicid, cylinder, corrugated, hexagon, epitrochoid, and square.

ION OPTICS.

The equation of trajectory of a charged particle in a two - dimensional electric field and a normal magnetic field has been derived by Naidu and Westphal ([19] and [20]) from the basic equation of motion. A series of paracial approximations reduced the non-lineal trajectory equation to a linear inhomogeneous ordinary differential equation.

The boundary value problem is solved by the Schwarz-Christoffel transformation and the field configuration is then found.

MATHEMATICAL THEORY OF ELASTICITY.

Only a few papers published after 1958 and which consequently have not been referenced in Goodier's excellent monograph will be briefly discussed in this section.

Florence and Goodier ([21]) have solved the thermoelastic problem of uniform heat flow disturbed by an insulated hole of ovaloid form. Deresiewicz ([22]) has extended the previous analysis to holes whose boundaries can be mapped conformally on a unit circle by means of polynomials.

The determination of stresses in beams subjected to pure bending and having holes of arbitrary shape has been analized in [23] and [24]. The cases of equilateral triangular, square, rectangular and regular polygonal holes have been examined in great detail and the circumferential normal stresses have been evaluated in each case as a function of the radius of curvature at the vertices ([23]).

The names of Wilson and Richardson must be, certainly, mentioned at this point since they have analyzed extremely complex shapes ([25],[26]). They have developed computer programs to find the corresponding mapping functions by solving an integral equation of the Fredholm type ([25]) or a system of coupled integral equations ([26]). A discussion on available methods for finding the mapping function has also been published ([48]).

A conformal mapping approach has also been used in predicting the stress distribution in rotating disks of non-circular shape ([27]). No numerical values are given.

Stress fields in plates with reinforced holes of several shapes have also been determined by several researchers ([28]-[30]).

References [31] and [32] deal with applications of conformal mapping to the analysis of stressed plates with inclusions.

Savin and his coworkers have followed a complex variable approach to determine stress fields around holes of arbitrary shape in physically nonlinear media ([33],[34]).

NON CLASSICAL APPLICATIONS OF CONFORMAL MAPPING TO FLUID FLOW PROBLEMS.

Bartels and Laporte ([35]) have developed a general method for treating the linearized equations of the supersonic flow past conical bodies. Their approach makes use of conformal mapping techniques.

Segel has shown that conformal mapping is a useful tool in obtaining the solution of certain unsteady two-dimensional perturbation problems involving the flow of a viscous incompressible fluid ([36]).

The case offlow between moving circular cylinders is solved by mapping the given eccentric - circular boundaries into concentric circles ([36]).

Yu and Chen ([37]) have solved the problem of an unsteady laminar flow of a viscous incompressible fluid in a conduit of arbitrary cross section due to a time dependent axial pressure gradient. The given region is transformed onto a unit circle to facilitate the choice of the coordinate functions. The stationary value problem in the circular region is then solved by the Rayleigh - Ritz method. Velocity profiles, friction factors and rates of energy dissipation factors are calculated for ducts of regular polygonal cross section.

SOLIDIFICATION PROBLEMS.

Siegel and coworkers ([38]-[40]) have developed a conformal mapping method for analyzing two-dimensional transient and steady - state solidification problems. The method has been applied to the solidification which takes place on a cold plate of finite width immersed in a flowing liquid and to the solidification inside of a cooled rectangular channel which containe a warm flowing liquid. The investigations have dealt with the transient and stead y - state shapes of the frozen regions.

It is interesting to point out that the transient shapes of the frozen region are found by mapping the region into a potential plane and then determining the time varying conformal transformation between the potential and physical planes.

THEORY OF ACOUSTICS.

Acoustic waveguides of complicated cross section have been studied using conformal mapping techniques by several authors ([41]-[44]), ([46]-[48]). It must be pointed out that the governing differential system is similar to that of microwave theory:

$$\nabla^2 \phi + k^2 \phi = 0 \ ; \quad \phi = 0 \quad \text{(soft walls, TM waves)}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{(rigid walls, TE waves)}$$

A method for solving three - dimensional axially symmetric problems related to the diffraction and radiation from a general class of bodies of revolution has recently been developed ([45]). The method depends on the conformal transformation of the region outside the meridian profile of the body onto the region outside a circle. The required boundary value problem is formulated in spherical coordinates in the transformal space. In this form, Galerkin's method can be applied to obtain a functional approximation for the solution of the boundary value problem.

THEORY OF PLATES.

Simply supported plates of rectilinear sides subjected to complex distributions of loading have been considered by Aggarwala in several papers ([49]-[51]). Aggarwala has derived a simple but very accurate formula which yields the central deflection of a simply supported centrally loaded rhombic plate. The formula, which depends only on the first coefficient obtained in the mapping of the rhombus onto a unit circle, gives results correct within about one per cent ([50]).

The approach is also valid to any simply supported plate of regular polygonal shape. Such functional relation is:

$$W_o \approx \frac{P}{8\pi D} \alpha_1^2$$

where W_o: deflection at the center, P: concentrated load, D:

flexural rigidity and α_1 : is the first coefficient of the mapping function given in series form:

$$z = \sum_{n=1}^{\infty} \alpha_n . \xi^n$$

Clamped and simply supported plates of complicated boundary shape have been considered by several authors ([52]-[54]).

In Ref.[55] Ramu determines the collapse load of plates of arbitrary boundary shape. The author uses a method similar to Musklelishvili for plane elasticity problems. By introducing a suitable stress function, the problem of determining a statically admissible stress field is reduced to finding a solution of the governing biharmonic equation in terms of analytic functions. It is assumed that the material obeys Von Mises yield condition. As pointed out by Laura and Shahady ([56]) solution of the eigenvalue problem governing the stability of a thin elastic plate subjected to hydrostatic in-plate loading is easily accomplished when the boundary configuration is natural to one of the common coordinates systems. Reference [56] shows that is convenient to conformally transform the given domain onto a simpler one, i.e. the unit circle. The boundary conditions can then be satisfied identically. Since the governing partial differential equation is not invariant under the transformation and becomes considerably more complicated, a variational method is used to solve it. The method has been illustrated in the case of clamped and simply supported plates of various configurations.

It has also been shown that the determination of an upper bound of the critical in-plane loading of simply supported plates of rectilinear sides is quite straightforward if use is made of a theorem by Szego ([56]-[57]).

Obviously the same approach is also valid when determining natural frequencies of vibrations of plates of complicated boundary shape ([58]-[62]).

VIBRATIONS OF SOLID PROPELLANT ROCKET MOTORS.

The grain of a solid propellant rocket motor usually takes the form of a circular cylinder bonded to a thin case. This grain quite commonly has a star-shaped internal perforation. The mathematical solution of any boundary or eigenvalue problem becomes quite complicated in view of the exotic geometric configuration. This difficulty can be alleviated to a large extent by conformally transforming the grain cross section into a simpler region such as a circle or annulus. Several studies have been performed on axial shear vibrations of solid and hollow bars making use of conformal mapping and variational or bounding techniques ([63]-[65]).

