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STRICTLY CYCLIC WEIGHTED SHIFTS

Domingo A. Herrero

ABSTRACT. A method is given to construct a strictly cyclic bilateral weighted shift on ℓ^2 from a strictly cyclic unilateral shift with non-increasing sequence of weights. It is also shown that a unilateral weighted shift in ℓ^p whose weights either decrease to 0, or decrease to 1 and satisfy the boundedness condition $\sum_0^\infty (a_1 a_2 \ldots a_n)^{-\varepsilon} < \infty$ for all $\varepsilon > 0$ is not necessarily stricly cyclic for any p, $1 < p < \infty$.

1. INTRODUCTION.

A *weighted shift* T in the complex Banach space $\ell^p(K)$, $1 \leq p < \infty$, is a (bounded linear) operator defined by the equations $Te_n = a_n e_{n+1}$, where $\{e_n\}$ is the canonical basis (i.e., e_n is the sequence $\{\delta_{nk}\}_{k\in K}$, where δ_{nk} denotes the Kronecker's delta function) and $\{a_n\}$ is a (necessarily bounded) sequence of positive reals. If n runs over the set $K = N$ of all non-negative integers ($K = Z$ of all integers), then T is called a *unilateral* (*bilateral*, resp.) weghted shift.

Given $A \in L(\ell^2)$ (= the algebra of all operators acting on ℓ^2), let $A(A)$ ($A^a(A)$) and $A'(A)$ denote the weak closure of the polynomials (the rational functions with poles off the spectrum $\Lambda(A)$ of A, resp.) in A and the commutant of A in $L(\ell^2)$, respectively. Assume that there exists a vector x in ℓ^2 such that $A(A)x = \{Lx: L \in A(A)\}$ ($A^a(A)x$) coincides with ℓ^2; then A is said to be a *strictly cyclic* (*analytically strictly cyclic*, resp.) operator. It is well known (see [7]) that a strictly cyclic unilateral weighted shift (analytically strictly cyclic bilateral weighted shift B) always satisfies the following: $A(T) = A^a(T) = A'(T)$ (B is invertible and $A(B) \neq A^a(B) = A'(B)$; moreover, $A'(B)$ is the norm closure of the polynomials in B and B^{-1}) and the Gelfand spectrum of the Banach algebra $A(T)$ ($A'(B)$) can be naturally identified with $\Lambda(T)$ ($\Lambda(B)$, respectively).

Strictly cyclic unilateral weighted shifts (SCUWS) have been analyzed by several authors (see, e.g., [1];[2];[4];[7];[8];[9];[10];[11];[12];[13]), but very little is known about analytically strictly cyclic bilateral weighted shifts (ASCBWS). In *section* 2 a method will be given to construct an ASCBWS in $\ell^2(Z)$ by using a SCUWS in $\ell^2(N)$ whose weights are bounded below from zero and satisfy a certain boundedness condition (This result applies, in particular, if the weights of the SCUWS form a non-increasing sequence converging to some positive number).

In *section* 3 a different kind of problem is analyzed: It is shown that there exist UWS in $\ell^p(N)$ whose weights either decrease to 0, or decrease to 1 and satisfy the condition $\sum_0^\infty (a_0 a_1 \ldots a_n)^{-\varepsilon} < \infty$, which are not strictly cyclic for any p, thus answering in the negative a question of A.L.Shields ([13]).

2. A CLASS OF SYMMETRIC ASCBWS.

Let B be a BWS with weight sequence $\{a_n\}_{n\in Z}$ and define $w_0 = 1$, $w_n = a_0 a_1 \ldots a_{n-1}$, $w_{-n} = (a_{-1} a_{-2} \ldots a_{-n})^{-1}$ for $n > 0$. For T a UWS, the sequence $\{w_n\}_{n\in N}$ is similarly defined.

THEOREM 1. *Let* T *be a SCUWS with weights* $\{a_n\}_{n\in N}$ *such that*

$$\text{spectral radius (T)} = \lim_{n\to\infty} (\sup_k w_{n+k}/w_k)^{1/n} = 1 \qquad (1)$$

and

$$w_n/w_{n+k} = (a_n a_{n+1} \ldots a_{n+k-1})^{-1} \leqslant C \qquad (2)$$

for some constant $C \geqslant 1$, *and for all* $n,k \in N$.

Define the BWS B *by*

$$Be_n = \begin{cases} a_n e_{n+1}, & \text{if } n \in N \\ (1/a_{-n}) e_{n+1}, & \text{if } n \in Z\setminus N \end{cases}$$

Then B *is analytically strictly cyclic and* $\Lambda(B)$ *is the boundary* ∂D *of the unit disc* D.

Proof. Clearly, $w_n(B) = w_{-n}(B) = w_n(T)$ for all $n \in N$. Let R be the unitary map defined by $Re_n = e_{1-n}$; then $R = R^{-1}$ and $B^{-1} = RBR$; in particular, B is invertible.

Let $A \in A'(B)$; then A is the strong limit of the sequence of Cesàro averages of a formal Laurent series $\sum_{n\in Z} c_n B^n$ (see [5];[6]). B is ASC if and only if, given $x = \sum_{n\in Z} b_n e_n$, there exists an A_x in $A'(B)$ such that $A_x e_0 = x$ (see [7]). This is equivalent to say that the central column of the matrix of A_x (with respect to the canonical basis) coincides with the column vector x. By using the fact that $w_n = w_{-n}$, the matrix of A = (formally) $\sum_{n\in Z} c_n B^n$ is equal to :

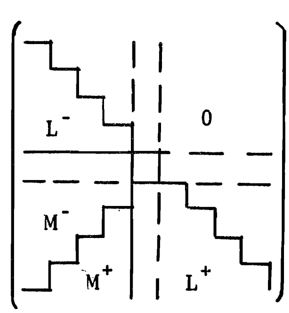
(i.e., the (j,k)-entry is equal to $c_{j-k} w_j/w_k$ for all $j,k \in Z$). The dotted lines remark the central column and the central row.

$$A = \left(\begin{array}{ccc|c|ccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & c_{-1}w_3/w_2 & c_{-2}w_3/w_1 & c_{-3}w_3 & c_{-4}w_3/w_1 & c_{-5}w_3/w_2 & \cdots \\
\cdots & c_0 & c_{-1}w_2/w_1 & c_{-2}w_2 & c_{-3}w_2/w_1 & c_{-4} & \cdots \\
\cdots & c_1w_1/w_2 & c_0 & c_{-1}w_1 & c_{-2} & c_{-3}w_1/w_2 & \cdots \\
\hline
\cdots & c_2/w_2 & c_1/w_1 & c_0 & c_{-1}/w_1 & c_{-2}/w_2 & \cdots \\
\hline
\cdots & c_3w_1/w_2 & c_2 & c_1w_1 & c_0 & c_{-1}w_1/w_2 & \cdots \\
\cdots & c_4 & c_3w_2/w_1 & c_2w_2 & c_1w_2/w_1 & c_0 & \cdots \\
\cdots & c_5w_3/w_2 & c_4w_3/w_1 & c_3w_3 & c_2w_3/w_1 & c_1w_3/w_2 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right) \qquad (3)$$

Let $A_+ = \sum_{n\in N} c_n B^n$, $A_- = \sum_{n\in Z\setminus N} c_n B^n$, $x_+ = \sum_{n\in N} b_n e_n$ and $x_- = \sum_{n\in Z\setminus N} b_n e_n$. Then $A =$ (formally) $A_+ + A_-$ and $x = x_+ + x_-$. Assume that A_+ and A_- actually define bounded linear maps; then it is clear that $A_+e_0 \in \ell^2(N)$, while $A_-e_0 \in \ell^2(Z\setminus N)$. Thus, in order to complete the proof, it suffices to show that if $c_n w_n = b_n$ for all $n \in Z$, then A_+ is a bounded linear map in $A(B)$ and A_- is a bounded linear map in $A(B^{-1})$, whence it readily follows that $A = A_+ + A_-$ and $Ae_0 = A_+e_0 + A_-e_0 = x_+ + x_- = x$.

The matrix of A_+ is obtained from (3) by replacing the c_n's by 0's for all negative n. Decompose this matrix according to the heavy lines of (3). Then $A_+ =$ (formally) $L^- + L^+ + M^- + M^-$, where $L^-(L^+)$ is the upper left (lower right, resp.) quarter of A_+ and $M^-(M^+)$ is the upper (lower, resp.) part of the lower left quarter of A_+ (and the remaining entries of L^-, L^+, M^- and M^+ are 0's).

Schematically, we have

$A_+ =$ 

$L^+:\ell^2(N) \to \ell^2(N)$, $L^-:\ell^2(Z\setminus N) \to \ell^2(Z\setminus N)$ and $M^-(M^+):\ell^2(Z\setminus N) \to \ell^2(N)$.

$L^+ = A_+|\ell^2(N) \in A'(T)$ (The vertical bar denotes "restriction") and therefore, since T is a SCUWS and the first column of L^+ belongs to $\ell^2(N)$, L^+ is actually bounded ([9]).

Condition (1) implies that $\Lambda(T) = D^- = \{z: |z| \leq 1\}$ = point spectrum of T*, the adjoint of T (see [3];[7];[9]). Thus, in particular, $\{1/w_n\}_{n\in N} \in \ell^2(N)$ and $|c_n| \leq \|x\|/w_n \leq C'$ for a suitable constant $C' \geq 1$ and for all $n \in Z$. Consider L^-; clearly, $\|L^-\| \leq \sum_{n\in N} \|L^-(n)\|$, where $L^-(n)$ is obtained from L^- by replacing c_m by 0 for every $m \neq n$, $n = 0,1,2,\dots$. Hence, by (2),

$$\|L^-\| \leq \sum_{n\in N} |c_n| \max\{w_j/w_{n+j};\ j \in N\} \leq C \sum_{n\in N} |c_n| \leq$$

$$\leq C\{\sum_{n\in N} (|c_n| w_n)^2\}^{1/2}\{\sum_{n\in N} w_n^{-2}\}^{1/2} < \infty$$

Similarly, by considering the formal decomposition $M^- = \sum_{n\in N\setminus\{0\}} M^-(n)$, where $M^-(n)$ is that part of the matrix M^- corresponding to a fixed coefficient c_n, it is not difficult to conclude that M^- is also bounded.

It is easy to see that M^+ has the same norm as the operator $M: \ell^2(N) \to \ell^2(N)$ defined by the matrix

$$M = \begin{pmatrix} c_2 & c_3 w_2/w_1 & c_4 w_3/w_1 & c_5 w_4/w_1 & \cdots \\ 0 & c_4 & c_5 w_3/w_2 & c_6 w_4/w_2 & \cdots \\ 0 & 0 & c_6 & c_7 w_4/w_3 & \cdots \\ 0 & 0 & 0 & c_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Let $y = \sum_{n\in N} d_n e_n \in \ell^2(N)$; then $My = \sum_{n\in N} \{\sum_{k=n+1}^{\infty} c_{k+n}\, y_{k-1}\, w_k/w_n\}\, e_n$.Hence

$$\|M^+\|^2 = \|M\|^2 = \sup_{\|y\|=1} \sum_{n\in N} |\sum_{k=n+1}^{\infty} c_{k+n}\, y_{k-1} w_k/w_n|^2 \leq$$

$$\leq \sup_{\|y\|=1} \sum_{n\in N} \{\sum_{k=n+1}^{\infty} (|c_{k+n}|\, w_k/w_n)^2\}\ \{\sum_{k=n}^{\infty} |y_k|^2\} \leq$$

$$\leq \sum_{n\in N} \sum_{m\in N} (|c_m|\, w_{m-n}/w_n)^2 \leq C^2\{\sum_{n\in N} w_n^{-2}\}\ \{\sum_{m\in N} (|c_m|\, w_m)^2\} < \infty$$

The boundedness of A_- follows by a completely symmetric argument. The details are left to the reader.

3. NON-SCUWS WITH NON-INCREASING SEQUENCES OF WEIGHTS.

In ([13], Question 15), A.L.Shields asked the following: If $a_n \searrow 1$ and $\sum w_n^{-2} < \infty$, must T be a SCUWS (in $\ell^2(N)$)?.If $a_n \searrow 0$, must T be strictly cyclic?. It is worth to recall that both questions have an affirmative answer in $\ell^1(N)$ ([3];[5]; see also [1]). Nevertheless, the answer is NO in $\ell^p(N)$ for $1 < p < \infty$. We shall need the following result:

THEOREM 2. (E.Kerlin and A.L.Lambert, [8], Theorem 3.2). *If* $\{a_n\}$ *is monotonically non-increasing and* $Te_n = a_n e_{n+1}$, *then* T *is strictly cyclic in* $\ell^p(N)$ *if and only if*

$$\sup_{n\in N} \sum_{k=0}^{n} (w_n/w_k w_{n-k})^q < \infty \quad , \quad \textit{where} \quad q = p/(p-1).$$

COROLLARY. (i) *There exists a sequence* $a_n \searrow 1$ *such that* $\sum w_n^{-\varepsilon} < \infty$ *for every* $\varepsilon > 0$, *but* $Te_n = a_n e_{n+1}$ *does not define a SCUWS in* $\ell^p(N)$ *for any* p, $1 < p < \infty$.

(ii) *There exists a sequence* $a_n \searrow 0$ *such that* $Te_n = a_n e_{n+1}$ *does not define a SCUWS in* $\ell^p(N)$ *for any* p, $1 < p < \infty$.

Proof. (i) Set $a_0 = 2$, $n_0 = 0$, $n_1 = 1$, $n_{j+1} > 3n_j$ (n_j to be defined) for all $j = 1,2,\ldots$ and $a_n = 1 + 2^{-j}$ for $n_j \leq n < n_{j+1}$; then $a_n \searrow 1$. If $2n_j < n \leq n_{j+1} - n_j$ and $n_j < k \leq n - n_j$, we have

$$w_n/w_k w_{n-k} = (1 + 2^{-j})^{n_j}/w_{n_j} = c_j > 0$$

so that we can inductively define the n_j's in such a way that

$$\sum_{k=0}^{n_{j+1}} (w_{n_{j+1}}/w_k w_{n_{j+1}-k})^j \geq (n_{j+1} - 2n_j)(c_j)^j > j \text{ for all } j=1,2,\ldots$$

Clearly, the above inequalities remain true if the n_j's are replaced by m_j's so that $m_j > n_j$ and $m_{j+1}/m_j > n_{j+1}/n_j$, $j = 1,2,\ldots$. Therefore, without loss of generality we can assume that the n_j's tend to ∞ fast enough so that $a_n > [(n+2)/(n+1)]^{n^{1/2}}$, whence it follows that $\sum w_n^{-\varepsilon} < \infty$ for every $\varepsilon > 0$.

(ii) Take n_j as above, except that now we take $a_n = 2^{-1}$ for $n_j \leq n < n_{j+1}$; then $a_n \searrow 0$. For n and k as above, $w_n/w_k w_{n-k} = 2^{-jn_j}/w_{n_j} = d_j > 0$, so that we can inductively define the n_j's in such a way that

$$\sum_{k=0}^{n_{j+1}} (w_{n_{j+1}}/w_k w_{n_{j+1}-k})^j \geq (n_{j+1}-2n_j)(d_j)^j > j, \quad j = 2,3,\ldots .$$

The conclusion is the same in both cases: By THEOREM 2, T cannot be a SCUWS in $\ell^p(N)$, for $1 < p < \infty$.

THEOREM 2 and its COROLLARY are actually true for the space $c_0(N)$ of all sequences converging to 0, under the maximum norm, if the exponent q is replaced by 1. (The details for these changes are left to the reader).

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SPHERICAL FUNCTIONS

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INTRODUCTION.

The fundamental properties of spherical functions have been established by R. Godement in a well known paper [1] in 1952. There he defines in a general manner the notion of spherical function associated to an irreducible representation of a locally compact unimodular group G. Moreover, he gives a characterization of such functions as characters of certain subalgebras of the algebra of all continuous functions on G with compact support. For certain purposes it is best not to work with the characters of such subalgebras but rather directly with their finite dimensional representations. This leads one to consider spherical functions with values in the endomorphism ring of a finite dimensional vector space and not just complex valued functions.

Despite the importance of the close connection between the spherical functions and the representations of G, and being of interest in their own right, it is desirable to have an intrinsic definition for the important notion of spherical function. Such definition is given and explored in §1.

In fact, it is possible to start from two different points which leads to the same concept. The reason of our choice is the existence of the general notion of μ-spherical function, where $\mu = (\mu_1,\mu_2)$ is a double representation of a compact subgroup K of G on a finite dimensional vector space E. By this one understands a continuous function ϕ from G to E such that

$$\phi(k_1gk_2) = \mu_1(k_1)\phi(g)\mu_2(k_2) \qquad (k_1,k_2 \in K;\ g \in G).$$

In §2 we establish the close connection between the spherical functions and the representations of certain algebras of functions on G, from which the most important properties of spherical functions follow. In §3 we discuss thoroughly the relation between the two different view points we mentioned above. In §4 we study the differential properties of spherical functions on Lie groups.

Since we have dropped every irreducibility assumption, some interesting questions naturally arise. For example, we don't know if any spherical function is associated to a representation of G. If G is a compact

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group, then we know that any spherical function is a direct sum of irreducibles, because if $\phi: G \rightarrow End(V)$ is spherical there is an inner product (,) on V such that

$$(\phi(g)v_1,v_2) = (v_1,\phi(g^{-1})v_2) \qquad (v_1,v_2 \in V \ ; \ g \in G)$$

In some other place we shall be concern with local spherical functions and complete reducibility of spherical functions on semi-simple Lie groups.

1. Throughout this paper we shall denote by G a locally compact unimodular group and by K a compact subgroup of G. We shall often use the following notation: if X denotes a group, then x will denote a generic element of X and e will denote the identity element of X.

Let $\hat{K}$ denote the set of all equivalence classes of finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field C of all complex numbers and by End(V) the space of all endomorphisms of V. Whenever we shall refer to a topology on such vector spaces we shall be talking about the unique Hausdorff linear topology on them.

By definition, a *zonal spherical function* φ on G is a continuous, complex valued function which satisfies $\varphi(e) = 1$ and

$$(1) \qquad \int_K \varphi(xky)dk = \varphi(x)\varphi(y) \qquad x,y \in G$$

A fruitful generalization of the functional equation above is the equation

$$(2) \qquad \int_K \chi_\delta(k^{-1})\phi(xky)dk = \phi(x)\phi(y) \qquad x,y \in G$$

whose End(V)-valued solutions will be called spherical functions on G.

The purpose of this paper, then, is to present in a systematic fashion the generalities which lie at the basis of the theory of spherical functions on those pairs (G,K) where G is a locally compact unimodular group, K a compact subgroup of G.

DEFINITION 1.1. *Let* $\delta \in \hat{K}$. *A spherical function* ϕ *(on* G*) of type* δ *is a continuous function on* G *with values in* End(V) *such that:*

(i) $\phi(e) = I$ (I = identity) ;

(ii) $\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk$ for all $x,y \in G$.

PROPOSITION 1.2. *If* $\phi: G \rightarrow End(V)$ *is a spherical function of type* δ *then:*

(i) $\phi(kgk') = \phi(k)\phi(g)\phi(k')$ *for all* $k,k' \in K$, $g \in G$;

(ii) $k \mapsto \phi(k)$ *is a representation of* K *such that any irreducible subrepresentation belongs to* δ.

Proof. (i) Let $k' \in K$ and $g \in G$. Then we have from the definition

$$\phi(k'g) = \phi(e)\phi(k'g) = \int_K \chi_\delta(k^{-1})\phi(kk'g)dk$$

by the symmetry of χ_δ we can interchange k and k'

$$\phi(k'g) = \int_K \chi_\delta(k^{-1})\phi(k'kg)dk = \phi(k')\phi(g) .$$

In a similar way it follows that $\phi(gk') = \phi(g)\phi(k')$, which completes the proof of (i).

(ii) Since $\phi(e) = I$, (i) implies that $\phi(kk') = \phi(k)\phi(k')$; that ϕ is continuous is obvious, therefore, $k \mapsto \phi(k)$ is a representation of K. Now,

$$I = \phi(e)\phi(e) = \int_K \chi_\delta(k^{-1})\phi(k)dk$$

but, it is well-known that the right hand side is a projection of V onto the space of all vectors which under $k \mapsto \phi(k)$ transform according to δ. This proves (ii).