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EXISTENCE OF SOLUTIONS FOR GENERALIZED CAUCHY-GOURSAT TYPE PROBLEMS FOR HYPERBOLIC EQUATIONS

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INTRODUCTION. Let X be a Banach space and R the set of real numbers. If $S \subset R^n$ is a Lebesgue measurable set we will denote by $L_q(S,X)$ the set of all Lebesgue-Bochner measurable functions with power q summable on the set S into the Banach space X. Let $a_i \in R$, $a_i > 0$ $(i= 1,2)$ and consider the closed intervals $I_i = \langle 0,a_i \rangle$ for $i= 1,2$.

Let the graphs of the functions $g_1:I_1 \to I_2$, $g_2:I_2 \to I_1$ represent two continuous non-decreasing curves with (0,0) as their only point in common. Denote by Δ the set of all points (x_1,x_2) in the x_1x_2-plane such that $g_1(x_1) \leqslant x_2 \leqslant a_2$ and $g_2(x_2) \leqslant x_1 \leqslant a_1$. Take $p_i \in \langle 1,\infty \rangle$ $(i= 0,1,2)$, $p_o \geqslant \max(p_1,p_2)$ and let $p_3=(p_o,p_1,p_2)$. In this paper the derivatives we understand in the sense of S.Sobolev (i.e., L. Schwartz derivatives representable by a Lebesgue-Bochner locally summable function).

In the first section we define a class of functions U_{p_3}. This class is a subset of the set of continuous functions u from $I_1 \times I_2$ into X which have S. Sobolev partial derivatives u_{x_1}, u_{x_2}, $u_{x_1x_2}$. We prove that the class U_{p_3} is linearly isomorphic to the product space $W_{p_3} = L_{p_o}(\Delta,X) \times L_{p_1}(I_1,X) \times L_{p_2}(I_2,X) \times X$. Thus the class U_{p_3} inherits a Banach type structure from the product space W_{p_3}.

In the sequel we shall be concerned with the following hyperbolic equation (0.1) $u_{x_1x_2}(x_1,x_2)= f(x_1,x_2)$ a.e. on Δ, where $f: \Delta \to X$ is a bounded Bochner measurable function on Δ .

Let Y= L(X,X) denote the collection of all linear continuous mappings from X into itself. Let $V= B(\Delta,X) \times L_p(I_1,Y) \times L_\infty(I_1,Y) \times L_p(I_1,X) \times L_p(I_2,Y) \times L_\infty(I_2,Y) \times L_p(I_2,X) \times X$ where $B(\Delta,X)$ is the space of bounded Bochner measurable functions with the supremum norm from Δ into X, and $p \in \langle 1,\infty \rangle$. Take $(f,\alpha_o,\alpha_1,\alpha_2,\beta_o,\beta_1,\beta_2,\gamma) \in V$ and let $\bar{p}= (\infty,p,p)$. By a solution of the generalized Cauchy-Goursat boundary problem in the class $U_{\bar{p}}$ for the hyperbolic equation we mean a function $u \in U_{\bar{p}}$ satisfying equation (0.1) and the boundary conditions (0.2)

$$u_{x_1}(\cdot,g_1(\cdot)) = \alpha_o(\cdot)\,.\,u(\cdot,g_1(\cdot)) + \alpha_1(\cdot)\,.\,u_{x_2}(\cdot,g_1(\cdot)) + \alpha_2(\cdot)$$

a.e. on I_1

$$u_{x_2}(g_2(\cdot),\cdot) = \beta_o(\cdot)\,.\,u(g_2(\cdot),\cdot) + \beta_1(\cdot)\,.\,u_{x_1}(g_2(\cdot),\cdot) + \beta_2(\cdot)$$

a.e. on I_2

$$u(0,0) = \gamma$$

In the third section we establish that the generalized Cauchy-Goursat boundary problem is meaningful, i.e., all the operations appearing in the definition of the problem make sense. Also we prove the existence and uniqueness of the solutions for the initial data from the product space V in the fourth section.

The continuity of the solutions on the initial data in the sense of the topology of the normed space V is also established.

1. DEFINITION OF THE CLASS U_{p_3} .

From now on when dealing with derivatives we will specify if they are to be taken in Sobolev sense, otherwise they will be taken in the usual sense.

DEFINITION 1.1. *A function* $u:I_1 \times I_2 \to X$ *belongs to the class* U_{p_3}, *if and only if,* u *is continuous on* $I_1 \times I_2$ *and there exist* $u_1 \in L_{p_1}(I_1 \times I_2, X)$, $u_2 \in L_{p_2}(I_1 \times I_2, X)$, $u_{12} \in L_{p_o}(\Delta, X)$ *such that:*

(a) $D_1 u = u_1$, $D_2 u = u_2$, $D_{12} u = u_{12}$ *where the derivatives are taken in Sobolev sense.*

(b) *There exists a set* $A_1 \subset I_1$ *of Lebesgue measure zero such that the function* $x_2 \to u_1(x_1,x_2)$ *is continuous on* I_2 *for every fixed* $x_1 \notin A_1$; *the function* $u_1(\cdot,g_1(\cdot)) \in L_{p_1}(I_1,X)$; $u_1(x_1,g_1(x_1)) = u_1(x_1,x_2)$ *for all* $x_2 \in \langle 0,g_1(x_1)\rangle$ *at each* $x_1 \in I_1$; $u_1(x_1,c_{x_1}) = u_1(x_1,x_2)$ *for all* $x_2 \in \langle c_{x_1},a_2\rangle$ *at each* $x_1 \in \langle 0,g_2(a_2)\rangle$, *where* $c_{x_1} = \sup\{x_2 \in I_2 : g_2(x_2) = x_1\}$.

(c) *Symmetrically, there exists a set* $A_2 \subset I_2$ *of measure zero such that the function* $x_1 \to u_2(x_1,x_2)$ *is continuous on* I_1 *for every fixed* $x_2 \notin A_2$; *the function* $u_2(g_2(\cdot),\cdot) \in L_{p_2}(I_2,X)$;

$u_2(g_2(x_2),x_2)= u_2(x_1,x_2)$ *for all* $x_1\in <0,g_2(x_2)>$ *at each* $x_2\in I_2$;
$u_2(c_{x_2},x_2)= u_2(x_1,x_2)$ *for all* $x_1\in <c_{x_2},a_1>$ *at each* $x_2\in <0,g_1(a_1)>$
where $c_{x_2}= \sup\ \{x_1\in I_1;g_1(x_1)= x_2\}$.

DEFINITION 1.2. *Let* $s\in L_q(I_1\times I_2,X)$, $q\geqslant 1$. *We* ***define*** *the operators* $J_i(i= 1,2)$ *by the formulas*

$$J_1 s.(x_1,x_2)= \int_0^{x_1} s(t,x_2)dt$$

$$J_2 s.(x_1,x_2)= \int_0^{x_2} s(x_1,r)dr \qquad \text{for all} \quad (x_1,x_2)\in I_1\times I_2$$

LEMMA 1.1. *The operators* $J_i(i= 1,2)$ *are well defined bounded linear operators on* $L_q(I_1\times I_2)$.

LEMMA 1.2. *The operator* T *given by the formula:*

$$T(s,\phi,\psi,\gamma)= J_2J_1\overline{s} + J_1\phi + J_1\psi + \gamma$$

where $\overline{s}= s$ *on* Δ *and* $\overline{s}= 0$ *on* $I_1\times I_2\setminus\Delta$, *is a well defined linear operator from the product* W_{p_3} *into the space* U_{p_3}.

Proof. Let $u= T(s,\phi,\psi,\gamma)$, where $(s,\phi,\psi,\gamma)\in W_{p_3}$. Clearly, u is continuous on $I_1\times I_2$ and $D_1u= J_2\overline{s} + \phi$, $D_2u= J_1\overline{s} + \psi$, $D_{12}u= \overline{s}$, where the derivatives are taken in the sense of Sobolev.

Letting $u_1= J_2\overline{s} + \phi$, $u_2= J_1\overline{s} + \psi$, $u_{12}= \overline{s}$ one can prove that u satisfies all the conditions specified in the definition of U_{p_3}. Thus, T is a well defined mapping.

From the linearity of the integral and the fact that U_{p_3} is a linear space follows the linearity of the operator T.

LEMMA 1.3. *Let the set* $A\subset I_1\times I_2$, $A_1\subset I_1$, $B_1\subset I_2$ *be of measure zero. Then the boundary value problem*

$$w_{12}(x_1,x_2)= 0 \qquad \text{if} \quad (x_1,x_2)\notin A$$

$$w_1(x_1,g_1(x_1))= 0 \quad \text{if} \quad x_1\notin A_1$$

$$w_2(g_2(x_2),x_2)= 0 \quad \text{if} \quad x_2\notin B_1$$

$$w(0,0)= 0$$

has a unique solution in the class U_{p_3}, *namely*, $w\equiv 0$, *where derivatives are taken in the sense of Sobolev*.

Proof. It is evident that $w\equiv 0$ satisfy the given boundary value problem. Suppose $w\in U_{p_3}$ is a solution of the boundary value problem. Then there exists a set $B_2\subset I_2$ of measure zero such that

$w_2(x_1,x_2)= J_1w_{12}\cdot(x_1,x_2) + w_2(0,x_2)$ if $x_2 \notin B_2$, $x_1 \in I_1$.

The equation $J_2J_1w_{12}\cdot(x_1,x_2)= 0$ for all $(x_1,x_2) \in I_1 \times I_2$ implies the existence of a set $B_3 \subset I_2$ of measure zero such that $J_1w_{12}\cdot(x_1,x_2)= 0$ if $x_2 \notin B_3$, $x_1 \in I_1$.

Hence $w_2(x_1,x_2)= w_2(0,x_2)= w_2(g_2(x_2),x_2)= 0$ if $x_1 \in I_1$, $x_2 \notin B_1 \cup B_2 \cup B_3$.

Similarly we obtain sets $A_2, A_3 \subset I_1$ of measure zero such that

$$w_1(x_1,x_2)= 0 \text{ if } x_1 \notin A_1 \cup A_2 \cup A_3, \quad x_2 \in I_2 .$$

Also there exist sets $A_4 \subset I_1$, $B_4 \subset I_2$ of measure zero such that

$$w(x_1,x_2)= J_1w_1\cdot(x_1,x_2) + w(0,x_2) \text{ if } x_2 \notin B_4, \ x_1 \in I_1$$

$$w(x_1,x_2)= J_2w_2\cdot(x_1,x_2) + w(x_1,0) \text{ if } x_1 \notin A_4, \ x_2 \in I_2$$

Hence,

$$w(x_1,x_2)= w(0,x_2) \text{ if } x_2 \notin B_4, \ x_1 \in I_1$$

$$w(x_1,x_2)= w(x_1,0) \text{ if } x_1 \notin A_4, \ x_2 \in I_2$$

The last two equalities and the continuity of w imply that there exists $k \in X$ such that $w(x_1,x_2)= k$ for all $(x_1,x_2) \in I_1 \times I_2$.

So, from $w(0,0)= 0$ we have that $w \equiv 0$ on $I_1 \times I_2$.

THEOREM 1.1. *The map* T *defined in Lemma 1.2 establishes a linear isomorphism between the product* W_{p_3} *and the space* U_{p_3}. *The inverse map* F *is given by the formulas*:

$$\begin{aligned} s &= u_{12} && \text{a.e. on } \Delta \\ \phi &= u_1(\cdot,g_1(\cdot)) && \text{a.e. on } I_1 \\ \psi &= u_2(g_2(\cdot),\cdot) && \text{a.e. on } I_2 \\ \gamma &= u(0,0) \end{aligned}$$

Proof. It is clear that F is a well defined linear map. Let $(s,\phi,\psi,\gamma) \in W_{p_3}$, $u= T(s,\phi,\psi,\gamma)$, and $F(u)= (\overline{s},\overline{\phi},\overline{\psi},\overline{\gamma})$.

By definition of the map F we have:

$$\begin{aligned} \overline{s} &= u_{12} && \text{a.e. on } \Delta \\ \overline{\phi} &= u_1(\cdot,g_1(\cdot)) && \text{a.e. on } I_1 \\ \overline{\psi} &= u_2(g_2(\cdot),\cdot) && \text{a.e. on } I_2 \\ \overline{\gamma} &= \gamma \end{aligned}$$

But, $u_{12}= s$, $u_1= J_2s + \phi$, $u_2= J_1s + \psi$. Hence, $s= \overline{s}$ a.e. on Δ, $\phi= \overline{\phi}$ a.e. on I_1, $\psi= \overline{\psi}$ a.e. on I_2, $\gamma= \overline{\gamma}$, or equivalently $F \circ T= I_{W_{p_3}}$ i.e. the identity map on W_{p_3}.

Let $v \in U_{p_3}$, $F(v)= (s,\phi,\psi,\gamma)$, $u= T(s,\phi,\psi,\gamma)$.

Letting w= u - v we obtain:

$$\begin{array}{ll} w \in U_{p_3} & \\ w_{12} = 0 & \text{a.e. on } \Delta \\ w_1(\cdot, g_1(\cdot)) = 0 & \text{a.e. on } I_1 \\ w_2(g_2(\cdot), \cdot) = 0 & \text{a.e. on } I_2 \\ w(0,0) = 0 & \end{array}$$

Therefore from Lemma 1.3 it follows that $w \equiv 0$ on $I_1 \times I_2$, or equivalently $T \circ F = I_{U_{p_3}}$. This completes the proof of the theorem.