Concerning the definition let us point out that the spherical function ϕ determines its type univocaly and let us say that the number of times that δ occurs in the representation $k \mapsto \phi(k)$ is called the *height* of ϕ.

Whenever K is a central subgroup of G (i.e. K is contained in the center of G) and ϕ is a spherical function, we have

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk = \int_K \chi_\delta(k^{-1})\phi(k)\phi(xy)dk = \phi(xy) \ , \ x,y \in G$$

in other words, ϕ is a representation of G. Therefore, if we take K reduced to the identity, the spherical functions are precisely the finite dimensional representations of G, and if G is abelian the spherical functions are the finite dimensional representations of G such that 1.2 (ii) is satisfied.

Another extreme case occurs when G is compact and K = G. In this case the spherical functions are also the finite dimensional representations of G, with all their irreducible subrepresentations equivalent.

The function $0: G \to End(V)$ identically zero satisfies the functional equation 1.1 (ii) for any $\delta \in \hat{K}$. If $K = \{e\}$ the functional equation reduces to $\phi(x,y) = \phi(x)\phi(y)$ which implies that $\phi(e)$ is a projection commuting with all $\phi(G)$. Let V_1 and V_2 be respectively the kernel and the image of $\phi(e)$. Then $V = V_1 \oplus V_2$; if we write $\phi = \phi_1 \oplus \phi_2$ accordingly, we have that ϕ_2 is spherical while ϕ_1 is identically

zero. For a moment one may think that something of this sort happens in general with the solutions of 1.1 (ii). But, the following example will show that this is not the case. Let G = R* be the multiplicative group of all non-zero real numbers and let K = {1,-1}. The two possible irreducible characters $\chi_\pm$ of K are given by $\chi_\pm(-1) = \pm 1$.
Let ϕ: R* $\longrightarrow M_2(C)$ be of the form

$$\phi(g) = \begin{pmatrix} 0 & f(g) \\ 0 & 0 \end{pmatrix}$$

where f: R* $\longrightarrow$ C is continuous. Then ϕ satisfies the functional equation with the character χ_+(resp. χ_-) if and only if f is an odd (resp. even) function.

Later on we shall prove (see Lemma 4.1) that, when G is a Lie group, every spherical function is C^∞ (moreover analytic). Therefore one cannot expect to "build up" the solutions of 1.1 (ii) out of spherical functions and "elementary functions".

Let φ be a complex valued continuous solution of the equation (1). If φ is not identically zero then $\varphi(e) = 1$ (cf. Helgason [1,p.399]). This result generalizes in the following way: we shall say that a function ϕ: G $\longrightarrow$ End(V) is *irreducible* whenever ϕ(G) is a non-trivial irreducible family of endomorphisms of V; then, we have

PROPOSITION 1.3. *Let* ϕ *be an* End(V)*-valued continuous solution of the equation* (2). *If* ϕ *is irreducible then* $\phi(e) = I$.

Proof. Let W_v denote the vector space spanned by $\{\phi(g)v: g \in G\}$. Now

$$\phi(x)\phi(y)v = \int_K \chi_\delta(k^{-1})\phi(xky)v\,dk \in W_v$$

which shows that W_v is ϕ(G)-invariant, therefore W_v is either {0} or V. We also have

$$\phi(x)\phi(e)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xk)\phi(y)\,dk = \int\int_{K\times K} \chi_\delta(k^{-1})\chi_\delta(k_1^{-1})\phi(xkk_1y)\,dk\,dk_1 =$$

$$= \int_K \Big(\int_K \chi_\delta(k^{-1})\chi_\delta(k_1^{-1}k)\,dk\Big)\phi(xk_1y)\,dk_1 =$$

$$= \int_K \chi_\delta(k_1^{-1})\phi(xk_1y)\,dk_1 = \phi(x)\phi(y)$$

where we have used that $\chi_\delta * \chi_\delta = \chi_\delta$ (orthogonality relations).
From this and what we observed before it follows that $\phi(g)\phi(e) = \phi(g) = \phi(e)\phi(g)$, all $g \in G$. Hence, $\phi(e)$ is a projection which commutes with every $\phi(g)$, therefore $\phi(e) = I$.

Spherical functions of type δ arise in a natural way upon consideration of representations of G. We recall that a continuous representation of G on a locally convex, Hausdorff, topological vector space E over C is a homomorphism $g \longmapsto U(g)$ of G into the group of topological automorphisms of E, such that, the map $(g,a) \mapsto U(g)a$ of $G \times E$ into E is continuous. We also want to be able to lift U in the well-known way to a homomorphism $\mu \mapsto U(\mu)$ of the algebra $M_c(G)$ of Radon measures on G with compact support, into the algebra of continuous linear operators on E. Thus we want that the integral

$$U(\mu)a = \int_G U(g)a \, d\mu(g)$$

defines an element in E for every $a \in E$. This will be the case if we assume for example that E is complete.

Let $P(\delta)$ be defined by

$$P(\delta) = U(\overline{\chi}_\delta) = \int_K \overline{\chi}_\delta(k)U(k)dk$$

$P(\delta)$ is a continuous projection of E onto $P(\delta)E = E(\delta)$; $E(\delta)$ consists of those vectors in E, the linear span of whose K-orbit is finite dimensional and splits into irreducible K-submodules of type δ. Whenever $E(\delta)$ is finite dimensional, the function $\phi: G \to \mathrm{End}(E(\delta))$ defined by $\phi(g)a = P(\delta)U(g)a$, $g \in G$, $a \in E(\delta)$, is spherical of type δ. In fact, if $a \in E(\delta)$ we have

$$\phi(x)\phi(y)a = P(\delta)U(x)P(\delta)U(y)a = \int \overline{\chi}_\delta(k)P(\delta)U(x)U(k)U(y)a \, dk =$$

$$= \left(\int_K \chi_\delta(k^{-1})\phi(xky)dk\right)a$$

($\overline{\chi}_\delta(k) = \chi_\delta(k^{-1})$ for all $k \in K$).

In the next paragraph we shall consider the question of seeing when a spherical function is obtained in this way.

There is an important class of pairs (G,K), namely those where K is a large compact subgroup of G, where the above construction works. A compact subgroup K of G is said to be large (in G) if for each $\delta \in \hat{K}$ there exists an integer $m(\delta) \geqslant 1$ such that $\dim E(\delta) \leqslant m(\delta)$ in every topologically completely irreducible Banach representation (E,U) of G. Examples of groups which admit large compact subgroups include the connected semisimple Lie groups with finite center and the motion groups (cf. Warner [1], §4.5).

If the representation $g \mapsto U(g)$ is topologically irreducible (i.e. U admits no non-trivial closed G-invariant subspace) then the associated spherical function ϕ is also irreducible. In fact, let W be a non-zero $\phi(G)$-invariant subspace of $E(\delta)$ and let $Q: E(\delta) \to W$ be a projection of $E(\delta)$ onto W. Then

$$0 = P(\delta)U(g)QP(\delta) - QP(\delta)U(g)QP(\delta) = (I-Q)P(\delta)U(g)QP(\delta)$$

(I = identity transformation of $E(\delta)$). Since the vectors $U(g)a$, $g \in G$, $a \in W$, span a dense subspace of E, it follows that I = Q which proves our assertion.

2. THE ALGEBRAS $C_{c,\delta}(G)$ AND THEIR REPRESENTATIONS.

We consider the given group G, its compact subgroup K and the function χ_δ, $\delta \in \hat{K}$, introduced before.

We shall denote by $M_c(G)$ (resp. $C_c(G)$) the algebra, with respect to convolution "*", of Radon measures (resp. continuous functions) on G with compact support, and by $M_\omega(G)$ (resp. $C_\omega(G)$) the space of Radon measures (resp. continuous functions) on G with support contained in the compact subset ω of G. We shall equip $M_c(G)$ (resp. $C_c(G)$) with the inductive limit of the topologies defined by the norm on the spaces $M_\omega(G)$ (resp. $C_\omega(G)$). We shall always identify a measure $\alpha \in M(K)$ on K with the measure $\alpha \in M_c(G)$ on G given by $f \mapsto \int_K f(k)d\alpha(k)$; in this way we get an isomorphism of the algebra M(K) into the algebra $M_c(G)$. We shall choose once and for all a left Haar measure on G, and we shall always identify every continuous function f(g) with the corresponding measure f(g)dg. In the same way, every continuous function on K will be identified with a measure on K, hence with a measure on G.

It is well-known that $C_c(G)$ is a two-sided ideal in $M_c(G)$, and it is clear that

$$(\alpha * f)(e) = (f * \alpha)(e)$$

for all $\alpha \in M_c(G)$ and all $f \in C_c(G)$. We shall also use for measures the operation $\alpha \rightarrow \check{\alpha}$; $\check{\alpha}$ is the transform of α under $g \mapsto g^{-1}$. In particular, if $f \in C_c(G)$, $\check{f}(g) = f(g^{-1})$ and $\check{\alpha}(f) = \alpha(\check{f})$ for all $\alpha \in M_c(G)$, $f \in C_c(G)$. Of course we have

$$(\alpha * \beta)^\vee = \check{\beta} * \check{\alpha}$$

for all $\alpha,\beta \in M_c(G)$.

Now, we may consider the set $C_{c,\delta}(G)$ of those $f \in C_c(G)$ which satisfy $\overline{\chi}_\delta * f = f = f * \overline{\chi}_\delta$; since $\chi_\delta * \chi_\delta = \chi_\delta$ (orthogonality relations), it is clear that $C_{c,\delta}(G)$ is a subalgebra of $C_c(G)$ and that $f \mapsto \overline{\chi}_\delta * f * \overline{\chi}_\delta$ is a continuous projection of $C_c(G)$ onto $C_{c,\delta}(G)$. We shall consider $C_{c,\delta}(G)$ as a topological subspace of $C_c(G)$.

We are in a position to take up a very important result, which establishes a close connection between spherical functions of type δ and representations of the algebra $C_{c,\delta}(G)$.

THEOREM 2.1. *If* Φ *is a spherical function on* G *of type* δ*, then the mapping*

$$\Phi : f \mapsto \int_G f(g)\Phi(g)dg$$

is a continuous finite dimensional representation of $C_{c,\delta}(G)$ *such that* $I \in \Phi(C_{c,\delta}(G))$. *Conversely, if* L *is a continuous finite dimensional representation of* $C_{c,\delta}(G)$ *such that* $I \in L(C_{c,\delta}(G))$ *then* L *is represen-*

ted as above by a spherical function of type δ.

Needless to say that if L is an irreducible finite dimensional representation of $C_{c,\delta}(G)$ then $I \in L(C_{c,\delta}(G))$ (Burnside's theorem).
The proof of this theorem requires the following proposition.

PROPOSITION 2.2. *Let* ϕ: $G \to \mathrm{End}(V)$ *be a continuous function such that* $\chi_\delta * \phi = \phi = \phi * \chi_\delta$. *Then* ϕ *satisfies the functional equation* 1. (2) *if and only if the mapping*

$$\phi: f \mapsto \int_G f(g)\phi(g)dg$$

is a representation of $C_{c,\delta}(G)$.

Proof. Let f,h be two functions in $C_c(G)$, then

$$\phi(f) = \int_G f(g)\phi(g)dg = (\phi * \check{f})(e)$$

Therefore

$$(1) \qquad \phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta) = (\phi * (\overline{\chi}_\delta * f * \overline{\chi}_\delta)\check{}\,)(e) = (\phi * \chi_\delta * \check{f} * \chi_\delta)(e) =$$
$$= (\phi * \check{f} * \chi_\delta)(e) = (\chi_\delta * \phi * \check{f})(e) = (\phi * \check{f})(e) = \phi(f)$$

we have used that $\overline{\chi}_\delta = \check{\chi}_\delta$, which is well-known. Now

$$(2) \qquad \phi((\overline{\chi}_\delta * f * \overline{\chi}_\delta) * (\overline{\chi}_\delta * h * \overline{\chi}_\delta)) = \phi(f * \overline{\chi}_\delta * h) = \int_G (f * \overline{\chi}_\delta * h)(y)\phi(y)dy =$$
$$= \int_G\int_G (f * \overline{\chi}_\delta)(x)h(x^{-1}y)\phi(y)dx\, dy =$$
$$= \int_G\int_G\int_K f(xk^{-1})\overline{\chi}_\delta(k)h(y)\phi(xy)dk\, dx\, dy =$$
$$= \iint_{G\times G} f(x)h(y)\left(\int_K \chi_\delta(k^{-1})\phi(xky)dk\right)dx\, dy .$$

On the other hand

$$(3) \qquad \phi(\overline{\chi}_\delta * f * \overline{\chi}_\delta)\phi(\overline{\chi}_\delta * h * \overline{\chi}_\delta) = \phi(f)\phi(h) = \iint_{G\times G} f(x)h(y)\phi(x)\phi(y)dx\, dy .$$

Considering (2) and (3), the proposition follows immediately.

Proof of Theorem 2.1. Let ϕ: $G \to \mathrm{End}(V)$ be a spherical function on G of type δ. Then, by Propositions 1.2 and 2.2 the mapping ϕ: $C_{c,\delta}(G) \to \mathrm{End}(V)$ is a representation of $C_{c,\delta}(G)$, which is obviously continuous.

In order to prove that $I \in \phi(C_{c,\delta}(G))$ we first notice that $\phi(C_{c,\delta}(G)) = \phi(C_c(G))$. The neighborhoods $\mathcal{O}$ of $g \in G$ form a directed system under inclusion, and if $f_{\mathcal{O}} \in C_c(G)$ is a nonnegative function with spt $f_{\mathcal{O}} \subset \mathcal{O}$

and satisfying $\int_G f_0(g)dg = 1$, then $f_0 * f \to \delta_g * f$ in $C_c(G)$ (δ_g is the Dirac measure at g). Then

$$(4)\qquad \phi(f_0) = (f_0 * \check{\phi})(e) \to (\delta_g * \check{\phi})(e) = \phi(g)$$

hence the linear span of $\{\phi(g): g \in G\}$ is contained in $\phi(C_c(G))$. Since the other inclusion is obvious we get

$$(5)\qquad \phi(C_{c,\delta}(G)) = \{\phi(g): g \in G\}_C$$

Now it is clear that $I \in \phi(C_{c,\delta}(G))$.

Conversely, let $L: C_{c,\delta}(G) \to \mathrm{End}(V)$ be a continuous representation of $C_{c,\delta}(G)$ such that $I \in L(C_{c,\delta}(G))$. The mapping $\phi: f \mapsto L(\overline{\chi}_\delta * f * \overline{\chi}_\delta)$ defines an End(V)-valued Radon measure on G. Let $h \in C_{c,\delta}(G)$ be an element such that $L(h) = I$, then

$$\phi(f) = L(\overline{\chi}_\delta * f * \overline{\chi}_\delta)L(h) = L(\overline{\chi}_\delta * f * h) = \phi(f * h) = (\phi * \check{h} * \check{f})(e) = (\phi * \check{h})(f)$$

for all $f \in C_c(G)$. Therefore $\phi = \phi * \check{h}$ is a continuous function on G which represents L. But we also have

$$(\chi_\delta * \phi * \chi_\delta)(f) = ((\chi_\delta * \phi * \chi_\delta) * \check{f})(e) = (\phi * (\overline{\chi}_\delta * f * \overline{\chi}_\delta)^{\vee})(e) = (\phi * \check{f})(e) = \phi(f)$$

if $f \in C_c(G)$, which implies that $\phi = \chi_\delta * \phi * \chi_\delta$. Hence by Proposition 2.2

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk .$$

In particular

$$(6)\qquad \phi(e)\phi(g) = (\chi_\delta * \phi)(g) = \phi(g) = (\phi * \chi_\delta)(g) = \phi(g)\phi(e)$$

hence $\phi(e)$ is an identity of $L(C_{c,\delta}(G))$ and therefore $\phi(e) = I$. This completes the proof of Theorem 2.1.

REMARK 2.3. Under the hypothesis of Proposition 2.2 the function ϕ is spherical of type δ if and only if the representation ϕ of the algebra $C_{c,\delta}(G)$ cannot be decomposed as a direct sum of two representations, one of which is the trivial zero representation. This follows at once from (6).

Let $\phi: G \to \mathrm{End}(V)$ be a spherical function of type δ. Then a direct consequence of (5) is that a subspace W of V is $\phi(G)$-invariant if and only if it is $\phi(C_{c,\delta}(G))$-invariant. In particular we have the following corollary:

COROLLARY 2.4. *A spherical function* $\phi: G \to \mathrm{End}(V)$ *is irreducible if and only if the linear span of* $\phi(G)$ *coincides with* End(V).

We shall say that the spherical functions $\phi: G \to \mathrm{End}(V)$ and

ϕ_1: G $\rightarrow$ End(V_1) are *equivalent* if there exists a linear isomorphism T of V onto V_1 such that $\phi_1(G) = T\phi(g)T^{-1}$ for all $g \in G$. It is clear that this equivalence relation preserves the type and the height of the spherical functions. Moreover we have

PROPOSITION 2.5. *The spherical functions* ϕ: G $\rightarrow$ End(V) *and* ϕ_1: G $\rightarrow$ End(V_1) *of type* δ *are equivalent, if and only if the corresponding representations* ϕ: $C_{c,\delta}(G) \rightarrow \mathrm{End}(V)$ *and* $\phi_1: C_{c,\delta}(G) \rightarrow \mathrm{End}(V_1)$ *are equivalent.*

Proof. Let T be an isomorphism of V onto V_1 such that $\phi_1(f) = T\phi(f)T^{-1}$ for all $f \in C_{c,\delta}(G)$. Then

$$\phi_1(f) = \phi_1(\overline{X}_\delta * f * \overline{X}_\delta) = T\phi(\overline{X}_\delta * f * \overline{X}_\delta)T^{-1} = T\phi(f)T^{-1}$$

for any $f \in C_c(G)$. Therefore $\phi_1(g) = T\phi(g)T^{-1}$, all $g \in G$.
The other assertion is obvious.

As a corollary of Theorem 2.1 and Proposition 2.5 we obtain the following result

THEOREM 2.6. *The irreducible spherical functions* ϕ *and* ϕ_1 *are equivalent if and only if* $\mathrm{tr}\,\phi(g) = \mathrm{tr}\,\phi_1(g)$ *for all* $g \in G$.

Proof. It is obvious that if ϕ and ϕ_1 are equivalent they have the same trace. Conversely, $\mathrm{tr}\,\phi(g) = \mathrm{tr}\,\phi_1(g)$, all $g \in G$, implies $\mathrm{tr}\,\phi(f) = \mathrm{tr}\,\phi_1(f)$ for all $f \in C_{c,\delta}(G)$. Since ϕ and ϕ_1 are two irreducible finite dimensional representations of an associative algebra over C with the same trace, they are equivalent. Hence, ϕ and ϕ_1 are equivalent.

REMARK 2.7. Theorem 2.6 does not hold in general if we drop the irreducibility hypothesis, because, it is not even true for finite dimensional representations. For example, the functions ϕ, ϕ_1: R $\rightarrow$ $M_2(C)$ defined by

$$\phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \phi_1(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x \in R$$

are two spherical functions of the pair (R,{0}) with the same trace which are not equivalent. But, as one can expect, when G is compact it is not necessary to assume that the spherical functions are irreducible for Theorem 2.6 to be true.

The possible heights of the various irreducible spherical functions ϕ on (G,K) are not entirely arbitrary. In order to clarify this, it is convenient to recall the following algebraic fact due to Kaplansky: let A be an associative algebra over C, and let n be a fixed integer; if there are enough representations of A of dimensions $\leq n$ to separate the points of A, then, every irreducible finite dimensional representation of A has dimension $\leq n$ (cf. Godement

[1], p.503). Some interesting examples of pairs (G,K) which have the property that $C_{c,\delta}(G)$ has a separating family of representations of dimensions $\leq n$ are:

(1) G is a motion group, i.e. G is the semi-direct product of a closed normal abelian subgroup H and a compact subgroup K;

(2) G is a connected semi-simple Lie group which admits a faithful finite dimensional representation and K is a maximal compact subgroup. In both cases it can be proved (cf. Godement [1], §1) that the integer n can be taken equal to $d(\delta)$. Therefore, the height of an irreducible spherical function ϕ on (G,K) ((G,K) as in (1) or (2)) of type δ is $\leq d(\delta)$.

Let us turn now to consider when a spherical function on (G,K) is obtained from a representation of G as described in §1. To avoid some technical troubles we shall now assume that our locally compact group G is, furthermore, countable at infinity. We shall also presuppose that the reader is familiar with inductive limits and strict inductive limits. A good reference is Horváth [1]. A space (E,τ) is called a *strict LF-space* if (E,τ) is the strict inductive limit of Fréchet spaces (τ being the topology on E); for example $C_\omega(G)$ (ω a compact subset of G) is a Fréchet space, while $C_c(G)$ is a strict LF-space. Thus, a strict LF-space is a locally convex, complete, Hausdorff, topological vector space. We shall be concerned with continuous representations of G on a strict LF-space E, and with the corresponding quotient representations of G on E/J, J being a closed G-stable subspace of E. Even if E/J is not complete, we can lift by integration the representation of G on E/J to a representation of $M_c(G)$.

Let ϕ: $G \to End(V)$ be an irreducible spherical function of type δ, and let L be a maximal left ideal in End(V). If I is the set of all $f \in C_{c,\delta}(G)$ such that $\phi(f) \in L$, then I is a closed regular maximal left ideal in $C_{c,\delta}(G)$. Now let J be the set of all $f \in C_c(G)$ such that

$$\overline{\chi}_\delta * h * f * \overline{\chi}_\delta \in I \quad \text{for every} \quad h \in C_c(G)$$

then J is a closed regular maximal left ideal in $C_c(G)$, $I = J \cap C_{c,\delta}(G)$, and we have $f * \overline{\chi}_\delta \equiv f \pmod J$ for all $f \in C_c(G)$ (for the proof see Godement [1], p.513).

Since J is a closed left ideal in $C_c(G)$ it is invariant under left translation by elements of G, otherwise said there is induced on the space $C_c(G)/J$ a natural representation U of G. The corresponding lift of U to $M_c(G)$ associates with each $\mu \in M_c(G)$ the operator which transforms the class of $f \in C_c(G)$ (mod J) into the class of $\mu * f$ (mod J); thus, its restriction to the ideal $C_c(G)$ is an algebraically irreducible (J is maximal) representation of $C_c(G)$. That is to say that U is an algebraically irreducible representation of G.

The projection operator $P(\delta) = U(\overline{\chi}_\delta)$ is given by $P(\delta)(f + J) = = \overline{\chi}_\delta * f + J$; on the other hand $f * \overline{\chi}_\delta \equiv f \pmod J$ for all $f \in C_c(G)$, hence $f \mapsto f + J$ is a mapping of $C_{c,\delta}(G)$ onto $E(\delta)$ (the range of $P(\delta)$). Since $I = J \cap C_{c,\delta}(G)$ it is clear that $\dim E(\delta) = = \dim C_{c,\delta}(G)/I = \dim End(V)/L = \dim V$. The associated spherical function ϕ_1: $G \to End(E(\delta))$ is equivalent to ϕ. To see this it is sufficient to show that the representations ϕ: $C_{c,\delta}(G) \to End(V)$ and ϕ_1: $C_{c,\delta}(G) \to End(E(\delta))$ of $C_{c,\delta}(G)$ are equivalent (Proposition 2.5). If $f \in C_{c,\delta}(G)$ is such that $\phi(f) = 0$ then $\phi(f*h) = 0$ for all $h \in C_{c,\delta}(G)$, and $\phi_1(f)(h+J) = f*h + J = 0$ ($f*h \in I \subset J$); therefore $\phi_1(f) = 0$. Consequently, since it is a question of finite dimensional irreducible representations of an associative algebra, it follows that ϕ and ϕ_1 are equivalent.

The preceding discussion serves to prove that any irreducible spherical function on G can be obtained from an algebraically irreducible representation U of G, U being a quotient of a representation of G on a strict LF-space. To complete this circle of ideas, it remains to show that if the irreducible spherical function ϕ, comes from a representation U of G as above, the construction of the representation U_ϕ of G out of ϕ, gets us back to U.

Let E and E_ϕ be the representation spaces of U and U_ϕ respectively, let $E(\delta)$ and $E_\phi(\delta)$ be the corresponding K-isotypic subspaces, $P(\delta)$ and $P_\phi(\delta)$ the corresponding projections. If ϕ_1 is the spherical function of type δ associated to U_ϕ, there exist non-zero vectors $v \in E(\delta)$, $v_1 \in E_\phi(\delta)$ such that $\phi(f)v = 0$ if and only if $\phi_1(f)v_1 = 0$, $f \in C_c(G)$ (ϕ and ϕ_1 are equivalent). Let S: $C_c(G) \to E$ and $S_1 . C_c(G) \to E_\phi$ be the linear maps defined by $S(f) = U(f)v$, $S_1(f) = = U_\phi(f)v_1$. Then $Ker(S) = Ker(S_1)$. In fact, if $f \in Ker(S)$ we have $\phi(h*f)v = U(\overline{\chi}_\delta)U(h)U(f)v = 0$ therefore $0 = \phi_1(h*f)v_1 = = U_\phi(\overline{\chi}_\delta)U_\phi(h)U_\phi(f)v_1$, $h \in C_c(G)$, which implies that $f \in Ker(S_1)$ (algebraic irreducibility). In the same way one proves that $Ker(S_1) \subset \subset Ker(S)$, and therefore they are equal. The maps S and S_1 are clearly continuous surjective linear maps,hence they are strict morphisms (cf. Horváth [1], Prop. 11, p.306). From this it follows that the continuous representations U and U_ϕ of G are equivalent, i.e. there exists a linear bicontinuous bijection Q: $E \to E_\phi$ such that $QU(g) = = U_\phi(g)Q$ for all $g \in G$.

One can play exactly the same game as before, but with Fréchet representations of G, and prove that any irreducible spherical function ϕ: $G \to End(V)$ can be obtained from a topologically irreducible representation of G on a Fréchet space E. For this, one writes $G = \cup K_n$ as a countable union of an increasing sequence of compact subsets K_n of G, and such that every compact subset of G is contained in some

K_n. Then, $\|f\|_n = \sup_{x\in K_n} \int_G \|\phi(xy)f(y)\|dy \qquad n = 1,2,\ldots$.

($\| \ \|$ a norm on End (V)) are semi-norms on $C_c(G)$, and $\|f\|_n = 0$ for every n is equivalent to f=0. Then the Fréchet space L(G) which is the completion of $C_c(G)$ by these semi-norms, plays the role of $C_c(G)$ in the construction of the representation of G (for the details see Shin'ya [1]).

If a spherical function ϕ is associated to a Banach representation U of G then

$$\|\phi(g)\| \leqslant \|U(g)\| \quad \text{for all } g \in G$$

($\| \ \|$ is the usual operator norm). The function

$$\rho(g) = \|U(g)\|$$

is a positive real valued lower semi-continuous function which is bounded on compact subsets of G and satisfies

$$\rho(xy) \leqslant \rho(x)\rho(y)$$

for all x,y $\in$ G; such a function is called a *semi-norm* on G.

A Banach space valued function f on G is said to be *quasi-bounded* if there exists a semi-norm ρ on G such that $\sup_{g\in G} \|f(g)\|/\rho(g) < \infty$. Thus, if a spherical function comes from a Banach representation of G it is quasi-bounded. Conversely, if ϕ is an irreducible quasi-bounded spherical function on G, then it is associated to an algebraically irreducible Banach representation of G. Let ρ be a semi-norm on G such that $\sup_{g\in G} \|\phi(g)\|/\rho(g) < \infty$. One constructs the Banach representation as before, but replacing the space $C_c(G)$ by the Banach algebra obtained by completing $C_c(G)$, with respect to the ρ-norm

$$\|f\|_\rho = \int_G |f(g)|\rho(g)dg \qquad (f \in C_c(G))$$

(cf. Godement [1]).

3. THE ALGEBRAS $I_{c,\delta}(G)$ AND THEIR REPRESENTATIONS.

In what follows we shall denote by $I_c(G)$ the set of functions $f \in C_c(G)$ which are K-central, i.e. invariant under $g \mapsto kgk^{-1}$; thus $I_c(G)$ is a subalgebra of $C_c(G)$ and the operator

$$f \mapsto f^0(g) = \int_K f(kgk^{-1})dk$$

is a continuous projection (in the inductive limit topology) of $C_c(G)$ onto $I_c(G)$. We shall put $I_{c,\delta}(G) = I_c(G) \cap C_{c,\delta}(G)$, this is also a subalgebra of $C_c(G)$ and $f \mapsto f^0$ maps $C_{c,\delta}(G)$ onto $I_{c,\delta}(G)$. If $f \in I_c(G)$ and if $\overline{\chi}_\delta * f = f$, then also $f = f * \overline{\chi}_\delta$; this means that the map $f \mapsto \overline{\chi}_\delta * f$ is a continuous projection of $I_c(G)$ onto $I_{c,\delta}(G)$.

The topology induced by $C_c(G)$ is the one we shall consider on $I_{c,\delta}(G)$. If α is a Radon measure on G then α^0 can be defined by the following "weak" integral: $\alpha^0 = \int_K (\delta_k * \alpha * \delta_{k^{-1}}) dk$ (δ_x denotes the Dirac measure at x). Observe that

$$(1) \qquad (\alpha^0 * \beta)^0 = (\alpha * \beta^0)^0 = \alpha^0 * \beta^0$$

whenever α or β has compact support.

Let ϕ: G $\rightarrow$ End(V) be a spherical function of type δ and height p. Then V is a K-module under π: k $\mapsto \phi(k)$. Let $End_K(V)$ be the commutator of the representation π. Now, since the representation π decomposes into p equivalent irreducible representations, it is clear that its commutator is isomorphic with the algebra $M_p(C)$ of all p x p matrices, and such isomorphism is unique up to an inner automorphism of $M_p(C)$. If $f \in I_c(G)$ then

$$\phi(f) = \int_G f(g)\phi(g)dg \in End_K(V) \quad , \quad \text{in fact}$$

$$\pi(k)\phi(f) = \int_G f(g)\phi(kg)dg = \int_G f(k^{-1}g)\phi(g)dg = \int_G f(gk^{-1})\phi(g)dg =$$

$$= \int_G f(g)\phi(gk)dg = \phi(f)\pi(k).$$

Therefore, we may view ϕ: $I_{c,\delta}(G) \rightarrow End_K(V)$ as p-dimensional representation of $I_{c,\delta}(G)$. Also note that if $\phi(f) = I$ then $\phi(f^0) = I$ ($f \in C_c(G)$). Hence, we have proved the first part of the following theorem:

THEOREM 3.1. *If* ϕ: G $\rightarrow$ End(V) *is a spherical function of type* δ *then* ϕ: $I_{c,\delta}(G) \rightarrow End_K(V)$ *gives a continuous representation of* $I_{c,\delta}(G)$ *such that* $I \in \phi(I_{c,\delta}(G))$. *Conversely, any continuous finite dimensional representation* L *of* $I_{c,\delta}(G)$ *such that* $I \in L(I_{c,\delta}(G))$ *is equivalent to one given by a spherical function* ϕ *of type* δ.

We shall prove the second part of this theorem in several steps.

PROPOSITION 3.2. *Let* ψ: G $\rightarrow$ End(V) *be a* K-*central continuous function such that* $\chi_\delta * \psi = \psi$. *Then* ψ *satisfies the functional equation*

$$\psi(x)\psi(y) = \int_K \psi(k\, x\, k^{-1} y) dk$$

if and only if the mapping ψ: f $\mapsto \int_G f(g)\psi(g)dg$ *is a representation of* $I_{c,\delta}(G)$.

Proof. In view of 2.(2) and 2.(3) we have

$$\psi((\overline{\chi}_\delta * f^0 * \overline{\chi}_\delta) * (\overline{\chi}_\delta * h^0 * \overline{\chi}_\delta)) = \int\int_{G\times G} f^0(x)h^0(y) \left(\int_K \chi_\delta(k^{-1})\psi(xky)dk\right)dxdy =$$

$$= \int\int_{G\times G} f^0(x)h^0(y)\psi(xy)dxdy = \int\int_{G\times G} f(x)h^0(y) \left(\int_K \psi(k\ x\ k^{-1}y)dk\right)dxdy =$$

$$= \int\int_{G\times G} f(x)h(y) \left(\int_K \psi(k\ x\ k^{-1}y)dk\right)dxdy$$

and

$$\psi(\overline{\chi}_\delta * f^0 * \overline{\chi}_\delta)\psi(\overline{\chi}_\delta * h^0 * \overline{\chi}_\delta) = \int\int_{G\times G} f^0(x)h^0(y)\psi(x)\psi(y)dxdy =$$

$$= \int\int_{G\times G} f(x)h(y)\psi(x)\psi(y)dxdy$$

for all $f,h \in C_c(G)$. Now, the proposition follows immediately.

Let (V,π) be a finite dimensional K-module such that any irreducible submodule belongs to δ.

Let $\mathcal{A}$ denote the vector space of all continuous functions $\phi: G \to \mathrm{End}(V)$ such that $\phi(k_1gk_2) = \pi(k_1)\phi(g)\pi(k_2)$ for all $k_1,k_2 \in K$, and let $\mathcal{B}$ denote the vector space of all continuous functions $\psi: G \to \mathrm{End}_K(V)$ such that ψ is K-central and $\chi_\delta * \psi = \psi$.

PROPOSITION 3.3. *Let* A *and* B *be the linear mappings defined by*

$$(A\phi)(g) = \int_K \pi(k)\phi(g)\pi(k^{-1})dk \qquad \textit{for} \quad \phi \in \mathcal{A},$$

$$(B\psi)(g) = d(\delta)^2 \int_K \pi(k)\psi(k^{-1}g)dk \qquad \textit{for} \quad \psi \in \mathcal{B},$$

Then A *is an isomorphism of* $\mathcal{A}$ *onto* $\mathcal{B}$ *and* B *is the inverse of* A.

Proof. It is clear that $(A\phi)(g) \in \mathrm{End}_K(V)$, $g \in G$, and that $A\phi$ is K-central. Let us check that $\chi_\delta * A\phi = A\phi$:

$$(\chi_\delta * A\phi)(g) = \int_K \int_K \chi_\delta(k)\pi(k_1)\phi(k^{-1}g)\pi(k_1^{-1})dk_1dk =$$

$$= \int_K \int_K \chi_\delta(k)\pi(k_1)\pi(k^{-1})\phi(g)\pi(k_1^{-1})dkdk_1 = (A\phi)(g)$$

since $\int_K \chi_\delta(k)\pi(k^{-1})dk = I$.

An obvious computation shows that B maps $\mathcal{B}$ into $\mathcal{A}$. For $\psi \in \mathcal{B}$ we have,

$$(A(B\psi))(g) = d(\delta)^2\int_K\int_K \pi(k)\pi(k_1)\psi(k_1^{-1}g)\pi(k^{-1})dk_1dk =$$

$$= d(\delta)^2 \int_K \int_K \pi(kk_1k^{-1})\psi(k_1^{-1}g)\,dk\,dk_1 = \psi(g)$$

since $d(\delta)^2 \int_K \pi(kk_1k^{-1})dk = \chi_\delta(k_1)I.$

In a similar way one proves that B is a left inverse of A, and this completes the proof of Proposition 3.3.

COROLLARY 3.4. *Let* $\psi \in \mathcal{B}$; *if* $\psi(e) = I$ *then* $\psi(k) = \chi_\delta(e)^{-1}\chi_\delta(k)I$ *for all* $k \in K$.

Proof. If $\phi \in A$ then $\phi(e) \in \mathrm{End}_K(V)$, since $\pi(k)\phi(e) = \phi(k) = \phi(e)\pi(k)$ Therefore $(A\phi)(e) = \phi(e)$.

Now let $\phi = B\psi$, then $\phi(e) = \psi(e) = I$ and $\phi(k) = \pi(k)$ for all $k \in K$. From this we get

$$\psi(k) = (A\phi)(k) = \int_K \pi(k_1)\phi(k)\pi(k_1^{-1})dk_1 = \chi_\delta(e)^{-1}\chi_\delta(k)I \;, \qquad k \in K.$$

It may be worthwhile to point out also the following corollary:

COROLLARY 3.5. *For any* $\phi \in A$ *we have* $\mathrm{tr}A\phi = \mathrm{tr}\phi$ *and for any* $\psi \in \mathcal{B}$ *we have* $\mathrm{tr}B\psi = \mathrm{tr}\psi$.

Proof. The first assertion is obvious and to prove the second let $\phi = B\psi$, then

$$\mathrm{tr}B\psi = \mathrm{tr}\phi = \mathrm{tr}A\phi = \mathrm{tr}\psi \;.$$

PROPOSITION 3.6. *Let* $\psi = A\phi$, $\phi \in A$. *Then* ϕ *satisfies*

$$\phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk \tag{2}$$

if and only if ψ *satisfies*

$$\psi(x)\psi(y) = \int_K \psi(k\,x\,k^{-1}y)dk \;. \tag{3}$$

Proof. If we assume (3) we have

$$\phi(x)\phi(y) = (B\psi)(x)(B\psi)(y) = d(\delta)^4 \int\int_{K\times K} \pi(k_1)\psi(k_1^{-1}x)\pi(k_2)\psi(k_2^{-1}y)dk_1dk_2 =$$

$$= d(\delta)^4 \int\int\int_{K\times K\times K} \pi(k_1k_2)\psi(kk_1^{-1}\,x\,k^{-1}k_2^{-1}y)dk\,dk_1dk_2 =$$

$$= d(\delta)^4 \int\int\int_{K\times K\times K} \pi(k_1k_2)\psi(k_2^{-1}kk_1^{-1}\,x\,k^{-1}y)dk\,dk_1dk_2 =$$

$$= d(\delta)^2 \int\int_{K\times K} \pi(k_1)\phi(kk_1^{-1}\,x\,k^{-1}y)dk\,dk_1 = \int_K \chi_\delta(k)\phi(xk^{-1}y)dk \;.$$

Conversely, if we assume (2) we have

$$\psi(x)\psi(y) = (A\phi)(x)(A\phi)(y) =$$

$$= \iint_{K\times K} \pi(k_1)\phi(x)\pi(k_1^{-1})\pi(k_2)\phi(y)\pi(k_2^{-1})dk_1dk_2 =$$

$$= \iint_{K\times K} \pi(k_2k_1)\phi(x)\pi(k_1^{-1})\phi(y)\pi(k_2^{-1})dk_1dk_2 =$$

$$= \iiint_{K\times K\times K} \pi(k_2)\chi_\delta(k^{-1})\phi(k_1\ x\ k_1^{-1}ky)\ \pi(k_2^{-1})dkdk_1dk_2 =$$

$$= \iiint_{K\times K\times K} \pi(k_2)\chi_\delta(k^{-1})\phi(kk_1\ x\ k_1^{-1}y)\pi(k_2^{-1})dkdk_1dk_2 =$$

$$= \int_K \psi(k_1\ x\ k_1^{-1}y)dk_1 \ .$$

PROPOSITION 3.7. *Let* ϕ: $G \to \mathrm{End}(V)$ *be a continuous function such that* $\phi(k_1gk_2) = \pi(k_1)\phi(g)\pi(k_2)$, *all* $k_1,k_2 \in K$. *Then* ϕ *satisfies the functional equation*

$$(4) \qquad \phi(x)\phi(y) = \int_K \chi_\delta(k^{-1})\phi(xky)dk$$

if and only if the mapping ϕ: $f \mapsto \int_G f(g)\phi(g)dg$ *is a representation of* $I_{c,\delta}(G)$.

Proof. That ϕ gives a representation of $I_{c,\delta}(G)$ whenever ϕ satisfies (4), it follows at once from Proposition 2.2.