COROLLARY 1.1. *The space* U_{p_3} *is a Banach space with the norm* $|\ |$ *defined by the formula*

$$|u| = \|F(u)\| \quad \textit{for all } u \in U_{p_3}$$

The operator T *establishes a linear isomorphism and isometry between the spaces* W_{p_3} *and* U_{p_3}.

2. A CAUCHY-GOURSAT TYPE PROBLEM IN THE CLASS $U_{\overline{p}}$.

We are going to enunciate a series of hypothesis which will be used throughout the remainder of this paper.

HYPOTHESIS (A_1). The functions g_1, g_2 are continuous, strictly increasing, $g_i(0) = 0$ for $i = 1,2$, and $x_2 = g_1(x_1)$, $x_1 = g_2(x_2)$ imply $x_1 = x_2 = 0$.

HYPOTHESIS (A_2). The functions g_1, g_2 satisfy hypothesis (A_1), g_i^{-1} ($i = 1,2$) are absolutely continuous functions on their domain of definition, and the derivatives $(g_i^{-1})'$ ($i = 1,2$) are essentially bounded functions.

HYPOTHESIS (A_3). The curves g_1, g_2 satisfy hypothesis (A_2) and they are absolutely continuous on their domain of definition.

HYPOTHESIS (A_4). The functions g_i ($i = 1,2$) are such that

$$g_i(x_i) \leqslant x_i \quad \text{for all} \quad x_i \in I_i \ (i = 1,2)$$

HYPOTHESIS (A_5). $(f, \alpha_o, \alpha_1, \alpha_2, \beta_o, \beta_1, \beta_2, \gamma) \in V$.

DEFINITION 2.1. *Under Hypothesis* (A_1) *and* (A_4) *we want to find a function* $u \in U_{\overline{p}}$ *satisfying equation* (0.1) *and the boundary conditions* (0.2) *where the derivatives are understood in the sense of Sobolev. Such a function* u, *if it exists, will be called a solution of the Cauchy-Goursat problem for equation* (0.1) *under the bounda-*

ry conditions (0.2).

3. THE CAUCHY-GOURSAT PROBLEM IS MEANINGFUL.

The following two lemmas will be needed in this section.

LEMMA 2.1. *If* $w_i \in L_\infty(I_i,Y)$, $\psi_j \in L_p(I_j,X)$ *and* g_i *satisfy Hypothesis* (A_2), *then the function* $x_i \to w_i(x_i)(\psi_j o g_i(x_i))$ *belongs to the space* $L_p(I_i,X)$, *where* $i,\ j \in \{1,2\}$, $i \neq j$.

LEMMA 2.2. *If* $f:\Delta \to X$ *is Bochner measurable and bounded on* Δ, $w_i \in L_\infty(I_i,Y)$ $(i=1,2)$, $g_i (i=1,2)$ *satisfy Hypothesis* (A_2), *then the function* $x_i \to w_i(x_i)(\int_o^{x_i} f(t,g_i(x_i))dt)$ *belongs to the space* $L_p(I_i,X)$.

THEOREM 2.1. *Under Hypothesis* (A_2) *and* (A_5) *the Cauchy-Goursat problem is meaningful.*

Proof. Because of Theorem 1.1 every $u \in U_{\bar{p}}$ has a representation of the form $u = J_2J_1s + J_1\phi + J_2\psi + \gamma$ where $(s,\phi,\psi,\gamma) \in W_{\bar{p}}$. For any function $G_i \in L_p(I_i,X)$ $(i=1,2)$ we are going to write $J_iG_i.(x_1,x_2) = J_iG_i.(x_i)$, for all $x_i \in I_i$.
To find $u \in U_{\bar{p}}$ satisfying the generalized Cauchy-Goursat boundary problem is equivalent to find $(s,\phi,\psi,\gamma) \in W_{\bar{p}}$ such that

$$(2.1)\quad \begin{cases} \phi(\cdot) = \alpha_o(\cdot)[J_1J_2f.(\cdot,g_1(\cdot)) + J_1\phi.(\cdot) + J_2\psi.(g_1(\cdot)) + \gamma] + \\ \qquad + \alpha_1(\cdot)[J_1f.(\cdot,g_1(\cdot)) + \psi(g_1(\cdot))] + \alpha_2(\cdot) \text{ a.e. on } I_1 \\ \psi(\cdot) = \beta_o(\cdot)[J_1J_2f.(g_2(\cdot),\cdot) + J_1\phi.(g_2(\cdot)) + J_2\psi.(\cdot) + \gamma] + \\ \qquad + \beta_1(\cdot)[J_2f.(g_2(\cdot),\cdot) + \phi(g_2(.))] + \beta_2(\cdot) \text{ a.e. on } I_2 \end{cases}$$

From Lemmas 2.1 and 2.2 it follows that the equations of system (2.1) are meaningful. This completes the proof of the theorem.

3. THE OPERATORS H AND J.

DEFINITION 3.1. *Under hypothesis* (A_2) *and* (A_5) *let us define the operators* H *and* J *from the space* $L_p(I_1,X) \times L_p(I_2,X)$ *into itself by the formulas*

$$H\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\alpha_1(x_1)(\psi(g_1(x_1))\\ \beta_1(x_2)(\phi(g_2(x_2))\end{pmatrix} \quad J\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\alpha_o(x_1)[J_1\phi.(x_1)+J_2\psi.(g_1(x_1))]\\ \beta_o(x_2)[J_1\phi.(g_2(x_2))+J_2\psi.(x_2)]\end{pmatrix}$$

LEMMA 3.1. *The operators* H *and* J *are well defined.*

For every $\phi \in L_p(I_1,X)$, $\psi \in L_p(I_2,X)$, let $\tau = \begin{pmatrix}\phi\\ \psi\end{pmatrix}$ and

$$\tau_o = \begin{pmatrix}\alpha_o(x_1)[J_1J_2f.(x_1,g_1(x_1))+\gamma]+\alpha_1(x_1)[J_1f.(x_1,g_1(x_1)]+\alpha_2(x_1)\\ \beta_o(x_2)[J_1J_2f.(g_2(x_2),x_2)+\gamma]+\beta_1(x_2)[J_2f.(g_2(x_2),x_2)]+\beta_2(x_2)\end{pmatrix} \tag{3.1}$$

assuming that Hypothesis (A_2) and (A_5) hold.

By means of the operators H and J equation (2.1) can be written

$$\tau = J\tau + H\tau + \tau_o \tag{3.2}$$

Thus to solve the Cauchy-Goursat problem is equivalent to find a solution τ of equation (3.2).