To prove the converse let $\psi = A\phi$ and observe that

$$(5) \qquad \phi(f) = \iint_{G\times K} f(g)\phi(kgk^{-1})dkdg = \int_G f(g)\psi(g)dg = \psi(f)$$

for all $f \in I_c(G)$. Therefore by Proposition 3.2 ψ satisfies (3) which in turn implies that ϕ satisfies (4).

Proof of Theorem 3.1. The first part was already proved. Now let L: $I_{c,\delta}(G) \to M_p(C)$ be a continuous representation such that $L(h) = I$ for some $h \in I_{c,\delta}(G)$. The composite map ψ: $C_c(G) \to M_p(C)$ defined by $\psi(f) = L(\overline{\chi}_\delta * f^0)$ is a $M_p(C)$-valued Radon measure on G. We have

$$\psi(f) = L(\overline{\chi}_\delta * f^0) = L(\overline{\chi}_\delta * f^0 * h) = L(\overline{\chi}_\delta * (f*h)^0) = \psi(f*h) = (\psi*\check{h})(f)$$

for all $f \in C_c(G)$. Therefore $\psi = \psi*\check{h}$ is a continuous function on G which represents L. Using once more (1) we get

$$\psi(f) = \psi(f^0) = \psi(\psi*f^{0\vee})(e) = (\psi*f^{\vee 0})^0(e) = (\psi^0*f^\vee)^0(e) = \psi^0(f)$$

for any $f \in C_c(G)$, which shows that ψ is K-central. In a similar way one also establishes that $\chi_\delta * \psi = \psi$.

Let (V,π) be a K-module which is the direct sum of p irreducible modules belonging to δ. If we identify $M_p(C)$ with $\mathrm{End}_K(V)$ the function $\psi \in \mathcal{B}$. Let $\phi = B\psi$ (see Proposition 3.3). Now, $L(f) = \psi(f) = \phi(f)$ for every $f \in I_{c,\delta}(G)$ (cf. (5)). Therefore, by Proposition 3.7, ϕ satisfies the functional equation (4). To finish the proof we have to show that $\phi(e) = I$. From the fact that ψ satisfies (3) we obtain $\psi(e)\psi(g) = \psi(g) = \psi(g)\psi(e)$, $g \in G$. Since $\psi(I_{c,\delta}(G))$ coincides with the linear span of $\{\psi(g)\colon g \in G\}$ it follows that $\psi(e) = I$, which implies $\phi(e) = I$ (see Corollary 3.4).

REMARK 3.8. If $V = V_\delta \oplus \ldots \oplus V_\delta$ (p-times) is a K-module as above, it is easy to verify that there is an algebra isomorphism $\iota : \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) \to \mathrm{End}(V)$ such that $\iota(T \otimes S) = (T\oplus \ldots \oplus T)S$. Let $I(\delta) = C(K) * \overline{\chi}_\delta$; of course $I(\delta)$ is a $*$-algebra isomorphic to $\mathrm{End}(V_\delta)$, more precisely, if we make use of the natural identification $\mathrm{End}(V_\delta) \simeq V_\delta \otimes V_\delta^*$ then an isomorphism ℓ can be described by $\ell(v \otimes \lambda)(k) = d(\delta)\lambda(k^{-1}.v)$ for all $v \in V_\delta$, $\lambda \in V_\delta^*$, $k \in K$. Now the relation between the linear maps $\phi\colon C_{c,\delta}(G) \to \mathrm{End}(V)$ and $\psi\colon I_{c,\delta}(G) \to \mathrm{End}_K(V)$, defined by $\phi \in A$ and $\psi = A\phi$, can be explained appealing to the following structural fact due to Dieudonné (cf. Dieudonné [1], p. 237): the bilinear map $(a,f) \to a*f$ of $I(\delta) \otimes I_{c,\delta}(G)$ into $C_c(G)$, establishes a $*$-algebra isomorphism of the tensor product $*$-algebra $I(\delta) \otimes I_{c,\delta}(G)$ with $C_{c,\delta}(G)$. Then

$$\begin{array}{ccc} I(\delta) \otimes I_{c,\delta}(G) & \xrightarrow{\simeq} & C_{c,\delta}(G) \\ \ell \otimes \psi \downarrow & & \downarrow \phi \\ \mathrm{End}(V_\delta) \otimes \mathrm{End}_K(V) & \xrightarrow[\iota]{} & \mathrm{End}(V) \end{array}$$

commutes. A simple and important consequence of this is the following: there exists a natural one-to-one correspondence between the equivalence classes of finite dimensional irreducible representations of $I_{c,\delta}(G)$ and those of $C_{c,\delta}(G)$.

REMARK 3.9. Let $\phi\colon G \to \mathrm{End}(V)$ be an irreducible spherical function of height p. Let $\varphi(g) = \mathrm{tr}\,\phi(g)$, $g \in G$, and put $\varphi_0 = d(\delta)^{-1}\varphi$. Then

$$\begin{array}{ccc} I_{c,\delta}(G) & \xrightarrow{\psi} & M_p(C) \\ & {}_{\varphi_0}\searrow \quad \swarrow{}_{tr} & \\ & C & \end{array}$$

is commutative, where $\varphi_0(f) = \int_G f(g)\varphi_0(g)dg$ and

$\psi(f) = \int_G f(g)(A\phi)(g)dg \in \mathrm{End}_K(V) \simeq M_p(C)$, $f \in I_{c,\delta}(G)$. According to Proposition 3.2 φ_0 satisfies (3) if and only if tr: $M_p(C) \to C$ is a homomorphism which corresponds to p=1. Therefore we have proved that φ satisfies

$$\varphi(x)\varphi(y) = d(\delta) \int_K \varphi(k\ x\ ky^{-1})dk$$

for arbitrary $x,y \in G$, if and only if p=1 (cf. Godement [1], p. 524).

For completeness we shall point out the following. Suppose that every topologically completely irreducible Banach representation of $C_{c,\delta}(G)$ is finite dimensional (see Warner [1], p. 228). Then the set of all irreducible spherical functions of type δ separates the points of $C_{c,\delta}(G)$. In fact, in virtue of the Gelfand-Raikov Theorem the set of all topologically irreducible unitary representations of G separates the points of $C_c(G)$. Let $f \in C_{c,\delta}(G)$, $f \neq 0$ and let U be a topologically irreducible unitary representation of G such that $U(f) \neq 0$. But $U(f) = U(\overline{X}_\delta * f * \overline{X}_\delta) = U(\overline{X}_\delta)U(f)U(\overline{X}_\delta)$, which says $\phi(f) \neq 0$, ϕ being the spherical function of type δ associated to U. As a consequence of this we have:

PROPOSITION 3.10. *The following properties are equivalent:*

(i) $I_{c,\delta}(G)$ *is commutative.*

(ii) *Every irreducible spherical function of type* δ *is of height one.*

(iii) $I_{c,\delta}(G)$ *is the center of* $C_{c,\delta}(G)$.

Proof. If (ii) holds, then $I_{c,\delta}(G)$ admits sufficiently many one dimensional representations, hence (i). Conversely, if $I_{c,\delta}(G)$ is commutative, then every finite dimensional irreducible representation of $I_{c,\delta}(G)$ is one dimensional so that every irreducible spherical function of type δ is of height one. It is clear that (iii) implies (i). To complete the proof it suffices to show that (ii) implies (iii). Let $f \in I_{c,\delta}(G)$, then for any $h \in C_{c,\delta}(G)$ and any irreducible spherical function ϕ of type δ we have $\phi(f*h) = \phi(f)\phi(h) = \phi(h)\phi(f) = \phi(h*f)$ since $\phi(f)$ is a scalar for every $f \in I_{c,\delta}(G)$. Therefore

$I_{c,\delta}(G)$ is contained in the center of $C_{c,\delta}(G)$. Furthermore, if f belongs to the center of $C_{c,\delta}(G)$ then $\phi(f)$ is a scalar in every irreducible spherical function of type δ, hence $\phi(f^0) = \phi(f)$, which proves that $f^0 = f$.

If ϕ: $G \to End(V)$ is a spherical function of type δ and height p, the function $A\phi = \psi$: $G \to End_K(V) \simeq M_p(C)$ should be considered as the other face of the same coin. Thus a spherical function ψ(on (G,K)) of type δ is also a continuous function on G with values in End(W) (W a finite dimensional vector space) such that:

(i) $\psi(e) = I$.

(ii) $\chi_\delta * \psi = \psi$.

(iii) $\psi(x)\psi(y) = \int_K \psi(k\,x\,k^{-1}y)dk$ for all $x,y \in G$.

The dimension of W is the height of ψ.

PROPOSITION 3.11. *Let* ψ: $G \to End(W)$ *be a continuous* K-*central function which satisfies* (iii). *Then* ψ *can be decomposed in a unique way as the direct sum* $\psi = 0 + \sum_\delta \psi_\delta$ *of a zero function and of spherical functions* ψ_δ *of type* δ.

Proof. Note that for any $g \in G$,

$$(\chi_\delta*\psi)(g) = \int_K \chi_\delta(k)\psi(k^{-1}g)dk = \int_K\int_K \chi_\delta(k)\psi(k_1 k^{-1} k_1^{-1} g)dk_1 dk =$$

$$= \int_K \chi_\delta(k)\psi(k^{-1})\psi(g)dk = (\chi_\delta*\psi)(e)\psi(g) .$$

Because ψ is K-central, $\chi_\delta*\psi = \psi(\chi_\delta*\psi)(e)$ follows as before. Consequently, $(\chi_\delta*\psi)(e)\psi = \chi_\delta*\psi = \psi(\chi_\delta*\psi)(e)$.

Given $\delta,\delta' \in \hat{K}$ we have

$$(\chi_\delta*\psi)(e)\ (\chi_{\delta'}*\psi)(e) = (\chi_{\delta'}*(\chi_\delta*\psi)(e)\psi)(e) = (\chi_{\delta'}*\chi_\delta*\psi)(e)$$

showing that $(\chi_\delta*\psi)(e)$, $\delta \in \hat{K}$, are orthogonal projections, and therefore they are zero for almost all $\delta \in \hat{K}$. Hence, $\psi(k) = \sum_\delta(\chi_\delta*\psi)(k)$ all $k \in K$, and in particular $\psi(e) = \sum_\delta(\chi_\delta*\psi)(e)$.

We also have $\psi(e)\psi(x) = \psi(x) = \psi(x)\psi(e)$ for all $x \in G$.

Therefore $\psi = (I-\psi(e))\psi + \sum_\delta(\chi_\delta*\psi)(e)\psi = (I-\psi(e))\psi + \sum_\delta \chi_\delta*\psi$, which clearly completes the proof of the proposition.

The K-central functions ψ: $G \to End(W)$ which satisfies (iii) are precisely those which give a representation of $I_c(G)$ on W.

4. DIFFERENTIAL PROPERTIES OF SPHERICAL FUNCTIONS. THE ALGEBRA $D_0(G)$ AND THEIR REPRESENTATIONS.

In this section, we assume that G is a connected Lie group.

LEMMA 4.1. *If* $\phi: G \to \mathrm{End}(V)$ *is a spherical function, then* ϕ *is differentiable* (C^∞).

Proof. Let $\| \ \|$ be a norm on End(V) such that $\|TS\| \leqslant \|T\| \, \|S\|$ for all $T,S \in \mathrm{End}(V)$. Now, it is well-known that if $\|T-1\| < 1$, $T \in \mathrm{End}(V)$, then T is invertible. Since ϕ is continuous we can choose a neighborhood $\mathcal{U}$ of the identity in G such that $\|I - \phi(g)\| < 1$ for all $g \in \mathcal{U}$.

Let f be a C^∞ real valued function with compact support contained in $\mathcal{U}$ such that $f \geqslant 0$ and $\int_G f(g)dg = 1$. Then $\int_G f(g)\phi(g)dg$ is an automorphism of V. In fact

$$\|I - \int_G f(g)\phi(g)dg\| = \|\int_G f(g)(I - \phi(g))dg\| \leqslant \int_G f(g)\|I - \phi(g)\|dg < 1.$$

Finally,

$$\phi(x)\int_G f(y)\phi(y)dy = \int_G f(y) \int_K \chi_\delta(k^{-1})\phi(xky)dkdy =$$

$$= \int_K \chi_\delta(k^{-1}) \int_G f(k^{-1}x^{-1}y)\phi(y)dydk =$$

$$= \int_G \left(\int_K \chi_\delta(k^{-1})f(k^{-1}x^{-1}y)dk\right)\phi(y)dy$$

which shows that ϕ is C^∞.

Let D(G) denote the algebra of all left invariant differential operators on G and let $D_0(G)$ denote the set of operators in D(G) which are invariant under all right translations from K. Of course $D_0(G)$ is a subalgebra of D(G).

LEMMA 4.2. *Let* ϕ *be a spherical function of type* δ. *Then*

$$[D\phi](g) = \phi(g)[D\phi](e)$$

for all $D \in D_0(G)$, $g \in G$.

Proof. For each $D \in D(G)$ we get from 1.(2)

$$\phi(x)[D\phi](y) = \int_K \chi_\delta(k^{-1})[D\phi](xky)dk.$$

Putting y=e we obtain

$$\phi(x)[D\phi](e) = \int_K \chi_\delta(k^{-1})[D\phi](xk)dk.$$

If $D \in D_0(G)$, then

$$[D\phi](gk) = [D(\phi*\delta_k)](g) = [D\phi](g)\phi(k)$$

for all $g \in G$. Therefore,

$$\phi(g)[D\phi](e) = \int_k \chi_\delta(k^{-1})[D\phi](g)\phi(k)dk = [D\phi](g)\phi(e)$$

which proves the lemma.

PROPOSITION 4.3. *Any spherical function on* G *is analytic.*

Proof. Suppose $f: G \to V$ is a C^∞ function such that $[Df](g) = Tf(g)$, all $g \in G$, for some $T \in End(V)$ and some $D \in D(G)$. We can find a basis $\{e_i\}$ of V so that T is given by a matrix of the form

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Let $S \in End(V)$ be the linear map defined by $Se_i = \lambda_i e_i$, $i=1,2,\ldots,n$. Then $(D-S)^n f = (T-S)^n f = 0$. Hence, if f_i denotes the i^{th}-component of f with respect to $\{e_i\}$ we have $(D-\lambda_i)^n f_i = 0$. If D is elliptic, by a theorem of S. Bernstein and induction on n, it follows that every solution of an equation $(D-\lambda)^n h = 0$ is analytic. Therefore, in this case our function f is analytic.

It is well-known that $D_0(G)$ contains elliptic operators (cf.Godement [1], p. 539) thus, the proposition follows now directly from Lemma 4.2.

We shall frequently use the following basic property:

LEMMA 4.4. *Let* f *be a* K-*central analytic function on* G; *then* f=0 *is equivalent to*

$$[Df](e) = 0 \quad \textit{for every} \quad D \in D_0(G).$$

Proof. Since f is analytic and since G is connected, it is clear that f=0 is equivalent to $[Df](e) = 0$ for all $D \in D(G)$. Let $D^{L(g)}$, $D^{R(g)}$ denote respectively the left and right translation by g of $D \in D_0(G)$. We can form the integral

$$D_0 = \int_K D^{R(k)}\, dk$$

which is an operator in $D_0(G)$. Since f is K-central we have

$$[D^{R(k)}f](e) = [D^{L(k)}f](e) = [Df](e) \quad , \quad \text{so}$$

$$[D_0f](e) = \int_K [D^{R(k)}f](e)dk = [Df](e)$$

This proves the lemma.

PROPOSITION 4.5. *Let* ψ: G $\longrightarrow$ End(V) *be a* K-*central analytic function. Then* ψ *satisfies the functional equation* 3.(3) *if and only if the mapping* ψ: D $\longrightarrow$ $[D\psi](e)$ *is a representation of* $D_0(G)$.

Proof. From 3.(3) one gets, in a completely similar way as we proved Lemma 4.2, $[D\psi](g) = \psi(g)[D\psi](e)$ for every $D \in D_0(G)$. Conversely, it is also clear in virtue of Lemma 4.4, that this implies 3.(3). Invoking once more Lemma 4.4 one sees that

$$[D\psi](g) = \psi(g)[D\psi](e) \quad \text{for every} \quad D \in D_0(G)$$

is equivalent to require that ψ: $D_0(G) \longrightarrow$ End(V) is a representation.

In the following proposition (V,π) will be a K-module as in Section 3.

PROPOSITION 4.6. *Let* ϕ: G $\longrightarrow$ End(V) *be an analytic function such that* $\phi(kgk_1) = \pi(k_1)\phi(g)\pi(k_2)$ *(all* $k,k_1 \in K$). *Then* ϕ *satisfies the functional equation* 1.(2) *if and only if the mapping* ϕ: D $\longrightarrow$ $[D\phi](e)$ *is a representation of* $D_0(G)$.

Proof. First of all let us observe that $[D\phi](e) \in End_K(V)$ for all $D \in D_0(G)$. In fact, if $D \in D_0(G)$ we have

$$[D\phi](e)\pi(k) = [D\phi^{R(k^{-1})}](e) = [D^{R(k)}\phi](k) = [D^{L(k)}\phi](k) =$$

$$= [D\phi^{L(k^{-1})}](e) = \pi(k)[D\phi](e).$$

Let $\psi = A\phi$ (see Proposition 3.3), then

$$\psi(D) = \int_K \pi(k)[D\phi](e)\pi(k^{-1})dk = \phi(D)$$

for every $D \in D_0(G)$. Therefore, the proposition follows at once from Propositions3.6 and 4.5.

REMARK 4.7. Of course, combining Proposition 3.3 and Lemma 4.4 one gets the following analogue of Lemma 4.4 for analytic functions ϕ: G $\longrightarrow$ End(V) which satisfies $\phi(kgk_1) = \pi(k)\phi(g)\pi(k_1)$, for all $k,k_1 \in K$, namely: $\phi = 0$ if and only if $[D\phi](e) = 0$ for all $D \in D_0(G)$.

We shall consider a topology on $D_0(G)$, introduced by Godement (cf. Godement [1], p. 538). We say that a variable $D \in D_0(G)$ converges to a given $D_0 \in D_0(G)$ if $[Df](e)$ converges to $[D_0f](e)$ for every analytic K- central function f. This topology is precisely the weak topology defined on $D_0(G)$ by the natural pairing of $D_0(G)$ and the vector space of all K-central analytic functions on G.

We are now in a position to prove the main result of this section which is an infinitesimal counterpart to Theorem 3.1.

We recall that if (V,π) is a finite dimensional K-module which is the direct sum of p irreducible equivalent submodules, we can identify $M_p(C)$ with $\mathrm{End}_K(V)$.

THEOREM 4.8. *If* ϕ: $G \to \mathrm{End}(V)$ *is a spherical function then* ϕ: $D \to [D\phi](e)$ *maps* $D_0(G)$ *into* $\mathrm{End}_K(V)$, *giving a continuous representation of* $D_0(G)$. *Conversely, any continuous finite dimensional representation of* $D_0(G)$ *is the direct sum of a zero representation and ones given by spherical functions.*

Proof. That $[D\phi](e) \in \mathrm{End}_K(V)$ for every $D \in D_0(G)$ was observed during the proof of Proposition 4.6. If we put $\psi = A\phi$ (see Proposition 3.3) we have $[D\phi](e) = [D\psi](e)$, which shows that ϕ: $D_0(G) \to \mathrm{End}_K(V)$ is continuous, by the very definition of the topology in $D_0(G)$. From Proposition 4.6 we get that ϕ defines a representation of $D_0(G)$.

To prove the second part, let us assume that L: $D_0(G) \to M_p(C)$ is a continuous representation. By weak duality such a linear map is defined by a K-central analytic function ψ: $G \to M_p(C)$; $L(D) = [D\psi](e)$. Now by Proposition 4.5 we know that ψ satisfies

$$\psi(x)\psi(y) = \int_K \psi(kxk^{-1}y)\,dk \qquad \text{all} \quad x,y \in G ,$$

which in turn implies our contention (cf. Proposition 3.11).

Naturally, a subspace $W \subset C^p$ is $\psi(G)$-invariant if and only if it is $\psi(D_0(G))$-invariant ($\psi(D) = [D\psi](e)$, $D \in D_0(G)$). This follows at once from Lemma 4.4. Thus, in particular, Theorem 4.8 establishes a one-to-one correspondence between the equivalence classes of continuous finite dimensional irreducible representations of $D_0(G)$ and the equivalence classes of irreducible spherical functions on G.

The relation between the spherical function ϕ: $G \to \mathrm{End}(V)$ and its associated representation of $D_0(G)$, is the exact generalization of the correspondence between a finite dimensional representation of G and the derived representation of the Lie algebra of G. In fact, if we take $K = \{e\}$ then $D_0(G)$ becomes the algebra D(G) of all left invariant differential operators on G, which is isomorphic to the universal enveloping algebra of the complexification of the Lie algebra of G. Since in this case the spherical functions are precisely the finite dimensional representations and moreover, there is a natural one-to-one correspondence between the set of all representations of a Lie algebra a on V and the set of all representations of the universal enveloping algebra of a on V, our assertion is clear.

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ON SMALL SUBMODULES IN THE TOTAL QUOTIENT RING OF A COMMUTATIVE RING

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As a generalization of Nakayama's lemma, we know the concept of small submodules and many authors have studied those submodules in projectives [1], [3], [7] and [8]. In [2] and [4], we have some applications of small submodules to hollow modules. In this short note, we shall study small submodules from a little different point of view.

Let R be a commutative ring with identity and Q the total quotient ring of R. In the first section, we shall show that R is a small submodule in Q as an R-module if and only if every maximal ideal in R contains a non zero-divisor. In the second section, we assume R is a Dedekind domain. Then we shall determine all small submodules in any direct sums of copies of Q as R-modules.

1. COMMUTATIVE RINGS.

Throughout this note we always assume that a ring R is commutative and has the identity. By Q and D(R) (or briefly D) we shall denote the total quotient ring of R and the set of zero-divisors, respectively. We call an element in R-D *regular*.

LEMMA 1. *Let* T *be an* R-*submodule in* Q *such that* $Q = R + T$ *and* $R \cap T$ *contains a regular element* x. *Then* $T = Q$.

Proof. Let $x^{-1} = r + t$; $r \in R$, $t \in T$. Then $1 = rx + tx \in T$ and so $T \supseteq R$ and $T = Q$.

THEOREM 2. *Let* R *be a commutative ring and* Q *its total quotient ring. Then* R *is small in* Q *as an* R-*module if and only if every maximal ideal in* R *contains a regular element.*

Proof. We assume R is not small in Q as an R-module. Then $Q = R + T$ for some R-module T ($\neq Q$). Since $T \cap R$ is an ideal contained in D(R) by Lemma 1, $TQ = Q$. Hence, we may assume that T is an ideal in Q. Let T' be a maximal ideal in Q containing T and put $M = T' \cap R$.

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Since $T' = MQ$, M is a prime ideal in R which is contained in D. If there exists an ideal M' in R such that $D \supseteq M' \supseteq M$, $Q \neq M'Q \supseteq MQ = T'$. Hence $M = M'$. Next, we shall show that M is a maximal ideal in R. Let N be an ideal in R, which contains properly M. Then N contains a regular element x from the above observation. Let $x^{-1} = r + p(b/a)$; $b,r \in R$, $a \in R-D$ and $p \in M$, since $Q = R + T'$. Then $(1-rx)a = pbx \in M$ and $a \notin M$. Hence, $(1-rx) \in M \subseteq N$, which implies $N = R$. Therefore, D contains the maximal ideal M. Conversely, we assume that D contains a maximal ideal B. Let a be any regular element. Then $R = (a) + B$ and so $1 = ra + b$; $r \in R$, $b \in B$. Hence, $a^{-1} = r + ba^{-1} \in R + BQ$. Therefore, $Q = R + BQ$ and $Q \neq BQ$ since $B \subseteq D$.

COROLLARY 3. *Let* R *be a noetherian ring with* Q *the total quotient ring. Let* $\{P_i\}$ *be the set of associated prime ideals of* (0) (see [9], p. 211). *Then* R *is an* R-*small submodule in* Q *if and only if every* P_i *is not maximal.*

COROLLARY 4. *Let* R *be a semi-local ring. Then* R *is small in* Q *if and only if* J(R) *contains a regular element, where* J(R) *is the Jacobson radical.*

The following proposition is a direct consequence from Theorem 2, however we shall give a proof, which is interesting itself.

PROPOSITION 5. R *is small in* Q *if one of the following conditions is satisfied*:

1) *There exists an ideal containing properly* D.

2) $Q \neq R$ *and* $J(R) \supseteq D$.

3) $J(R) \not\subseteq D$.

Proof. We assume $Q = R + T$ for some R-submodule T. Let a be regular and $a^{-1} = r + t$; $r \in R$, $t \in T$. Then $1 = ra + ta$ and $ta = 1-ra \in R \cap T$.

1) Let a be in A-D, where A is an ideal containing D. If ta is in D, $1 = ra + ta \in A$, which is a contradiction. Therefore, ta is regular and $Q = T$ from Lemma 1.

2) Since $Q \neq R$, there exists a regular element a such that $a^{-1} \notin R$. If ta is in D, $ra = 1-ta$ has the inverse in R from the assumption. Hence, so does a, which is a contradiction. Therefore, $Q = T$ from Lemma 1.

3) Let a be in J(R)-D. Then $ta = 1-ra$ is an unit in R. Hence, $Q = T$.

COROLLARY 6. *If* $Q \neq R$ *and* R *is one of the followings*
1) R *is domain*, 2) R *is local and* 3) (0) *is primary, then* R *is small in* Q.

Proof. It is clear from the above.

We note that all conditions in Proposition 5 are independent each other and if R is artinian, then Q = R.

PROPOSITION 7. *We assume* R *is small in* Q *and* B *is an ideal containing a regular element. Then* $B^{-n} = \{x \in Q \mid B^n x \subseteq R\}$ *is small in* Q.

Proof. Let $Q = B^{-n} + T$ for some R-module T. Since QB = Q, $Q = B^n B^{-n} + B^n T \subseteq R + T \subseteq Q$. Hence, Q = T.

2. DEDEKIND DOMAINS.

In this section, we assume R is a Dedekind domain.

LEMMA 8. *Let* R *be a Dedekind domain. Then every* R-*small submodule in* Q *is contained in a small submodule of a form* $\sum P_i^{-n_i}$*, where* P_i *runs through all maximal ideals and the* n_i *is a natural number for all* i.

Proof. Let S be an R-small submodule in Q. Then so is R + S by Corollary 6. Hence, we may assume $S \supseteq R$. Now $Q/R = \sum_P \oplus \sum_n P^{-n}/R$ by [5]. Since the $\sum_n P^{-n}/R$ is an uni-serial module, every proper submodule is small in it. Hence, $S/R = \sum_{P_i} \oplus P_i^{-n_i}/R$. Since R is small in Q, $\sum P_i^{-n_i}$ is small in Q from the above.

THEOREM 9. *Let* R *be a Dedekind domain with* Q *the quotient field. We put* $Q^{(I)} = \sum_I \oplus Q_\alpha$; $Q_\alpha = Q$ *and* I *is any index set. Then every* R-*small submodule in* $Q^{(I)}$ *is contained in* $\sum \oplus S_{\alpha_i}$, *where the* S_{α_i} *is a small submodule given in Lemma 8 and the converse* (cf. [3], Proposition 1 and Remark 3).

Proof. We assume $S \not\subseteq \sum_J \oplus Q_{\alpha_i}$ for any finite subset J of I and show a contradiction. We put $Q^{(I)} = \sum_{i=1}^{\infty} \oplus Q_i \oplus \sum \oplus Q_\beta$. When we consider the projection of $Q^{(I)}$ to $\sum \oplus Q_i$, we may assume $Q^{(I)} = \sum_{i=1}^{\infty} \oplus Q_i$. Let p_i be the projection of $Q^{(I)}$ to ith component, we may assume $S \subseteq \sum \oplus R_i$ and $p_i(S) \neq 0$ for all i. First, we consider $S_1 = \{s_1 \in R \mid$ there exists $s \in S$ such that $s = s_1 + s_2 + \dots\}$. Since S_1 is an ideal in R, $S_1 = s_1^{(1)}R + s_1^{(2)}R + \dots + s_1^{(t_1)}R$, $(s_1^{(i)} \neq 0)$. Let $s^{(1,i)} = s_1^{(i)} + s_2' + \dots + s_{n(1,i)}'$ be in S. Put $m_1' = \max\{n(1,i)\}$ and take m_2 such that $m_1' < m_2$. Let $S_{m_2} = \{s_{m_2} \in R \mid$ there exists $s \in S$ such that $s = o + s_2 + \dots + s_{m_2} + \dots\}$. Since $m_1' < m_2$ and $p_{m_2}(S) \neq 0$, $S_{m_2} \neq 0$. Let $S_{m_2} = s_{m_2}^{(1)}R + s_{m_2}^{(2)}R + \dots + s_{m_2}^{(t_2)}R$ and $s^{(2,i)} =$

$= o + \dots + s_{m_2}^{(i)} + \dots + s_{n(2,i)}'$, $(s_{m_2}^{(i)} \neq 0)$. Put $m_2' =$ $= \max\{n(2,i)\}$ and take m_3 such that $m_3 > m_2'$. Let $S_{m_3} = \{s_{m_3} \in R \mid$ there exists $s \in S$ such that $s = o + s_2' + \dots + s_{m_2} + \dots + s_{m_3} + \dots\}$.

Repeating those arguments, we obtain a sequence $m_1 = 1, m_2, m_3, \dots$ such that $p(S) \supseteq \sum_{i=1}^{\infty} \oplus S_{m_i} \supseteq \sum_{i=1}^{\infty} \oplus s_{m_i}^{(1)}R$, where $p = \sum p_{m_i}$. Then $\sum_{i=1}^{\infty} \oplus s_{m_i}R$ must be small in $p(Q^{(I)}) = \sum_{i=1}^{\infty} \oplus Q_{m_i}$. Now, we define $f: \sum \oplus Q_{m_i} \to Q$ by setting $f(q_{m_i}) = x^{-i}(s_{m_i}^{(1)})^{-1}q_{m_i}$, where $x \in A\text{-}A^2$ for a fixed prime A. Then $f(\sum_{i=1}^{\infty} \oplus s_{m_i}^{(1)}R)$ is not small in Q by Lemma 8, which contradicts the assumption that S is small. Therefore, $S \subseteq \sum_J \oplus Q_\alpha$, for some finite subset J of I. The remaining parts are clear from Lemma 8.

We note that if R is a noetherian U.F.D., then we can obtain a similar results to the above.

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MEROMORPHIC DIFFERENTIAL OPERATORS

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ABSTRACT. Meromorphic differential operators on a reduced locally irreducible complex analytic space are studied in this paper. Conditions are established for regularity of such operators.

1. INTRODUCTION.

This section contains some basic definitions and facts about meromorphic functions on complex analytic spaces.

Let X be a connected complex analytic space and $\mathcal{O}_X$ be the sheaf of germs of analytic functions on X. It will be assumed that X is reduced and locally irreducible. This means that each $x \in X$ has arbitrarily small neighborhoods U such that $(U, \mathcal{O}_X|U)$ is isomorphic as a ringed space to $(V, \mathcal{O}_W/\mathcal{J}_V)$ where W is the unit polydisc in $\mathbf{C}^N$ for some N, V is a closed irreducible subvariety of W, and $\mathcal{J}_V$ is the sheaf of ideals of analytic functions vanishing on V. In this case the stalks $\mathcal{O}_{X,x}$ are integral domains and $\mathcal{M}_X$, the sheaf of quotients of $\mathcal{O}_X$, is a sheaf of fields. $\mathcal{M}_X$ is the *sheaf of germs of meromorphic functions* on X. $\mathcal{O}_X$ can be considered a subsheaf of $\mathcal{M}_X$.

The results in this paper are of a local nature and the proofs are most conveniently carried out for germs. If X is an analytic space or a subvariety of an analytic space and $x \in X$, then X_x denotes the germ of X at x. For $f \in \mathcal{O}_{X,x}$, let f(x) denote the value of f at x, that is, the residue class modulo the maximal ideal. For a function $m \in \Gamma(U,\mathcal{M}_X)$ and a point $x \in U$, let m_x denote the germ of m at x.

Let U be an open subset of X and $m \in \Gamma(U,\mathcal{M}_X)$. The *singular set* of m, sing(m), is defined to be $\{x \in U: m_x \notin \mathcal{O}_{X,x}\}$. Since $m_x \in \mathcal{M}_{X,x}$ can be represented (not necessarily uniquely) by $m_x = f/g$ where $f,g \in \mathcal{O}_{X,x}$, x will be in sing(m) if and only if $g(x) = 0$ for all possible representations $m_x = f/g$. Because of this, sing(m) is an analytic subvariety of U. A germ $m_x \in \mathcal{M}_{X,x}$ can always be represented by a function m in a neighborhood of x, so $\mathrm{sing}(m_x)$

can be defined as $(\mathrm{sing}(m))_x$.

A more thorough discussion of meromorphic functions can be found in Narasimhan [2], p. 88.

Certain facts about extensions and restrictions of holomorphic differential operators will be assumed; these can be found in [1].

2. MEROMORPHIC DIFFERENTIAL OPERATORS.

Let Diff_X^n be the sheaf of germs of holomorphic differential operators of order n on X. Diff_X^n is a coherent sheaf of $\mathcal{O}_X$-modules. A holomorphic differential operator $D \in \Gamma(X, \mathrm{Diff}_X^n)$ can be considered to be a C-homomorphism $D\colon \Gamma(X,\mathcal{O}_X) \longrightarrow \Gamma(X,\mathcal{O}_X)$ or a C-linear sheaf homomorphism $D\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X$. A germ of a differential operator, $D_x \in \mathrm{Diff}_{X,x}^n$, defines a C-homomorphism $D_x\colon \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}$.

The *sheaf of germs of* n^{th} *order meromorphic differential operators* on X is defined to be $M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}_X^n$. An n^{th} order meromorphic differential operator on X is a section $D \in \Gamma(X, M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}_X^n)$ and can be considered to be a C-homomorphism $D\colon \Gamma(X,\mathcal{O}_X) \longrightarrow \Gamma(X,M_X)$ or a C-linear sheaf homomorphism $D\colon \mathcal{O}_X \longrightarrow M_X$. A germ $D_x \in (M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}_X^n)_x$ defines a C-homomorphism $D_x\colon \mathcal{O}_{X,x} \longrightarrow M_{X,x}$.

Diff_X^n can be considered a subsheaf of $M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}_X^n$.

A meromorphic differential operator D on X is *singular* at $x \in X$ if its germ at x is not in $\mathrm{Diff}_{X,x}^n$. We denote the set of such singular points by sing(D). The set sing(D) is a subvariety of X. For a germ $D_x \in (M_X \otimes_{\mathcal{O}_X} \mathrm{Diff}_X^n)_x$, $\mathrm{sing}(D_x)$ can be defined as was done for meromorphic functions.

PROPOSITION 2.1. *Let* X *be an analytic space and* D *be a meromorphic differential operator on* X. *If* $D(\mathcal{O}_X) \subset \mathcal{O}_X$ *then* D *is holomorphic.*

Proof. Let n be the order of D. It suffices to prove this proposition locally, so it can assumed that $X \subset \mathbf{C}^N$ as a closed analytic subvariety. For $x \in X$, $\mathrm{Diff}_{X,x}^n$ is finitely generated as an $\mathcal{O}_{X,x}$-module since Diff_X^n is coherent.

Let D denote the germ of D at x. Then

$$D = m_1 D_1 + \dots + m_k D_k$$

where $D_1,\dots,D_k \in \mathrm{Diff}_{X,x}^n$ are the generators at x and $m_1,\dots,m_k \in M_{X,x}$. Each m_i can be represented by $m_i = f_i/g_i$ with $f_i, g_i \in \mathcal{O}_{X,x}$. Let $g = g_1 \cdots g_k \in \mathcal{O}_{X,x}$. Then the operator

$$D' = gD = f_1 D_1 + \dots + f_k D_k \in \mathrm{Diff}_{X,x}^n$$

and $D = 1/g\ D'$. Let $m = 1/g \in M_{X,x}$.
g can be extended to $\tilde{g} \in \mathcal{O}_{C^N,x}$ and thus m can be extended to $\tilde{m} = 1/\tilde{g} \in M_{C^N,x}$. Furthermore D' can be extended to $\tilde{D}' \in \mathrm{Diff}^n_{C^N,x}$ and the operator $\tilde{D} = \tilde{m}\ \tilde{D}' \in (M_{C^N} \otimes_{\mathcal{O}_{C^N}} \mathrm{Diff}^n_{C^N})_x$.
It is clear that $\mathrm{sing}(\tilde{D}) \subset \mathrm{sing}(\tilde{m})$ and since $\tilde{m}$ is an extension of $m \in M_{X,x}$, $\mathrm{sing}(\tilde{m}) \not\supset X_x$.
Let $\tilde{\varphi} \in \mathcal{O}_{C^N,x}$ and let $\varphi = \tilde{\varphi}|X \in \mathcal{O}_{X,x}$.
By hypothesis $D(\varphi) = m\ D'(\varphi) \in \mathcal{O}_{X,x}$. Furthermore $\tilde{D}(\tilde{\varphi}) \in M_{C^N,x}$ and $\tilde{D}(\tilde{\varphi})|X \in M_{X,x}$ because $\mathrm{sing}(\tilde{D}(\tilde{\varphi})) \subset \mathrm{sing}(\tilde{D})$ and $\mathrm{sing}(\tilde{D}) \not\supset X_x$.
Now, $\tilde{D}(\tilde{\varphi})|X = [\tilde{m}\ \tilde{D}'(\tilde{\varphi})]|X = \tilde{m}|X.\tilde{D}'(\tilde{\varphi})|X = m\ D'(\varphi) = D(\varphi)$ so if $\tilde{\varphi} \in \mathcal{O}_{C^N,x}$ then $\tilde{D}(\tilde{\varphi})|X \in \mathcal{O}_{X,x}$.
$\tilde{D}'$ can be expressed in terms of the generators of $\mathrm{Diff}^n_{C^N}$ as

$$D' = \sum a_{i_1 \dots i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}}$$

with $a_{i_1 \dots i_N} \in \mathcal{O}_{C^N,x}$, so

$$\tilde{D} = \sum \tilde{m}\ a_{i_1} \dots a_{i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}} .$$

The functions $z_1^{j_1} \dots z_N^{j_N}$ are all in $\mathcal{O}_{C^N,x}$ so $\tilde{D}(z_1^{j_1} \dots z_N^{j_N})|X \in \mathcal{O}_{X,x}$. Since all the $\tilde{m}\ a_{i_1 \dots i_N}$ are linear combinations of the $\tilde{D}(z_1^{j_1} \dots z_N^{j_N})$, $\tilde{m}\ a_{i_1 \dots i_N}|X \in \mathcal{O}_{X,x}$

Let $b_{i_1 \dots i_N} = (\tilde{m}\ a_{i_1 \dots i_N})|X \in \mathcal{O}_{X,x}$.

Let $\tilde{b}_{i_1 \dots i_N}$ be an extension of $b_{i_1 \dots i_N}$ to $\mathcal{O}_{C^N,x}$.

Let $$\bar{D} = \sum \tilde{b}_{i_1 \dots i_N} \frac{\partial^{i_1 + \dots + i_N}}{\partial z_1^{i_1} \dots \partial z_N^{i_N}} .$$

The operators $\tilde{D}$ and $\bar{D}$ agree on X_x, so for $\psi \in \mathcal{O}_{C^N,x}$

$$\tilde{D}(\psi)|X = \bar{D}(\psi)|X .$$

If $\psi \in d_{X,x}$ then $\psi|X = 0 \in \mathcal{O}_{X,x}$ so $\tilde{D}(\psi)|X = 0$.

Thus $\bar{D}(\psi)|X = 0$ and since $\bar{D}$ is a holomorphic operator, $\bar{D}(d_{X,x}) \subset d_{X,x}$ and $\bar{D}$ induces a holomorphic differential operator on X_x. Since $\bar{D}$ and $\tilde{D}$ agree on X_x, the induced operator must coincide with D and thus D is holomorphic.