DEFINITION 3.2. *Under Hypothesis* (A_1) *define the functions* $\lambda_i^n: I_i \to I_i$, $(i=1,2)$, n *a non-negative integer, by the formulas*

$$\lambda_i^o(x_i) = x_i \qquad \textit{for all } x_i \in I_i$$

$$\lambda_i^1(x_i) = \lambda_i(x_i) = g_j o g_i(x_i) \qquad \textit{for all } x_i \in I_i,\ (j=1,2,\ j\neq i)$$

$$\textit{and} \quad \lambda_i^n(x_i) = \lambda_i(\lambda_i^{n-1}(x_i)) \qquad \textit{for all } x_i \in I_i,\ n > 1$$

LEMMA 3.2. *If* $g_i (i=1,2)$ *satisfy the Hypothesis* (A_3), *then* $(\lambda_i^n)^{-1}$ $(i=1,2)$ *are strictly increasing absolutely continuous functions on* $<0,\lambda_i^n(a_i)>$, *and the derivatives* $((\lambda_i^n)^{-1})'$ *are essentially bounded, where* $n=0,1,2,\dots$.

LEMMA 3.3. *If* g_i $(i=1,2)$ *satisfy Hypothesis* (A_1), *then the sequences* λ_i^n $(i=1,2)$ *are non-increasing sequences converging uniformly toward zero in* I_i.

This Lemma is proven by J. Kisynski and M. Bielecki in [3].

DEFINITION 3.3. *Under Hypothesis* (A_2) *and* (A_5) *let us define the functions*:

$$\mu_1(x_1) = \alpha_1(x_1)\beta_1(g_1(x_1)) \qquad \textit{for all} \quad x_1 \in I_1$$

$$\mu_{1n}(x_1) = \mu_1(x_1) \ \ldots \ \mu_1(\lambda_1^{n-1}(x_1)) \quad \textit{for all} \quad x_1 \in I_1, \ n > 1$$

$$\mu_2(x_2) = \beta_1(x_2)\alpha_1(g_2(x_2)) \quad \textit{for all} \quad x_2 \in I_2$$

$$\mu_{2n}(x_2) = \mu_2(x_2) \ \ldots \ \mu_2(\lambda_2^{n-1}(x_2)) \quad \textit{for all} \quad x_2 \in I_2, \ n > 1$$

LEMMA 3.4. *Let* $\overline{\alpha}_1 = \alpha_1/(g_1')^{1/p}$, $\overline{\beta}_1 = \beta_1/(g_2')^{1/p}$, *where* $\alpha_1 : I_1 \to Y$, $\beta_1 : I_2 \to Y$. *If* $\overline{\alpha}_1$ *and* $\overline{\beta}_1$ *are essentially bounded functions, and* g_i *satisfy Hypothesis* (A_3), *then* $\mu_{in}/((\lambda_i^n)')^{1/p}$ $(i = 1,2)$ *are essentially bounded functions on* I_i *for every natural number* n.

DEFINITION 3.4. *Let* $f \in L_p(\Delta,X)$, $p \geq 1$. *Define* $\|f\|_k =$

$$= \sup \{e^{-k(x_1+x_2)} \left(\int_0^{x_1} \int_0^{x_2} \|\overline{f}\|^p \right)^{1/p} : (x_1,x_2) \in \Delta\}$$ *where* $k > 0$ *and*

$\overline{f} = f$ *on* Δ, $\overline{f} = 0$ *on* $I_1 \times I_2 \setminus \Delta$. *From now on we will write* f *instead of* $\overline{f}$.

One can prove that $(L_p(\Delta,X), \| \ \|_k)$ is a Banach space for $k > 0$. This type of norm was introduced by M.A. Bielecki in [2].

DEFINITION 3.5. *If* $(\phi,\psi) \in L_p(I_1,X) \times L_p(I_2,X)$, $p \geq 1$, *we define* $\|(\phi,\psi)\|_k = \max(\|\phi\|_k, \|\psi\|_k)$, $k > 0$. *It is known that* $L_p(I_1,X) \times L_p(I_2,X)$ *is complete under the above defined norm.*

HYPOTHESIS (A_6). The functions $g_i (i = 1,2)$ satisfy Hypothesis (A_3) and (A_4). The functions $\alpha_1 \in L_\infty(I_1,Y)$ and $\beta_1 \in L_\infty(I_2,Y)$ are such that $\overline{\alpha}_1 \in L_\infty(I_1,Y)$, $\overline{\beta}_1 \in L_\infty(I_2,Y)$, and $\lim_{A_1 \not\ni x_1 \to 0^+} \overline{\alpha}_1(x_1) = \overline{\alpha}_1(0)$, $\lim_{A_2 \not\ni x_2 \to 0^+} \overline{\beta}_1(x_2) = \overline{\beta}_1(0)$ exist in the sense of the norm of Y where $\overline{\alpha}_1 = \dfrac{\alpha_1}{(g_1')^{1/p}}$, $\overline{\beta}_1 = \dfrac{\beta_1}{(g_2')^{1/p}}$ and $A_i \subset I_i$ $(i = 1,2)$ are of Lebesgue measure zero. Finally, $\|\overline{\alpha}_1(0)\overline{\beta}_1(0)\| < 1$.

LEMMA 3.5. *Under Hypothesis* (A_6) *the operator* $A = (I-H)^{-1}$, *from* $L_p(I_1 x X) \times L_p(I_2 x X)$ *into itself, is bounded and linear. Moreover, there exists* M *independent of* $k > 0$ *such that* $\|A\|_k \leq M$.

Proof. It is clear that H is a well defined linear operator. We have (3.3) $\|H\|_k \leq M_1$ where $M_1 = \max(\|\alpha_1\|_\infty \|(g_1^{-1})'\|_\infty^{1/p}, \|\beta_1\|_\infty \|(g_2^{-1})'\|_\infty^{1/p})$ is independent of $k > 0$.

From the definition of the operator H it follows that

$$H^{2n}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\mu_{1n}(x_1).\phi(\lambda_1^n(x_1))\\ \mu_{2n}(x_2).\psi(\lambda_2^n(x_2))\end{pmatrix}$$

for all $\phi \in L_p(I_1,X)$, $\psi \in L_p(I_2,X)$, $n= 1,2,...$

We have the following inequality for every natural number n:

$$(3.4)\qquad \int_0^{x_1}\int_0^{x_2} \|\mu_{1n}(\xi).\phi(\lambda_1^n(\xi))\|^p d\eta d\xi \leqslant$$
$$\leqslant \int_0^{x_1}\int_0^{x_2} \|((\lambda_1^n)^{-1})')^{1/p}(t)\mu_{1n}((\lambda_1^n)^{-1}(t))\|^p\|\phi(t)\|^p d\eta dt$$

Let us note that $\lim_{C\not\ni x_1\to 0^+} \mu_1(x_1)/(\lambda_1'(x_1))^{1/p} = \bar{\alpha}_1(0)\bar{\beta}_1(0)$ where

$C= A_1 U g_1^{-1}(A_2)$ is of Lebesgue measure zero. Thus, given $q > 0$ such that $\|\bar{\alpha}_1(0)\bar{\beta}_1(0)\| < q^2 < 1$ there exists $\delta > 0$ such that $\|\mu_1(x_1)/(\lambda_1'(x_1))^{1/p}\| < q^2$ if $x_1 \notin C$, $x_1 \in I_1$, $0 < x_1 < \delta$. Since by Lemma 3.3 there exists n_o such that $0 \leqslant \lambda_1^{n-1}(x_1) < \delta$ for all $x_1 \in I_1$, $n \geqslant n_o$ we have

$$(3.5)\qquad \|\mu_1(\lambda_1^{n-1}(x_1))/(\lambda_1'(\lambda_1^{n-1}(x_1)))^{1/p}\| < q^2 \text{ for all } n \geqslant n_o \text{ and}$$

$x_1 \notin (\lambda_1^{n-1})^{-1}(C)$, which is a set of Lebesgue measure zero.