3. MEROMORPHIC OPERATORS WITH STABLE IDEALS.

Let D be a differential operator on an analytic space X and let V be an analytic subvariety of X with associated sheaf of ideals d_V.

DEFINITION 3.1. d_V *is stable under* D *if for all* $x \in V$, $D(d_{V,x}) \subset d_{V,x}$.

Let V and W be analytic subvarieties of X.

DEFINITION 3.2. V *is transversal to* W *if there is an* $x \in V \cap W$ *such that* $V_x \not\subset W_x$ *and* $W_x \not\subset V_x$ *(in which case* V *is transversal to* W *at* x).

The proposition and corollary that follow deal with varieties transversal to sing(D).

PROPOSITION 3.3. *Let* D *be a meromorphic differential operator of order* n *on* X. *Let* $x \in X$. *Suppose that there is a germ* $f \in \mathcal{O}_{X,x}$ $f \neq 0$, *such that for all* $g \in \mathcal{O}_{X,x}$, $D(gf) \in \mathcal{O}_{X,x}$. *Then for all* $g \in \mathcal{O}_{X,x}$ *there is a representation* $D(g) = \frac{\varphi}{f^{n+1}}$ *where* $\varphi \in \mathcal{O}_{X,x}$.

Proof. The proof is by induction on n, the order of D. For n=1, $D = D' + h$ where D' is a meromorphic derivation and $h \in M_{X,x}$. For $g \in \mathcal{O}_{X,x}$

$$D(fg) = D'(fg) + hfg \quad \in \mathcal{O}_{X,x}$$

$$= f\,D'(g) + g\,D'(f) + hfg\ .$$

But $g\,D'(f) + hfg = g\,D(f) \in \mathcal{O}_{X,x}$ so $f\,D'(g) = k \in \mathcal{O}_{X,x}$ and $D'(g) = k/f$.

Now $\quad D(f^2) = 2f\,D'(f) + hf^2 \quad \in \mathcal{O}_{X,x}$.

But $2f\,D'(f) \in \mathcal{O}_{X,x}$ as above, so $hf^2 = k' \in \mathcal{O}_{X,x}$ and $h = \frac{k'}{f^2}$.

Therefore $\quad D(g) = D'(g) + hg = \frac{k}{f} + \frac{k'g}{f^2} = \frac{\varphi}{f^2}$ where $\varphi \in \mathcal{O}_{X,x}$.

Now suppose that order D = n and that the proposition is proved for differential operators of order $\leqslant$ n-1.

Let $g \in \mathcal{O}_{X,x}$. The operator D_f defined by

$$D_f(g) = D(fg) - f\,D(g)$$

has order $\leqslant n-1$ and satisfies the hypothesis $D_f(gf) \in \mathcal{O}_{X,x}$ for all $g \in \mathcal{O}_{X,x}$. Therefore

$$D_f(g) = \frac{\psi}{f^n} \text{ with } \psi \in \mathcal{O}_{X,x} .$$

Now $D_f(g) = D(fg) - f\,D(g)$ with $D(fg) = k \in \mathcal{O}_{X,x}$ so $f\,D(g) = k - \frac{\psi}{f^n}$.

Thus $D(g) = \frac{\varphi}{f^{n+1}}$ where $\varphi = kf^n - \psi \in \mathcal{O}_{X,x}$.

COROLLARY 3.4. *Let* D *be a meromorphic differential operator on* X. *Let* V *be an analytic subvariety with sheaf of ideals* d_V. *If* d_V *is stable under* D *then* sing(D) *is not transversal to* V.

Proof. By hypothesis, for all $x \in V$, $D(d_{V,x}) \subset d_{V,x}$. Then for $f \in d_{V,x}$ and $g \in \mathcal{O}_{X,x}$, $D(gf) \in \mathcal{O}_{X,x}$ so by Proposition 3.3 $D(g) = \varphi f^{-N}$ with $\varphi \in \mathcal{O}_{X,x}$. Thus $d_{V,x} \subset d_{\text{sing}(D),x}$ so $V_x \supset \text{sing}(D_x)$. Therefore sing(D) is not transversal to V at x, and this holds for all $x \in V$.

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CYCLIC RINGS

Joseph Neggers

ABSTRACT. Suppose that A is a ring with identity. Then A is cyclic if all modules M with a generating set $\{\ell_i | i \in w\}$ such that $\ell_i = \ell_{i+1}x_i$, $x_i \in A$, are in fact cyclic modules. Perfect rings are cyclic rings. All our examples of cyclic rings are perfect. Several properties of cyclic rings are established including other ways of characterizing cyclic rings. We believe that cyclic rings if not identical to the class of rings whose modules have minimal generating sets are very closely related to this class.

CYCLIC RINGS.

In the discussion below, all rings A have an identity and all modules are right unitary. A Steinitz ring is a local ring A whose Jacobson radical is T-nilpotent, i.e., given any sequence of elements $\{x_i | i \in w\}$, where $x_i \in R$, the Jacobson radical of A, there is some integer n such that $x_n \ldots x_1 = 0$. A perfect ring is a ring A such that the Jacobson radical R of A is T-nilpotent and such that A/R is a semi-simple Artinian ring. For more information concerning these rings, see [2], [3], [4].

For a ring A, if $\{x_i | i \in w\}$ is any sequence of elements of A, then we define $F(\{x_i | i \in w\})$ to be the quotient of the free module generated by the countable set $\{u_i | i \in w\}$ modulo the free module generated by the countable set $\{v_i | i \in w\}$, where $v_i = u_i - u_{i+1}x_i$. A sequence $\{x_i | i \in w\}$ is T-nilpotent provided the module $F(\{x_i | i \in w\}) = 0$. A subset of A is seen to be T-nilpotent if and only if every sequence $\{x_i | i \in w\}$ with $x_i \in A$ for all $i \in w$, is T-nilpotent.

Thus one can certainly describe some properties of rings by giving properties of some (or all) of the modules $F(\{x_i | i \in w\})$ where $\{x_i | i \in w\}$ is a sequence in A. If $F(\{x_i | i \in w\})$ is always a cyclic module, then we shall call A a cyclic ring. It is the purpose of this paper to establish some properties of cyclic rings. We note that as we are actually talking about right Steinitz rings, right perfect rings so we are talking about right cyclic rings. With the modules taken to be right unitary we shall, suppress the terminology right cyclic ring and use the terminology cyclic ring instead.

Our main classification is contained in the following theorem

THEOREM 1. *A ring* A *is cyclic if and only if for any sequence* $\{x_i | i \in w\}$ *of elements of* A *there is an index* i_0, *such that for all* $i \geqslant i_0$ *there is an index* $n \geqslant i+1$ *and an element* m_i *of* A *for which* $x_n \ldots x_{i+1} = x_n \ldots x_{i+1} x_i m_i$.

From this it follows almost immediately that the epimorphic image of a cyclic ring is a cyclic ring and a direct sum of two cyclic rings is a cyclic ring. All Steinitz rings are cyclic rings. To enlarge the class of examples further one shows that

THEOREM 2. *If* A *is a ring for which there exists an integer* k *such that every properly ascending chain of principal right ideals contains at most* k *terms, then* A *is cyclic.*

From this we see that if A is any algebra over a division ring D such that A is finite dimensional as a right vector space over D, then A is a cyclic ring since right ideals are subspaces.

From this we find that any semi-simple Artinian ring is a cyclic ring. We show that over a perfect ring A, a module M has m generators if and only if M/MR has m generators, where R is the Jacobson radical of A, and so perfect rings are cyclic as well. One also shows without much difficulty that

THEOREM 3. *If* A *is a cyclic ring, then the Jacobson radical* R *of* A *is* T-*nilpotent, and* A *satisfies the ascending chain condition on right principal ideals. If* $xy = 0$ *implies* $y = 0$, *then* x *is a unit. Also, if* $xy = 1$, *then* $yx = 1$.

Another way of identifying perfect rings is by stating that A is perfect provided the descending chain condition on left principal ideals holds. Thus rings of the type described in theorem 2 are not only cyclic but also left perfect. If D is a division ring and if G is a finite group, then D G = A is an algebra over D which is finite dimensional as a right and left vector space and hence the descending chain condition on right and left principal ideals holds with an upper bound $k = |G|$. This way we can construct perfect (cyclic) rings which are neither semi-simple Artinian nor Steinitz. Indeed, let D = GF(p), the field with p elements and let G be a finite group whose order $|G|$ is divisible by p and a prime q with $(p,q) = 1$. Since G is not a p-group D G is not a Steinitz ring [1] and since G contains elements of order p, D G is not a semi-simple Artinian ring by Maschke's theorem. If A = B G is the group ring of the group G with coefficients in the ring B, then if A is a cyclic ring, B is also cyclic since it is an epimorphic image of A by the norm homomorphism (For relevant information see [5], pp. 86-87).

A local cyclic ring is a Steinitz ring. Steinitz rings are those local rings whose modules have minimal generating sets. It is a question of some interest to give conditions identifying those rings whose modules have minimal generating sets. A necessary condition is that the Jacobson radical be T-nilpotent. If we require that every generating set of a module M over A contain a minimal generating set, then the ring A is in fact cyclic, since a module $F(\{x_i | i \in w\})$ contains a minimal generating set which is a subset of $\{h_i | i \in w\}$ with $h_i = u_i + V$, V generated by $\{v_i | i \in w\}$, if and only if it is cyclic. This suggests that possibly cyclic rings are those rings for which modules have minimal generating sets. This last property would make cyclic rings a very interesting class of rings indeed.

If we let E(A) be the collection of idempotents of the ring A, and if $e \leqslant f$ provided $ef = fe = e$, then it follows easily from theorem 3 that

THEOREM 4. *If* A *is a cyclic ring, then* E(A) *equipped with the partial order* $\leqslant$ *satisfies both the ascending chain condition and the descending chain condition.*

From this we find that every cyclic ring A is in fact a unique direct sum $A = A_1 + \ldots + A_n$ of cyclic rings A_i, where A_i contains only central idempotents 0 and 1. Furthermore it follows that a commutative ring is cyclic if and only if it is a finite direct sum of Steinitz rings which is so if and only if the ring is in fact perfect. One also shows that cyclic regular rings satisfy the descending chain condition on left principal ideals and are thus perfect. Finally, we have no examples of cyclic rings which are not also perfect.

Proof of theorem 1 and consequences. Suppose that the conditions stated in theorem 1 hold, and that $\{x_i | i \in w\}$ is any sequence of A. Let $F\{x_i | i \in w\}$ be the corresponding module and suppose $i \geqslant i_0$. Let h_i be the image of u_i in $F(\{x_i | i \in w\})$, as above. Then $h_{n+1}x_n \ldots x_{i+1} = h_{n+1}x_n \ldots x_i m_i$ implies $h_{i+1} = h_i m_i$ and since $h_i = h_{i+1}x_i$, it follows that $h_{i+1}A = h_i A$, whence, since this is so for all $i \geqslant i_0$, $F(\{x_i | i \in w\})$ is generated by $\{h_1, \ldots, h_{i_0}\}$ and thus by $\{h_{i_0}\}$. Hence A is cyclic.

On the other hand, if A is cyclic, then $F(\{x_i | i \in w\}) = gA$, and since $g = h_{i_0}a$ for some i_0, we may take $g = h_{i_0}$. Then, if we use the fact that $F(\{x_i | i \in w\}) = h_{i_0}A = h_iA$ for $i \geq i_0$, letting $h_{i+1} = h_i m_i$, we have $h_{i+1}(1 - x_i m_i) = 0$. If $h_j a = 0$, then $u_j a$ is an element of V, whence $u_j a = \sum_{i=1}^{n} v_i a_i$, i.e., $a_1 = \ldots = a_{j-1} = 0$, $a_j = a$, $x_n \ldots x_j a = 0$. Applying this to $a = (1 - x_i m_i)$ with $j = i+1$, we obtain the statement $x_n \ldots x_{i+1}(1 - x_i m_i) = 0$, which is precisely the condition given in the theorem.

Since the conditions of theorem 1 are preserved under homomorphism, it follows that the epimorphic image of a cyclic ring is also cyclic. Similarly, if A and B are cyclic rings, and if $\{(x_i, y_i) | i \in w\}$ is a sequence in A + B, then if $x_s \ldots x_{i+1}(1 - x_i m_i) = 0$ and $y_t \ldots y_{j+1}(1 - y_j n_j) = 0$ for all $i \geq i_0$, $j \geq j_0$ and for suitable $s \geq i+1$, $t \geq j+1$, selecting $k_0 = \max(i_0, j_0)$, $i, j \geq k_0$ and $r = \max(s, t)$, we have

$(x_r, y_r) \ldots (x_{i+1}, y_{i+1})(1 - (x_i, y_i)(m_i, n_i)) = 0$ and A+B is also cyclic.

If A is a Steinitz ring, then A is a local ring with a T-nilpotent Jacobson radical R. Thus, if $\{x_i | i \in w\}$ is any sequence of elements of A, then either there is an index i_0 such that $i \geq i_0$ implies x_i is a unit, or the sequence and all segments are themselves T-nilpotent. In the first case take $m_i = x_i^{-1}$ for $i \geq i_0$, in the second case select n such that $x_n \ldots x_{i+1} = 0$. It follows that A is a cyclic ring.

Proof of theorem 2. Suppose that A is not a cyclic ring. Then there is a sequence $\{x_i | i \in w\}$ of elements of A, such that for all i there is a $j(i) \geq i$ with $x_n \ldots x_{j(i)+1}(1 - x_{j(i)}m) \neq 0$ for all $m \in A$ and all $n \geq j(i) + 1$. This means that $x_n \ldots x_{j(i)} \notin x_n \ldots x_{j(i)+1}x_{j(i)}A$, and thus $x_n \ldots x_{j(i)+1}A \subset x_n \ldots x_{j(i)}A$ (proper containment).

Suppose now we select $i_1 = 1$, $i_2 = j(i_1), \ldots, i_\ell = j(i_{\ell-1})$, and $n \geq i_\ell + 1$. Then let $y_s = x_n \ldots x_{i_{s+1}}$. It follows readily that we have a proper ascending sequence $x_n \ldots x_{i_2}A \subset y_2A \subset \ldots \subset y_\ell A$ containing ℓ elements.

Hence if we let $\ell \geq k + 1$, we obtain a contradiction. From this the claims made above, following the statement of theorem 2, are virtually immediate.

PERFECT RINGS ARE CYCLIC.

Suppose now that M is a right A-module with R a T-nilpotent ideal. Then if M/MR has generators g_1+MR,...,g_k+MR (as an A-module or an A/R module), it follows that if $m_0 \in M$, then some linear combination $g_1+a_{10}+\ldots+g_k a_{k0}$ is congruent to m_0 module MR. Thus $m_0-(g_1 a_{10}+\ldots+g_k a_{k0}) = m_1 r_1$. Repeating this process with respect to m_1, we find $m_0-(g_1(a_{10}+a_{11})+\ldots+g_k(a_{k0}+a_{k1})) = m_2 r_2 r_1$. It is easy to see that we may in this way generate sequences $\{r_i\}$ $(r_i \in R)$ $\{m_i\}$ and $\{b_{1i},\ldots,b_{ki}\}$ such that $m_0-(g_1 b_{1i}+\ldots+g_k b_{ki}) =$ $= m_i r_i \ldots r_1$. Since R is T-nilpotent, taking i such that $r_i \ldots r_1 = 0$, it follows that M is generated by $\{g_1,\ldots,g_k\}$. Clearly if $\{g_1,\ldots,g_k\}$ generates M, then $\{g_1+MR,\ldots,g_k+MR\}$ generates M/MR.
Now if A is perfect and if R is its Jacobson radical, then R is T-nilpotent and A/R is a semi-simple Artininian ring, i.e., A/R is a cyclic ring. If we consider the module $F(\{x_i | i \in w\})/F(\{x_i | i \in w\})R$, then it is an A/R-module, and as an A/R-module it is isomorphic to the cyclic module $F(\{x_i+R | i \in w\})$. Since this latter module is cyclic, it follows that $F(\{x_i | i \in w\})$ is a cyclic module and thus A is also a cyclic ring.

Proof of theorem 3. An A-module M is quasi-cyclic if and only if it has a generating set $\{\ell_i | i \in w\}$ with $\ell_i = \ell_{i+1} x_i$ for some $x_i \in A$. It follows that there is a canonical epimorphism $F(\{x_i\}) \to M$ given by $h_i \to \ell_i$ for each quasi-cyclic module M. Hence A is cyclic if and only if all quasi-cyclic modules are cyclic.
Now suppose $\ell_1 A \subset \ell_2 A \subset \ldots \subset \ell_i A \subset \ell_{i+1} A \ldots$ is an ascending chain of principal right ideals. Then $\ell_i = \ell_{i+1} x_i$, i.e., the right ideal $\cup \ell_i A$ generated by $\{\ell_i | i \in w\}$ is a quasi-cyclic module and hence cyclic with generator $\ell_{i_0} A$. Hence A satisfied the ascending chain condition on principal right ideals.
Next, if x is not a left zero-divisor of A, then $x^\ell(1 - mx) = 0$ and $x^\ell(1 - xm) = 0$ implies $xm = mx = 1$. Also, if $x^\ell = x^{\ell+1}m$, then $x^{\ell+1} = x^{\ell+1}mx$, so that the first condition is a consequence of the second, for suitable ℓ. Hence if x has a left inverse it is a unit. Thus if x has a right inverse x' then x' is a unit and x is a unit. If we let U = {x | 1 - mx and 1 - xm is a non-unit for some m} , then $x \notin U$ provided for each $m \in A$, 1 - mx or 1 - xm is a unit. Thus the Jacobson radical of A is precisely the complement of U.
Now, if $\{x_i | i \in w\}$ is any sequence of elements of R, then

$x_n \ldots x_{i+1}(1 - x_i m) = 0$ for $i \geqslant i_0$, implies $x_n \ldots x_{i+1} = 0$ and $\{x_i | i \in w\}$ is a T-nilpotent sequence. But then R is a T-nilpotent set as asserted.

Proof of theorem 4. If $e \leqslant f$, then $ef = fe = e$, and thus $eA = feA \leqslant fA$, while if $eA = fA$, then $f = ex$, whence $ef = ex = e = f$. Hence a properly ascending chain of idempotents $e_1 < e_2 < \ldots < e_k$ implies a properly ascending chain of right principal ideals $e_1A < e_2A < \ldots < e_kA$. Since A satisfies the ascending chain condition for right principal ideals, E(A) satisfies the ascending chain condition as a partially ordered set.

Also, if $e \leqslant f$, then $(1 - e)(1 - f) = 1 - e - f + ef = 1 - f = (1 - f)(1 - e)$, so that $(1 - f) \leqslant (1 - e)$. Since an ascending chain $e_1 < \ldots < e_k$ gives rise to a descending chain $(1 - e_1) > (1 - e_2) > \ldots > (1 - e_k)$ and conversely, it follows that E(A) also satisfies the descending chain condition.

If we have an infinite orthogonal set of central idempotents, say $\{e_i | i \in w\}$, let $f_i = e_1 + \ldots + e_i$. Then $f_i f_{i+1} = f_{i+1} f_i = f_i$, and $f_1 < f_2 < \ldots$ is an infinite ascending chain in E(A), an impossibility if A is cyclic. Thus there exist minimal central idempotents, and these form a finite orthogonal set, say $\{e_1, \ldots, e_n\}$. It follows that $A = A_1 + \ldots + A_n$, where $A_i = Ae_i$, and that A_i is a cyclic ring with no central idempotents other than 0 or 1. The uniqueness of the decomposition follows from the uniqueness of the minimal central idempotents.

If A is a commutative cyclic ring, then $A = A_1 + \ldots + A_n$, where A_i is a commutative, cyclic and has no idempotents other than 0 or 1. Suppose $A = A_1$. If $x^\ell(1 - mx) = 0$, then $(x(1 - mx))^\ell = 0$, i.e., $x(1 - mx)$ is nilpotent. In particular $x(1 - mx)$ is an element of the prime radical R of A, and thus every prime ideal is maximal, while A/R is a regular ring. Hence, since $A = A_1$, it follows that A/R is a field and that R is the Jacobson radical of A. Hence $A = A_1$ is a Steinitz ring. Thus commutative cyclic rings are perfect.