From (3.4), (3.5) we obtain

$$(3.6)\qquad \|\mu_{1n}(x_1).\phi(\lambda_1^n(x_1))\|_k \leqslant (\|\bar{\alpha}_1\|_\infty\|\bar{\beta}_1\|_\infty)^{n_o-1} q^{2(n-n_o+1)}$$

Similarly there exists n_1 such that

$$(3.7)\qquad \|\mu_{2n}(x_2).\psi(\lambda_2^n(x_2))\|_k \leqslant (\|\bar{\alpha}_1\|_\infty\|\bar{\beta}_1\|_\infty)^{n_1-1} q^{2(n-n_1+1)}$$

Hence $\|H^{2n}\|_k \leqslant M_2 q^{2n}$ for all $n \geqslant \max(n_o,n_1)$, where M_2 independent of k is defined in an obvious way.

Noting that $A= (I + H)\cdot B$, where $B= I + H^2 + H^4 + \ldots + H^{2n} + \ldots$ and using (3.3), (3.6), and (3.7) we can obtain the desired result.

DEFINITION 3.6. *Let* $C(I_i)$ $(i= 1,2)$ *denote the set of all continuous functions* $f:I_i \to X$. *For every* $f \in C(I_i)$ *define* $\|f\|_{kc}^{(i)} =$ $= \sup\{e^{-kx_i}\|f(x_i)\| : x_i \in I_i\}$.

It is known that $(C(I_i), \|\ \|_{kc}^{(i)})$ is a Banach space.

LEMMA 3.6. *The operators* T_i (i= 1,2) *from the space* $(L_p(I_i,X),\| \|_k)$ *into the space* $(C(I_i),\| \|_{kc}^{(i)})$ *defined by the formula*

$$(T_i\phi)(x_i)= \int_0^{x_i} \phi(t)\, dt \quad \textit{for all} \quad x_i \in I_i$$

are well defined bounded linear operators and $\|T_i\|_k=$ $= (kep)^{1/p}\, a_i^{(p-1)/p}$ *if* $1/pk \leqslant \min(a_1,a_2)$, $p \geqslant 1$, $k > 0$.

LEMMA 3.7. *Let* $\alpha:I_1\to Y$, $\beta:I_2\to Y$ *be* p-*Bochner summable functions on* I_1 *and* I_2 *respectively. Then, the operators*

$$H_i:(C(I_i),\| \|_{kc}^{(i)}) \to (L_p(I_i,X),\| \|_k) \qquad (i= 1,2)$$

defined by the formulas

$$(H_1f)(x_1)= \alpha(x_1).f(x_1) \quad \textit{for all} \quad x_1 \in I_1,\ f \in C(I_1)$$

$$(H_2g)(x_2)= \beta(x_2).g(x_2) \quad \textit{for all} \quad x_2 \in I_2,\ g \in C(I_2)$$

are well defined bounded linear operators. Moreover, for any $\varepsilon > 0$ *there exists* k_o *such that* $\|H_i\|_k\leqslant (\varepsilon/pke)^{1/p}$ *for all* $k \geqslant k_o$ (i=1,2).

Proof. It is clear that H_i (i= 1,2) are well defined and linear. We have also

$$(3.8)\qquad \left(\int_0^{x_1} \int_0^{x_2} \|\alpha(t).f(t)\|^p dr\, dt\right)^{1/p} \leqslant$$

$$\leqslant \|f\|_{kc}^{(1)} \left(\int_0^{x_1} \int_0^{x_2} \|\alpha(t)\|^p e^{kpt} dr\, dt\right)^{1/p}$$

Let k_1 be such that $\frac{1}{pk} \leqslant \min(a_1,a_2)$ for all $k \geqslant k_1$. For any $k > k_1$ we have:

$$(3.9)\qquad e^{-pk(x_1+x_2)} \int_0^{x_1} \int_0^{x_2} \|\alpha(t)\|^p e^{kpt} dr\, dt \leqslant$$

$$\leqslant (pke)^{-1} e^{-kpx_1} \int_0^{x_1} \|\alpha(t)\|^p e^{kpt} dt\ , \quad (x_1,x_2) \in I_1\times I_2$$

Let $s \geqslant 0$ be a simple function defined on I_1 such that $\int_0^{a_1} | \|\alpha(t)\|^p - s(t)| dt < \varepsilon/2$ where $\varepsilon > 0$ is given. So,

$$(3.10)\qquad \int_0^{x_1} \|\alpha(t)\|^p e^{-kp(x_1-t)} dt < \frac{\varepsilon}{2} + \frac{\|s\|_\infty}{kp}$$

From inequalities (3.8), (3.9),(3.10) it follows that

$$\|H_1\|_k \leqslant (pke)^{-1/p}[\frac{\varepsilon}{2} + \frac{\|s\|_\infty}{kp}]^{1/p}$$

Similarly we obtain $\|H_2\|_k \leqslant (pke)^{-1/p}[\frac{\varepsilon}{2} + \frac{\|\bar{s}\|_\infty}{kp}]^{1/p}$ for some simple function $\bar{s}$ defined on I_2. Let k_2 be such that $\frac{\|s\|_\infty}{k_2 p} < \frac{\varepsilon}{2}$, $\frac{\|\bar{s}\|_\infty}{k_2 p} < \frac{\varepsilon}{2}$. Thus for all $k > k_o = \max(k_1,k_2)$ we have $\|H_i\|_k \leqslant$ $\leqslant (\varepsilon/pke)^{1/p}$ for $i = 1,2$.

LEMMA 3.8. *Assume* g_1, g_2 *satisfy Hypothesis* (A_1) *and* (A_4). *The operators* $T_j : (L_p(I_i,X), \| \; \|_k) \to (C(I_m), \| \; \|_{kc}^{(m)})$ (*where* $p \geqslant 1, i,m \in \{1,2\}$, $i \neq m$, $j = i+2$) *defined by the formulas* $(T_j\phi)(x_m) = \int_0^{g_m(x_m)} \phi(t)\, dt$, *for all* $x_m \in I_m$ *are well defined bounded linear operators. Moreover, if* $\frac{1}{kp} \leqslant \min(a_1,a_2)$ *then* $\|T_j\|_k \leqslant (kep)^{1/p} a_i^{(p-1)/p}$.