If A is a regular cyclic ring, then given any element x of A, there is an element x' such that $xx'x = x$. From this, one concludes that $xx' = e$ is idempotent, and that $xA = eA$. Similarly, $Ax = Ae$ where $x'x = e$ is an idempotent. Thus, suppose $eA \leqslant fA$, where e and f are idempotents. For the idempotents $1 - e$ and $1 - f$ we have a relation $(1 - f)(1 - e) = 1 - e - f + fe = 1 - f$ since $e = fx$ implies $fe = e$, and thus $A(1 - e) \geqslant A(1 - f)$. In particular, if A does not satisfy the ascending chain condition on right principal ideals, then A is not cyclic. Hence if A is cyclic it satisfies the descending chain condition on left principal ideals. But this means that A is perfect. More directly, if

A is a regular cyclic ring, then if $x \in R$, $(x'x)^n = 0$ for some n, whence $x = x(x'x)^n = 0$ as well, i.e., A is semi-simple and perfect, i.e., A is semi-simple Artinian.

Thus it seems not at all impossible that all cyclic rings are perfect. On the other hand cyclic rings, as we mentioned above, may well be those rings whose modules have minimal generating sets. So, in conclusion, we conjecture that the class of rings whose modules have minimal generating sets is the class of cyclic rings and that this class of rings is precisely the class of perfect rings.

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EVOLUCION DEL CONCEPTO DE DIFERENCIAL *

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El concepto de diferencial se remonta, como es sabido, a la época en que nació el cálculo infinitesimal; para las funciones reales de una variable real el uso de la notación dy/dx fue de gran utilidad pero enmascaró la esencia del concepto de diferencial. No hace demasiado tiempo era corriente el siguiente lenguaje: dx, cuando x es la variable, representa el incremento infinitamente pequeño dado a la variable, y cuando x es una función, es el término de primer orden de su incremento; puede admitirse que lo esencial está, pero desde luego que no está bien explicitado. En lo que respecta a las diferenciales de orden superior la explicación era, en casi todos los textos y hasta hace relativamente pocos años, bastante confusa.

En lo que respecta a las funciones reales de varias variables reales era corriente, en los buenos textos de comienzos del siglo, y aún hasta 1940, decir que una función f era diferenciable en (a,b) si existían las derivadas parciales en ese punto y se decía después que la diferencial era:

$$df = f'_x(a,b)dx + f'_y(a,b)dy$$

en donde dx y dy representaban los incrementos infinitamente pequeños de las variables.

Desde luego que esta definición es inadecuada, y no pone de manifiesto la propiedad de mejor aproximación de la diferencial. Por otra parte, como era de esperar, con la sola hipótesis de existencia de las dos derivadas primeras no se obtenía ningún resultado. ¿Qué hacían entonces los tratadistas para obtener los teoremas esenciales?. Tomemos por ejemplo uno de los mejores tratados de análisis que se han escrito, el libro de Goursat en su primera edición; para obtener los resultados fundamentales, por ejemplo: continuidad de la función diferenciable, diferenciación de funciones compuestas, condiciones de monogeneidad de las funciones complejas de variable compleja, se añadían hipótesis suplementarias, la más empleada era: las derivadas parciales existen en un entor-

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no del punto y son continuas en él.

La primera definición correcta de la diferencial para funciones de varias variables es la de Stolz dada en su libro: "Grundzüge der Differential und Integral Rechnung" aparecido en 1893: la función f es diferenciable en el punto (a,b) si existen, en dicho punto, las dos derivadas primeras y si además:

$$f(a+h,b+k) - f(a,b) = (f'_x(a,b) + \varepsilon)h + (f'_y(a,b) + \mu)k$$

en donde ε y μ son funciones de h y de k que tienden a cero cuando h y k tienden a cero. La extensión a n variables es inmediata.

Stolz prueba que con su definición se pueden obtener los resultados del cálculo diferencial de varias variables que se obtenían con la hipótesis de existencia de derivadas parciales en un entorno y continuidad de las mismas en el punto, probó además que esta última hipótesis implicaba la diferenciabilidad, pero dió un contraejemplo para probar que la recíproca no era cierta; dicho contraejemplo es la función

$$f(x,y) = (x^2 + y^2).\text{sen } (x^2 + y^2)^{-1/2} \quad ; \quad f(0,0) = 0$$

que es diferenciable en (0,0) sin que haya continuidad de las derivadas parciales en ese punto.

Otro contraejemplo es la función:

$$F(x,y) = (x^2 + y^2)\ g((x^2 + y^2)^{1/2})$$

en donde g es una función continua sin derivada en ningún punto. La función F es diferenciable en (0,0) pero no hay derivadas parciales en el entorno de (0,0).

En lo que respecta a las condiciones de monogeneidad el primero que demostró que se necesitaban solamente las condiciones de Cauchy Riemann y la hipótesis de diferenciabilidad de u y v fue Fréchet en 1919.
Stolz observó que la hipótesis de existencia de las derivadas parciales era superflua, pero no insistió sobre este punto que tiene importancia en las extensiones de la teoría de la diferencial.

La definición de Stolz tardó mucho en llegar a los textos de enseñanza y pasó aún bastante más tiempo antes de que se le diera a la definición una forma intrínseca:

Si Ω es un abierto de R^n, f una aplicación de Ω en R y P un punto de Ω, la diferencial de f en P es un vector df(P) con la propiedad:

$$f(P + H) - f(P) = \langle df(P),H\rangle + \|H\| r(H); \quad \lim_{H\to 0} r(H) = 0$$

Para las aplicaciones de R en R^n no se presentaron dificultades, ya que una tal aplicación es diferenciable si, y sólo si, lo son sus componentes.

En lo que respecta a las aplicaciones de R^n en R^m se hizo durante mucho tiempo su estudio mediante un análisis complicado de matrices jacobianas sin poner de manifiesto que el problema era de la misma naturaleza que el de la diferenciación de una función real de variables reales, es decir que la diferencial en P es una aplicación lineal, df(P) de R^n en R^m, que se podrá representar, dada una base, como una matriz jacobiana

$$df(P) = \left(\frac{\partial f_i}{\partial x_k}\right)$$

y tal que:

$$f(P + H) - f(P) = df(P)[H] + \|H\| r(H); \quad \lim_{H \to 0} r(H) = 0$$

en donde el primer sumando del segundo miembro es el vector de R^m que se obtiene al aplicar la diferencial al vector H de R^n. (Usaremos siempre la notación [ ] en ese sentido).

Naturalmente, si m=1, la diferencial es un vector del espacio dual y df(P)[H] es el producto escalar $\langle df(P), H \rangle$, si es n=1, la diferencial es un vector y df(P)[H] es el producto del escalar H por el vector df(P).

Es importante señalar que, para el caso de funciones reales de varias variables reales, Hadamard dió, en 1923, una nueva definición de la diferencial basada en el teorema de diferenciación de funciones compuestas. La memoria se titula: "La notion de différentielle dans l'enseignement" lo que muestra que el autor pensaba que el interés era sólo didáctico pero posteriores desarrollos probaron que esta definición era también interesante en otros aspectos. Severi dió también una nueva definición de la diferencial que no tuvo mayor repercusión.

Vamos a ver ahora cómo se produjo la extensión de la diferenciación a las aplicaciones entre espacios de dimensión infinita cuyo estudio empezó bastante antes de quedar bien establecida la definición para espacios de dimensión finita.

El precursor de la teoría fue Volterra, que ya en 1887 introdujo el concepto de función de línea, siguieron los trabajos de varios matemáticos entre los cuales el más importante, antes de la primera guerra mundial, fue el de Gateaux; posteriormente Paul Levy obtuvo resultados de interés, pero puede considerarse como el creador de la teoría moderna a Fréchet; su memoria: "Sur la notion de différentielle dans l'analyse générale" (Ann. Ec. Normale Sup., 1925) es la primera exposición de conjunto de la teoría en forma clara y con la mayor parte de los resultados básicos. Este trabajo fue completado por dos importantes memorias, una de Graves: "Riemann Integration and Taylor's Theorems in General Analysis" y otra de Hildebrandt y Graves: "Implicit Functions and their Differentials in General Analysis", ambas publicadas en 1929 en las "Transactions of the A.M.S.".

La teoría de la diferencial para aplicaciones entre espacios normados, iniciada con la memoria de Fréchet que acabamos de mencionar, es hoy día una teoría clásica que se enseña sistemáticamente en muchas universidades y son numerosos los textos en los que está expuesta. Uno de los primeros autores que incluyó esta teoría en un texto relativamente elemental fue Dieudonné en su libro Fundamentos del Análisis Moderno, aparecido en 1960.

Recordemos brevemente la formulación de esta teoría. Sean X e Y dos espacios normados reales , L(X,Y) el espacio de las aplicaciones lineales y contínuas de X en Y, Ω un abierto de X, x_o un punto de Ω y f una aplicación de Ω en Y. Se dice que f es *diferenciable Fréchet* en x_o si existe un elemento de L(X,Y), que se dice que es la diferencial de f en x_o y se representa con la notación $df(x_o)$, tal que:

$$f(x_o + h) - f(x_o) = df(x_o)[h] + \|h\| r(h); \quad \lim_{h\to 0} r(h) = 0$$

Se obtienen así todos los resultados clásicos: unicidad, linealidad, continuidad, diferenciación de productos (que se definen como aplicaciones bilineales continuas), el teorema básico de diferenciación de funciones compuestas, la fórmula del incremento finito, la determinación de máximos y mínimos cuando Y = R (con la que se obtiene una deducción elegante de las ecuaciones de Euler del Cálculo de Variaciones), los teoremas de diferenciación de sucesiones así como los teoremas de función inversa y de funciones implícitas.

También se pueden extender los teoremas sobre diferenciales exactas en la forma siguiente, que esbozaremos, por ser menos frecuente su inclusión en los textos: sean X e Y dos espacios de Banach reales, Γ una curva orientada simple en X, α una aplicación de Γ en L(X,Y). Se define la integral curvilínea en la forma habitual:

$$\int_\Gamma \alpha(x)[dx] = \lim \sum \alpha(t_i)[x_i - x_{i-1}] .$$

y se demuestra que el límite existe cuando α es continua y Γ rectificable, hipótesis que mantendremos en lo que sigue. Se tienen los siguientes resultados sobre integración de diferenciales exactas:

Si Ω es un abierto conexo de X, Γ una curva orientada simple, contenida en Ω, de extremos a y b y si f es una aplicación de Ω en Y continuamente diferenciable en Ω, se tiene:

$$\int_{\Gamma} df(x)[dx] = f(b) - f(a)$$

Si Ω es un abierto conexo en X, α es una aplicación continua de Ω en L(X,Y), tal que las integrales de α sobre dos poligonales con los mismos orígenes y extremos sean iguales, entonces es posible definir:

$$f(x) = \int_a^x \alpha(u)[du]$$

y esta función es diferenciable en Ω y $df(x) = \alpha(x)$.

Hay también otro enfoque posible para definir la diferencial de una aplicación entre dos espacios normados el cual está basado en la memoria antes mencionada de Hadamard, cuya idea esencial es primero la de definir, en la forma habitual, la derivada de una aplicación de R en un normado y definir después la diferencial usando la regla de diferenciación de funciones compuestas. Precisando:

Sean X e Y dos espacios normados reales, Ω un abierto de X, x_o un punto de Ω y f una aplicación de Ω en Y. Se dice que f es diferenciable Hadamard en x_o si existe un elemento de L(X,Y), que se dice que es la diferencial de f en x_o, y que representaremos con la notación $df(x_o)$, con la siguiente propiedad:

Cualquiera que sea la aplicación g de un intervalo real en Ω, derivable en λ_o con $g(\lambda_o) = x_o$, entonces la aplicación compuesta $G(\lambda) = f(g(\lambda))$ es derivable en λ_o y se tiene:

$$G'(\lambda_o) = df(x_o)[g'(\lambda_o)].$$

Hadamard probó que si $X = R$ e $Y = R^m$, la definición de Stolz y la suya coincidían y es fácil ver que lo mismo ocurre si se hace $X = R^n$.

Para el caso de dimensión cualquiera, finita o infinita, es claro que una aplicación diferenciable en el sentido de Fréchet lo es en el de Hadamard y esta implicación subsistirá con cualquier definición de la diferencial que cumpla la ley de diferenciación de funciones compuestas y que se reduzca a la derivada ordinaria para $X = R$.

En cambio con dimensión infinita existen aplicaciones diferenciables Hadamard que no son diferenciables Fréchet. Esbozaremos el contraejemplo; sea X el espacio de las funciones continuas en un intervalo con la norma del supremo y M la aplicación de X en R definida por $M(f) = \max\{f(x)\}$. Sea f_o un vector de X; es necesario para que M sea diferenciable en f_o que esta función alcance su máximo en un solo punto, ya que se prueba facilmente que si M es diferenciable en f_o y si c es un punto en el cual f_o alcanza su máximo, entonces la diferencial, que suponíamos existente, tiene que ser la delta de Dirac en c, luego por la unicidad de la diferencial, c tiene que ser el único punto de máximo de f_o.

Ahora bien puede probarse que si f_o alcanza su máximo en un solo punto, M es diferenciable Hadamard en el punto pero no es nunca diferenciable Fréchet.

Otra forma de encarar la extensión de la diferencial fue iniciada por Gateaux en 1913 y desarrollada en su memoria póstuma: "Fonctions d'une infinité de variables indépendantes" publicada en el Bull. de la Soc. Math. de France en 1919. La teoría de Gateaux presenta interés y tiene importantes aplicaciones. La diferencial se define de la siguiente manera:

Sean X e Y dos espacios normados, Ω un abierto de X, x_o un punto de Ω y f una aplicación de Ω en Y. Tomemos h perteneciente a X y distinto del cero, definimos:

$$F(t) = f(x_o + th) \quad \text{con } x_o + th \in \Omega$$

y supongamos que existe el límite:

$$F'(0) = \lim_{t\to 0} \frac{F(t) - F(0)}{t}$$

entonces se dice que F'(0) es la variación de Gateaux de f en x_o respecto del incremento, que se designa con la notación $Vf(x_o;h)$; se ve que esta definición es la extensión natural del concepto de derivada en una dirección.

Si $Vf(x_o;h)$ existe para todo h y si considerada como función de h es lineal y continua se dice que V es la diferencial de Gateaux, o diferencial débil de f en x_o.

La definición de Gateaux es muy general ya que se pueden dar ejemplos de aplicaciones de R^2 en R que son diferenciables en el sentido de Gateaux y no lo son en el de Stolz.

Desarrollada la teoría de la diferencial dentro del dominio de los espacios normados se planteó la posibilidad de extenderla a un cuadro más general. Fréchet,en su memoria de 1937, fue el primero que planteó la necesidad de ir mas allá del cuadro de los espacios normados y sugirió que la definición de Hadamard sería la más adecuada para hacer esta generalización.

Naturalmente la extensión a un dominio más amplio que el de los espacios normados necesitaba una buena definición de ese dominio, que en 1937 no existía aún. Cuando la teoría de espacios vectoriales topológicos tomó forma definitiva fue posible desarrollar el cálculo diferencial en dichos espacios.

Comenzaremos por un caso simple; sea Y un espacio vectorial topológico y f una aplicación de un intervalo real en Y. Para que una definición de la diferencial sea de interés debe conducir en este caso a definir la diferencial en x_o por la fórmula: $df(x_o)[t] = t.f'(x_o)$, en donde t es escalar real y $f'(x_o)$ es el vector derivada:

$$\lim_{h\to 0} \frac{f(x_o + h) - f(x_o)}{h}$$

y parece conveniente que se deba cumplir el teorema siguiente: si f'(x) es nula en todo el intervalo entonces f es una constante.

Ahora bien, esta propiedad puede no cumplirse para el caso en que Y sea un espacio no localmente convexo; en efecto sea Y el espacio de las funciones reales medibles en (0,1) con la convergencia en medida, es decir que una base de entornos del cero está formada por la familia $V(\varepsilon,\delta)$ definida por la propiedad f pertenece a $V(\varepsilon,\delta)$ si la medida del conjunto de puntos en los cuales es $|f(x)| > \varepsilon$ es menor que δ. Es conocido que así se obtiene un espacio vectorial topológico metrizable y no localmente convexo.

Consideremos la aplicación F de (0,1) en Y definida así: $F(t) = H_t$ en donde la función H_t se define por $H_t(x) = 0$ para $x \leqslant t$ y $H_t(x) = 1$ para $x > t$. El cociente de incrementos,

$$\frac{F(t+h) - F(t)}{h}$$

es una función que es nula salvo en un intervalo de longitud h, luego dado $V(\varepsilon,\delta)$, basta tomar $h < \delta$ para que el cociente de incrementos esté en el entorno, lo que implica $F'(t) = 0$ y sin embargo F no es constante.

Este resultado parece indicar que el dominio indicado para una buena teoría debe ser el de los espacios localmente convexos al menos si se buscan propiedades de tipo global; las propiedades de tipo local, que sólo dependen de la diferenciabilidad en un punto, son válidas aún cuando no se haga la hipótesis de convexidad local.

Las formas de extensión a los espacios localmente convexos del concepto de diferencial presentan características diferentes según que se quiera generalizar la definición de Hadamard (o la de Gateaux) o la definición de Fréchet.

En el primer caso la definición dada para los espacios normados se extiende en forma inmediata, no así las demostraciones que exigen técnicas y razonamientos diferentes. La teoría fue desarrollada por Balanzat (Revista de la U.M.A., 1962, Anais da Academia Brasileira de Ciências, 1962 y Mathematicae Notae , 1964).

La extensión para la diferencial de Fréchet implica obtener una condición equivalente a la

$$\frac{f(x_o+h) - f(x_o) - df(x_o)[h]}{\|h\|} \xrightarrow[h\to 0]{} 0$$

en la que interviene de manera esencial la norma.

Para obtener esta generalización se han elaborado numerosas teorías, en su mayoría a partir de 1960, citaremos entre los principales autores: Michal, Hyers, Lamadrid, Sebastiao e Silva, Marinescu, Vainberg, Engel, Bastiani, Keller y otros. No es posible, por razones de tiempo en esta conferencia hacer un estudio detallado y comparativo de las diferentes definiciones, nos remitimos a la excelente exposición sobre el tema: Averbukh and Smolyanov, "The various definitions of the derivative in linear topological spaces", Russian Mathematical Surveys, vol. 23, 1968, pgs. 67-113. La tabla de las distintas definiciones ocupa tres páginas y hay además dos diagramas sobre sus relaciones mutuas.

Parece indicado señalar la existencia de otras dos publicaciones en donde se hace una exposición al día de la teoría y se da además una extensa bibliografía, que recomendamos a los que quieran ampliar los resultados, forzosamente someros, de esta conferencia. Las dos publicaciones son: la de Averbukh and Smolyanov, "The theory of differentiation in linear topological spaces", Russian Mathematical Surveys", vol. 22, 1967, pgs. 201-258, y el artículo de M.Z. Nashed: "Differentiability and Related Properties of Nonlinear Operators; some Aspects of the Role of Differentials in Nonlinear Functional Analysis" publicado en el libro "Non Linear Functional Analysis and Applications", Academic Press,1971,pgs.103-310.

Daremos ahora una definición de la diferenciabilidad Fréchet que, abarca, haciendo modificaciones de detalle y particularizaciones, una gran parte de las definiciones dadas por los distintos autores.

Sean X e Y dos espacios vectoriales topológicos, σ una familia de subconjuntos de X que debe cumplir un cierto número de condiciones, análogas pero no idénticas, a las empleadas para definir las σ-topologías en L(X,Y).

Sea Ω un abierto de X, x_o un punto de Ω y f una aplicación de Ω en Y. Se dice que f es σ-diferenciable en x_o si existe $df(x_o) \in L(X,Y)$, que es la diferencial, tal que para $S \in \sigma$ y $h \in S$, se tenga:

$$\lim_{t\to 0} \frac{f(x_o+th) - f(x_o)}{h} = df(x_o)[h],$$

con convergencia uniforme en S.