LEMMA 3.9. *Under Hypothesis* (A_1), (A_4) *and* (A_5) *the operator* J (*Definition* 3.1) *is a well defined bounded linear operator. Moreover, for any given* $\varepsilon > 0$ *there exists* k_o *such that* $\|J\|_k \leqslant \varepsilon$ *for all* $k \geqslant k_o$.

The proof follows easily from Lemmas 3.6, 3.7 and 3.8.

4. EXISTENCE THEOREMS FOR THE CAUCHY-GOURSAT PROBLEM IN THE CLASS $\overline{U}_p$.

Under Hypothesis (A_5) and (A_6) equation (3.2) can be written:

(4.1) $\tau = (I-H)^{-1}J\tau + (I-H)^{-1}\tau_o$. Let us define the operator F_1 by the formula: (4.2) $F_1\tau = AJ\tau + A\tau_o$. Clearly F_1 is a well defined operator from the space $L_p(I_1,X) \times L_p(I_2,X)$ into itself because of Lemmas 3.5 and 3.9.

Moreover, from equation (4.1) it follows that to find a solution of the Cauchy-Goursat problem in the class $U_{\bar{p}}$ is equivalent to find a fixed point of the operator F_1.

THEOREM 4.1. *Under Hypothesis* (A_5) *and* (A_6) *the Cauchy-Goursat problem has a unique solution in the class* $U_{\bar{p}}$.

Proof. Note that $\|F_1\tau_1 - F_1\tau_2\|_k \leq M\|J\|_k\|\tau_1 - \tau_2\|_k$ where M is as in Lemma 3.5. From Lemma 3.9 it follows that there exists k_o such that $\|F_1\tau_1 - F_1\tau_2\|_k \leq \frac{1}{2}\|\tau_1 - \tau_2\|_k$ for all $k > k_o$.

Thus for any fixed $k > k_o$ the operator F_1 has a unique fixed point because of Banach Fixed Point Theorem.

THEOREM 4.2. *Under Hypothesis* (A_1), (A_4) *and* (A_5) *the boundary value problem* $u_{12}= f$ *a.e. in* Δ; $u_1(\cdot,g_1(\cdot))= \alpha_o(\cdot).u(\cdot,g_1(\cdot)) + \alpha_2(\cdot)$ *a.e. in* I_1; $u_2(g_2(\cdot),\cdot)= \beta_o(\cdot).u(g_2(\cdot),\cdot) + \beta_2(\cdot)$ *a.e. in* I_2; $u(0,0)= \gamma$, *has a unique solution in the class* $U_{\bar{p}}$.

Proof. This theorem is proven as the preceding one considering the operator $F_2\tau= J\tau + \tau_o$ from the space $L_p(I_1,X) \times L_p(I_2,X)$ into itself, where

$$\tau_o=\begin{pmatrix}\alpha_o(x_1)(J_1J_2f.(x_1,g_1(x_1)) + \gamma) + \alpha_2(x_1)\\ \beta_o(x_2)(J_1J_2f.(g_2(x_2),x_2) + \gamma) + \beta_2(x_2)\end{pmatrix}$$

REMARK. If in the Cauchy-Goursat problem we let $\alpha_o= 0$, $\beta_o= 0$, then we cannot weaken the conditions on g_i(i= 1,2) as we did in Theorem 4.2.

THEOREM 4.3. *If* g_i(i=1,2) *are non-decreasing functions, and* f, α_2, β_2, γ *are as in Hypothesis* (A_5) *then the boundary value problem* $u_{12}= f$ *a.e. in* Δ, $u_1(\cdot,g_1(\cdot))= \alpha_2(\cdot)$ *a.e. in* I_1, $u_2(g_2(\cdot),\cdot)=\beta_2(\cdot)$ *a.e. in* I_2, $u(0,0)= \gamma$, *has a unique solution in the class* $U_{\bar{p}}$.

Proof. From the isomorphism of the spaces $U_{\bar{p}}$ and $W_{\bar{p}}$ it follows that the unique solution of our boundary value problem in the class $U_{\bar{p}}$ is $u= J_1J_2f + J_1\alpha_2 + J_2\beta_2 + \gamma$.

5. CONTINUOUS DEPENDENCE OF THE SOLUTION ON THE INITIAL DATA FOR THE CAUCHY-GOURSAT PROBLEM IN THE CLASS $U_{\bar{p}}$.

Throughout this section we assume that the functions g_i(i=1,2) satisfy Hypothesis (A_3) and (A_4). Let V_1 be the subset of V such that the coordinates α_1 and β_1 satisfy the conditions specified in Hypothesis (A_6).

DEFINITION 5.1. *We define the operator* $S:V_1 \to U_{\bar{p}}$ *as follows* :

$S(v)= u$, $v \in V_1$, *if and only if,* u *is the unique solution of the Cauchy-Goursat problem corresponding to the initial data* v.

Note that S is a well defined operator because of Theorem 4.1. It is easy to show that V_1 is a normed space. In $U_{\bar{p}}$ consider the norm introduced in Corollary 1.1.

To prove continuous dependence of the solution on the initial data for the Cauchy-Goursat problem in the class $U_{\bar{p}}$ is equivalent to show that the operator S is continuous.

Take $v_n= (f,\alpha_{on},\alpha_{1n},\alpha_{2n},\beta_{on},\beta_{1n},\beta_{2n},\gamma_n)$ in V_1 and $v= (f,\alpha_o,\alpha_1,\alpha_2,\beta_o,\beta_1,\beta_2,\gamma)$ in V_1 such that $|v_n-v| \to 0$ as $n \to \infty$. Let $S(v_n)= u_n$, $S(v)= u$. We know that there exists $(s_n,\phi_n,\psi_n,\bar{\gamma}_n) \in W_{\bar{p}}$, $(s,\phi,\psi,\bar{\gamma}) \in W_{\bar{p}}$ such that $F(u_n)= (s_n,\phi_n,\psi_n,\bar{\gamma}_n)= (f_n,\phi_n,\psi_n,\gamma_n)$, $F(u)= (s,\phi,\psi,\bar{\gamma})= (f,\phi,\psi,\gamma)$. Taking $|u_n-u|= \max(|f_n-f|_\infty, |\phi_n-\phi|_p, |\psi_n-\psi|_p, \|\gamma_n-\gamma\|)$ it is evident that the operator S is continuous if $|\phi_n-\phi|_p \to 0$, $|\psi_n-\psi|_p \to 0$ as $n \to \infty$.

For each $(f_n,\phi_n,\psi_n,\gamma_n)$, $n= 1,2,....$, we can write an equation of the form (4.1).

Letting τ_n, τ_{on} be as in equation (3.1) we have (5.1): $\tau_n= (\dot{I}-H)^{-1}J\tau_n + (\dot{I}-H)^{-1}\tau_{on}$. Similarly for (f,ϕ,ψ,γ) we have τ and τ_o such that (5.2): $\tau= (\dot{I}-H)^{-1}J\tau + (\dot{I}-H)^{-1}\tau_o$.

LEMMA 5.1. *If* $|\tau_{on}-\tau_o|_p \to 0$ *as* $n \to \infty$ *then* $|\tau_n-\tau|_p \to 0$ *as* $n \to \infty$.

Proof. From equations (5.1) and (5.2) we obtain $\tau_n-\tau=(\dot{I}-H)^{-1}J(\tau_n-\tau)+$ $+ (\dot{I}-H)^{-1}(\tau_{on}-\tau_o)$.
Using the properties of the operators $(\dot{I}-H)^{-1}$ and J already established the lemma is proven.

LEMMA 5.2. $|\tau_{on}-\tau_o|_p \to 0$ *as* $n \to \infty$.

Proof. Let $\tau_{on}-\tau_o= \begin{pmatrix} \bar{\phi}_n \\ \bar{\psi}_n \end{pmatrix}$. If $v_n \to v$ as $n \to \infty$, then $|\bar{\phi}_n|_p$ and $|\bar{\psi}_n|_p$ converge toward 0 as $n \to \infty$. Hence, $|\tau_{on}-\tau_o|_p= \max(|\bar{\phi}_n|_p,|\bar{\psi}_n|_p) \to 0$ as $n \to \infty$.

THEOREM 5.1. *The operator* S *is continuous. Moreover for all* $\varepsilon > 0$

there exists $\delta > 0$ *such that if* u, $\bar{u}$ *are solutions of the Cauchy-Goursat boundary problem corresponding to initial datas* v, $\bar{v} \in V_1$ *respectively, then* $|u(x_1,x_2)-\bar{u}(x_1,x_2)| < \varepsilon$ *for all* $(x_1,x_2) \in I_1 \times I_2$ *if* $|v-\bar{v}| < \delta$.

Proof. The continuity of the operator S follows from the considerations made in this section and Lemmas 5.1 and 5.2.

Let $(s,\phi,\psi,\gamma) \in W_{\bar{p}}$, $(\bar{s},\bar{\phi},\bar{\psi},\bar{\gamma}) \in W_{\bar{p}}$ be such that $F(u) = (s,\phi,\psi,\gamma) = (f,\phi,\psi,\gamma)$; $F(\bar{u}) = (\bar{s},\bar{\phi},\bar{\psi},\bar{\gamma}) = (\bar{f},\bar{\phi},\bar{\psi},\bar{\gamma})$. One can prove that

(5.3) $|u(x_1,x_2)-\bar{u}(x_1,x_2)| \leqslant K|u-\bar{u}|$ for all $(x_1,x_2) \in I_1 \times I_2$

where $K = a_1 a_2 + a_1^{(p-1)/p} + a_2^{(p-1)/p} + 1$.

From (5.3) it follows that $\|u-\bar{u}\|_\infty \leqslant K|u-\bar{u}|$. The continuity of the operator S implies that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|S(v)-S(\bar{v})| = |u-\bar{u}| < \varepsilon/K$ if $|v-\bar{v}| < \delta$. Hence $\|u-\bar{u}\|_\infty \leqslant K|u-\bar{u}| < \varepsilon$ if $|v-\bar{v}| < \delta$. This completes the proof of the theorem.

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CORRIGENDUM: "NORM LIMITS OF NILPOTENT OPERATORS AND WEIGHTED SPECTRA IN NON-SEPARABLE HILBERT SPACES"

Domingo A. Herrero

There is a mistake in the proof of *Theorem* 5 and its non-separable analog *Corollary* 8 of [1]. The author was unable to obtain fair proofs of these two results. However, the other results of the paper remain true, after a few minor modifications of their proofs.

Proof of Corollary 7. Replace the sentence "By ([1],Theorem 2.2), ... it can be obtained that $\Pi_{\aleph_o}(T_{\nu k,2}) = \Lambda(N_{\nu k})$." by the following:

By ([1],Theorem 2.2), there exist operators $T_{\nu k}' \in \mathcal{L}(\mathcal{H}_{\nu k})$ such that $T_{\nu k}-T_{\nu k}'$ is a compact operator of norm smaller than $\varepsilon/4$ and $\mathcal{H}_{\nu k}$ admits the decompositions $\mathcal{H}_{\nu k} = \mathcal{H}_{\nu k,1} \oplus \mathcal{H}_{\nu k,2}' \oplus \mathcal{H}_{\nu k,2}'' = \mathcal{H}_{\nu k,1} \oplus \mathcal{H}_{\nu k,2}$ with respect to which

$$T_{\nu k}' = \begin{pmatrix} N_{\nu k} & 0 & T_{\nu k,1} \\ 0 & N_{\nu k} & T_{\nu k,2}' \\ 0 & 0 & T_{\nu k,2}'' \end{pmatrix} = \begin{pmatrix} N_{\nu k} & T_{\nu k,1} \\ 0 & T_{\nu k,2} \end{pmatrix}$$

where $N_{\nu k}$ is a normal operator such that $\Lambda(N_{\nu k}) = E(N_{\nu k}) = \Pi_{\aleph_o}(T) = \Pi_{\aleph_o}(T_{\nu k,2})$, and

$$T_{\nu k,2} = \begin{pmatrix} N_{\nu k} & T_{\nu k,2}' \\ 0 & T_{\nu k,2}'' \end{pmatrix}$$

with respect to the decomposition $\mathcal{H}_{\nu k,2} = \mathcal{H}_{\nu k,2}' \oplus \mathcal{H}_{\nu k,2}''$.

Also, replace the last two sentences "Moreover, Theorem 5 ... for all λ." by:

Moreover, it is clear that, in this case, $T_{\nu k,2}$ is also bi-quasi-triangular and $E(T_{\nu k,2}) = E(N_{\nu k})$. Then, our previous arguments show that T' actually satisfies the condition (vi) too. In fact, $\text{ind}(\lambda - T_2) = 0$ for all λ.

Finally, replace the section "*Sufficiency for the case* (ii)" by this new one:

Sufficiency for the case (ii). In this case the proof follows exactly as in the case (i). We only have to observe that if $\text{ind}(\lambda - A) = 0$ for all complex λ and $\Lambda(A)$ and $\Lambda_\alpha(A)$, $\aleph_0 \leq \alpha \leq h$, are connected sets containing the origin, then the A' of (i') satisfies the following property: $\Lambda(N_j) = \Lambda_h(N_j)$ is connected and contains the origin. Hence, N_j (and a fortiori $N_j \oplus D_j$) can be uniformly approximated by nilpotent operators in $\mathcal{L}(\mathcal{H}_j)$, $j=1,2,3,4$, whence it follows exactly as in the *Proof* of (i') that $A' \in$ $\in \underset{\sim}{N}(\mathcal{H})^-$. Therefore, A also belongs to $\underset{\sim}{N}(\mathcal{H})^-$.

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