En particular si σ es la familia de los finitos se dice que se tiene la diferencial débil (ligada con la de Gateaux), si es la familia de los compactos se dice que se tiene la diferencial compacta, (ligada con la de Hadamard) y finalmente si se toma la familia de los acotados se habla de la diferencial acotada.

Las distintas teorías de la diferenciación permiten obtener bastantes propiedades de la teoría para los normados, aún cuando algunas de ellas tienen una forma algo diferente. Hay, por ejemplo, varias formas de extender el teorema de los incrementos finitos para espacios localmemte convexos de las cuales mencionaremos la siguiente: Sea f una aplicación diferenciable Hadamard en un abierto convexo y supongamos que el conjunto de las diferenciales en los puntos de dicho conjunto sea equicontinuo; entonces dado un entorno V de cero en Y que sea cerrado y absolutamente convexo, existe un entorno U de cero en X, tal que: si a y b están en el abierto y $(b-a) \in tU$, entonces $f(b) - f(a) \in tV$.

La definición de las integrales curvilíneas se hace como para el caso de los espacios normados y siguen valiendo los teoremas sobre diferenciales exactas.

Puede igualmente hacerse un estudio de las diferenciales de orden superior, que aquí nos limitaremos a mencionar.

Por otra parte si bien algunas propiedades se generalizan, hay otras que no se pueden extender, lo que marca diferencias importantes entre la teoría para espacios normados y para espacios vectoriales topológicos, por ejemplo en este último dominio la diferenciabilidad en un punto no implica la continuidad en dicho punto y hay incluso resultados que prueban que esto no es demasiado excepcional, aclaremos este punto.

Recordemos que un espacio vectorial topológico es sucesional, cuando $x \in A$, implica la existencia de una sucesión (x_n) de elementos de A tales que $x = \lim x_n$. Naturalmente todos los espacios metrizables son sucesionales pero se puede dar ejemplos de espacios vectoriales topológicos sucesionales y no metrizables. Se obtienen los resultados siguientes:

Sea X un espacio sucesional e Y un espacio vectorial topológico, cualquier aplicación de X en Y diferenciable Hadamard en un punto es continua en dicho punto.

Sea X un espacio no sucesional e Y un espacio vectorial topológico, siempre se puede definir una aplicación de X en Y diferenciable Hadamard y discontinua en un punto.

Sean X e Y dos espacios vectoriales topológicos y f una aplicación de X en Y; suponemos que se toma una definición de σ-diferenciabilidad tal que la familia σ contenga todas las sucesiones convergentes. Entonces la condición necesaria y suficiente para que cada aplicación de X en Y que sea σ-diferenciable en x_o sea también continua en x_o, es que X sea sucesional.

Cuando se considera la diferenciabilidad en un abierto el problema cambia y hay distintos resultados de los que citaremos el siguiente:

Si f es diferenciable Hadamard en un entorno de x_o y si el conjunto de las diferenciales en ese entorno es equicontinuo, entonces f es continua en x_o.

Otra diferencia de interés con el caso de los normados es que no se conocen teoremas buenos de existencia para las funciones implícitas. El enunciado del teorema de existencia y diferenciabilidad es facilmente traducible al caso de los espacios vectoriales topológicos, pero no así la demostración, es más,la traducción literal lleva a un resultado falso, aún en el caso de los espacios metrizables.

Queda la posibilidad de obtener algunos resultados con traducciones menos literales o con nuevas definiciones de la diferencial, hay algunos resultados parciales de los cuales nos limitaremos a señalar el de ver Eecck "Sur le calcul différentiel dans les espaces vectoriels topologiques", Cahiers de Topologie et Géométrie Différentielle, 1974, en donde con, una nueva forma de definir la diferencial,obtiene la diferenciabilidad cuando se admite la existencia de la función implícita.

Para terminar mencionaremos la existencia de otros trabajos en los que se estudia la diferenciación con estructuras algo distintas de las de los espacios vectoriales topológicos, este aspecto no lo podemos desarrollar por falta de tiempo, nos limitaremos a señalar para estructuras pseudo-topológicasel texto de Frölicher and Bucher: "Calculus in Vector Spaces without Norm", Lectures Notes n° 30 y para las estructuras bornológicas los trabajos de Colombeau, en particular la memoria: "Sur quelques particularités du calcul différentiel dans les espaces bornologiques ou topologiques", Revue Roumaine de Mathématiques Pures et Appliquées, vol. 17, 1973.

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XXVI REUNION ANUAL DE LA U.M.A.

Auspiciada por la Universidad Nacional de San Luis, entre los días 16 y 17 de setiembre de 1976 se realizó la XXVI Reunión Anual de la Unión Matemática Argentina. La sesión inaugural tuvo lugar el día 16 a las 17 hs. con la alocución del señor rector de la Universidad Nacional de San Luis, Dr. Genaro Neme. A continuación usaron de la palabra el Prof. Modesto González y el Dr. Luis A. Santaló. El mismo día se dictó la conferencia "Julio Rey Pastor", que en esta oportunidad estuvo a cargo del Dr. M.A. Balanzat quien disertó sobre el tema "Evolución del concepto de diferencial". El día 17, en horas de la mañana expuso el Dr. Eduardo Zarantonello sobre el tema "Contracciones extremales en el espacio de Hilbert", mientras que en horas de la tarde le correspondió al Prof. Raymon Gerard dictar una conferencia sobre el tema "Convergencia de soluciones formales de ecuaciones diferenciales". Las sesiones de comunicaciones tuvieron lugar en horas de la mañana y tarde. El día 17, a las 11 hs. se realizó un concierto a cargo del Coro Universitario.

Como culminación de la XXVI Reunión Anual de la U.M.A., el día sábado serealizó la Asamblea General Ordinaria, con elección de autoridades y aprobación de memoria y balance. A propuesta del Prof.Gaspar se decidió nombrar Miembro Honorario al Dr. Alberto González Domínguez, moción que se recibió con gran entusiasmo y fue aprobada por unanimidad. Asimismo se realizó la exposición del balance, decidiéndose aumentar las cuotas, fijándose en $ 600 la cuota social para los socios titulares y $ 300 para los socios adherentes. A continuación se efectuó la elección de la nueva Junta Directiva de acuerdo con lo dispuesto por el Art. 11 del Estatuto. De acuerdo con los cómputos, la nueva junta directiva quedó constituída de la siguiente manera: Presidente: Orlando E. Villamayor; Vicepresidente 1°: Luis A. Santaló; Vicepresidente 2°: E. Gaspar; Secretario: Carlos Germán D. Gregorio; Prosecretario: Nicolás D. Patetta; Tesorero: Adrián Paenza; Protesorero: Ricardo Noriega; Director de publicaciones: Darío Juan Picco.

Posteriormente a la asamblea se recibieron los votos para vocales regionales, obteniéndose los siguientes resultados: Buenos Aires: Juan J. Martínez; Centro: Humberto Alagia; Cuyo: María R. Berraondo; Litoral: Carlos Meritano; Nordeste: Héctor Tamburini; Noroeste: Raúl Lucioni; Sur: María L. Gurmendi.

Se agradeció a la Universidad Nacional de San Luis por haber auspiciado esta reunión y especialmente a la Lic. María Rosa Berraondo por su colaboración en la organización de la misma.

Las comunicaciones presentadas fueron las siguientes:

ALAGIA, H. R. y SANCHEZ, C. U. (I.M.A.F.): *Estructuras spin en variedades pseudoriemannianas.*

La noción de estructura spin para variedades Riemannianas se generaliza a variedades M con una métrica indefinida de signatura (p,q). Para tales variedades se define el concepto de (p,q)-orientabilidad y se introduce el grupo Spin (p,q). Una estructura Spin (p,q) sobre M es un fibrado principal sobre M, con grupo de estructura Spin(p,q) que satisface ciertas condiciones. El principal resultado que se demuestra, es que la existencia de tal estructura equivale a la anulación de las segundas clases de Stieffel-Whitney de dos subfibrados complementarios del fibrado tangente. Una clase de ejemplos se obtiene considerando variedades de la forma G/T, G grupo de Lie compacto, T un toro maximal. Esta comunicación completa y amplía el informe preliminar presentado en la reunión de la U.M.A. en 1975.

ALVAREZ ALONSO, J. (U.N.B.A.): *Existencia de un cálculo funcional sobre ciertas álgebras de operadores seudodiferenciales.*

Sea A un operador lineal, acotado y autoadjunto, definido sobre un espacio de Hilbert H; si P es el álgebra de los polinomios en una variable con coeficientes reales, se sabe que la aplicación

$$P \longrightarrow L(H,H)$$
$$P \longrightarrow P(A)$$

es continua, cuando se da a P la topología de la convergencia uniforme sobre los compactos de R. Esto permite definir f(A), para una función f: R $\longrightarrow$ R continua. Queda así determinado lo que suele llamarse un cálculo funcional sobre la subálgebra real de $L(H,H)$, formada por los operadores autoadjuntos. Partiendo de esta observación, se intenta aquí, en primer lugar, definir con precisión el concepto de cálculo funcional sobre un álgebra de operadores, poniendo énfasis en el espacio de funciones que actúan. Luego se introducen ciertas álgebras de clases de operadores seudodiferenciales y se analiza la existencia de un cálculo funcional en ellas.

BORGHI, O. S. P. (U.N. San Luis): *Extensión de una función aditiva definida sobre una semi-álgebra difusa.*

PROPOSICION. Sea X un conjunto habitual. Sea J una semi-álgebra difusa de partes difusas de X. Para toda función aditiva P: $\longrightarrow$ [0,1] tal que P(X) = 1, entonces $P'(A) \overset{d}{=} \sum_I P(S_i)$ define sin ambigüedad la única prolongación P' de P que sea aditiva sobre a = [J], (álgebra difusa generada por la semi álgebra difusa). Si P es σ-aditiva sobre J, entonces P' es σ-aditiva sobre a.

BOUILLET, J. (U.N. Salta): *Un teorema de comparación para ciertas ecuaciones diferenciales de tipo parabólico.*

Sea la ecuación $(D(u)\,u_x)_x - (a(u))_t = 0$, donde $D(s) \geqslant 0$ es localmente integrable y a(s) es estrictamente creciente. Se supone que las funciones admisibles u, v a comparar son continuas en un dominio acotado G cuya proyección sobre el eje t es el intervalo (0,T), tienen flujo $D(u)u_x$ absolutamente continuo hasta la frontera parabólica $\partial_p G$ en la variable x, tal que $(D(u)u_x)_x$ sea integrable donde $u < v$, y que a(u) es absolutamente continua en la variable t y $(a(u))_t$ sea integrable donde $u < v$. Finalmente, se supone que las funciones u, v satisfacen la ecuación en casi todo punto de G. Entonces si $u \geqslant v$ en casi todo punto de $\partial_p G$, es $u \geqslant v$ en G. La demostración se basa en la positividad del funcional

$$g \longrightarrow \iint_{Q_r} \{(g(D(u)u_x - D(v)v_x))_x - (g(a(u) - a(v)))_t\} \, dx \, dt$$

donde $Q_r = \{(t,x)/t \leqslant r \text{ y } u < v\}$. Se trata de una versión parabólica de una técnica usada por Douglas, Du Pont & Serrin (Arch. Rat. Mech. Anal., V.42 n°3, 1971) para el caso elíptico. Es posible generalizar el resultado anterior en varias direcciones, por ejemplo aplicarlo a condiciones de contorno de segunda especie. Bajo hipótesis convenientes es también aplicable al caso de dominios no acotados.

CORACH, G. y LAROTONDA, A. (U.N.B.A.): *Sobre el álgebra de funciones de una variedad diferenciable.*

En la presente nota se establecen condiciones necesarias y suficientes para que un álgebra topológica A sea isomorfa al álgebra de aplicaciones de clase C^∞ de una variedad diferenciable X, de dimensión finita. Se establece de esta forma una correspondencia funtorial entre la categoría de variedades diferenciables y una categoría de álgebras topológicas, correspondencia que deviene una equivalencia categorística. Esta equivalencia tiene como fundamento una "algebrización" de la geometría diferencial (o al menos, de parte de ella), de modo análogo al caso de "variedades algebraicas afines" y "álgebras afines", o bien "espacios compactos" y "C*-álgebras conmutativas"; en tal sentido hay puntos de contacto con la teoría de álgebras de funciones holomorfas en variedades de Stein.

DAMKÖHLER, W. (San Luis): *Ampliación convexa de cuerpos convexos.*

Primero se bosqueja la demostración del teorema siguiente:
TEOREMA. Sea en el espacio $\{\xi_\alpha/\alpha = 1,2,\ldots,m\}$, de m dimensiones, $F(\dot{x}_\alpha)$ la "función soporte" (Stützfunktion) de un convexo completamente redondo y dos veces continuamente diferenciable

$$\Gamma_F: \xi_\alpha = F_{\dot{x}_\alpha}(\dot{x}_\beta) \quad (\alpha,\beta = 1,2,3,\ldots,m)$$

en el cual los $\dot{x}_\alpha$ designan las componentes del vector normal exterior de Γ_F; y sea $C_F: \dot{x}_{m+1} = -F_\mu(\dot{x}_\alpha)$ $(\alpha = 1,2,\ldots,m)$ una cinta cualquiera torcida a través de Γ_F con la única restricción de ser 2 veces continuamente diferenciable y tocar en ningún punto rozando el espacio portador de Γ_F, sino penetrar en todos sus puntos al espacio circundante de (m+1) dimensiones. Entonces existe siempre un cuerpo convexo, completamente redondo y dos veces continuamente diferenciable Γ_ϕ en ese espacio circundante de (m+1) dimensiones, que pasa por Γ_F y toca allí rozando la cinta torcida C_F.

DAMKÖHLER, W. (San Luis): *Aplicación del teorema anterior al cálculo de variaciones.*

Este teorema es útil en la construcción de un método para tratar "directamente" extremales estacionarias de un problema variacional F sin la necesidad de recurrir a la teoría de M. Morse: Se acopla para tal fin a $F(x_\alpha, \dot{x}_\alpha)$, que opera en un espacio de los $\{x_\alpha/\alpha = 1,2,\ldots,m\}$ de m dimensiones, otro $\phi(x_\alpha, \dot{x}_i)$ en un espacio de $i = 1,2,3,\ldots,n>m$ dimensiones, tal que todas las extremales F resultan proyecciones de extremales ϕ. Por adecuada elección de $\phi(x_\alpha, \dot{x}_i)$, es posible conseguir que el número de puntos conjugados a lo largo de un arco de la extremal ϕ sea menor que a lo largo de su proyección F. Y se aproxima así más a una extremal minimizante, que es caracterizada por la ausencia total de puntos conjugados.

D'ATTELIS, C.E. (C.N.E.A.): *Una dilatación J-isométrica y causal para cierta clase de operadores.*

Las dilataciones de operadores definidos en espacios de Hilbert tienen aplicación en problemas de síntesis de circuitos. En este trabajo se construye una dilatación para operadores lineales pertenecientes a una clase que contiene a todos los operadores contractivos (sistemas pasivos) y a los operadores simplemente activos (o expansivos) que verifican cierta acotación. La dilatación se construye de tal manera que resulta J-isométrica, es decir isométrica en un espacio de Krein, y también causal, lo que es esencial para considerar las aplicaciones físicas mencionadas.

FERRO FONTÁN, C., GRATTON, J. y GRATTON, R. (U.N.B.A.): *Solución autosimilar para el colapso de una cáscara gaseosa.*

Las ecuaciones en derivadas parciales que describen el flujo de un gas ideal.poseen soluciones invariantes ante el grupo de los cambios de escala. Es sabido que ellas son válidas físicamente en ciertos límites asintóticos. En esta comunicación se presenta una familia de soluciones de este tipo que describe la compresión óptima (isoentrópica) de una cáscara fina y esférica de gas ideal. Las soluciones son analíticas durante todo el proceso salvo una onda de choque generada al tiempo de colapso, y el grado de compresión final es arbitrario. También se discute el espectro de autovalores del grupo dimensional que corresponde a estas soluciones.

GNAVI, G. (U.N.B.A.): *Factorización de operadores J-bicontractivos reales meromorfos en el semiplano de la derecha.*

Sea H un espacio de Hilbert separable en el cual está definida una operación de conjugación. Si A es un operador lineal acotado definido en H, A* denota el operador adjunto y $\bar{A}$ el operador conjugado de A. Sea P_+ un ortoproyector y $P_- \equiv I - P_+$. El operador $J \equiv P_+ - P_-$ satisface a las relaciones $J^* = J$ y $J^2 = I$. Un operador A se denomina J-contractivo si $A^*JA \leqslant J$. Es J-bicontractivo si A y A* son J-contractivos. Si $A^*JA = J$ se denomina J-unitario. Diremos que el operador S(p) pertenece a la clase M_J^R, con $J = \bar{J}$, si satisface a las siguientes condiciones: i) S(p) y $S^{-1}(p)$ son operadores meromorfos en Re $p > 0$; ii) S(p), en cada punto de holomorfismo, es igual a un operador J-bicontractivo; iii) $S(\bar{p}) = \overline{S(p)}$.

TEOREMA 1. Si $S(p) \in M_J^R$ tiene un polo de orden s_0 en p_0, vale la fórmula $S(p) = S_2(p).S_1(p)$ donde $S_2(p) \in M_J^R$ tiene en p_0 un polo de orden s_0-1 y $S_1(p) \in M_J^R$ es un operador J-unitario en Re $p = 0$, cuyas únicas singularidades son polos simples en p_0 y $\bar{p}_0$.

TEOREMA 2. Sea $S(p) \in M_J^R$ holomorfo y J- unitario en un segmento Σ del eje imaginario con excepción del punto p_0, interior a Σ, en el cual S(p) tiene un polo de orden s_0. Bajo tales condiciones vale una fórmula de factorización similar a la del Teorema 1.

El operador A es nuclear sii $\|A\| \equiv tr(A^*A) < \infty$. Hemos demostrado anteriormente (cf. comunicación a la Reunión Anual de la U.M.A., Salta, 1975) que bajo ciertas condiciones el producto infinito correspondiente a la extracción de todos los polos del operador S(p) que pertenecen al semiplano de la derecha converge uniformemente en la norma nuclear. El Teorema 1 conjuntamente con este resultado da respuesta a un problema de la teoría cuántica de campos no relativista planteado por J.S. Toll (cf. "Two, three and four point functions in quantum field theory", Seminar on high-energy physics and elementary particles, Trieste, 1965, 5-6). El Teorema 1 generaliza, a una clase más amplia de operadores reales, resultados obtenidos anteriormente (cf. G.Gnavi, Trabajos de Matemática, I.A.M., 5, 21). El segundo teorema muestra la existencia de factorizaciones reales para polos ubicados en el contorno (Re p = 0), caso que no había sido estudiado. Estos resultados tienen aplicación a la síntesis por factorización de puertos de Hilbert de operador característico prefijado.

GONZALEZ, R. L. (U.N.R.): *Sobre la existencia de una solución máxima de la ecuación de Hamilton-Jacobi.*

En este trabajo se caracteriza a la función "costo óptimo" de un problema de control como el elemento máximo de un conjunto apropiadamente definido. Este conjunto contiene todas las soluciones de la ecuación de Hamilton-Jacobi. En una segunda etapa se extiende este resultado a diversos casos no contemplados en el primer análisis.

GONZALEZ DOMINGUEZ, A. y TRIONE, S. E. (U.N.B.A.): *Sobre la transformada de Laplace de funciones retardadas.*

Notaciones: Con t designamos un punto de R^n de coordenadas $t_0, t_1, \ldots, t_{n-1}$; $dt = dt_0 dt_1 \ldots dt_{n-1}$; $z_\nu = x_\nu + iy_\nu$, $x_\nu \in R^n$, $y_\nu \in R^n$; z es un punto de C^n, de componentes z_ν; $\langle t,z \rangle = \sum_{\nu=0}^{n} t_\nu z_\nu$; $S^2 = t_0^2 - t_1^2 - \ldots - t_{n-1}^2$; $\overline{V}^+$ es el volumen cónico de ecuaciones $t_0 \geqslant 0$, $S^2 \geqslant 0$; W^+ es el volumen cónico de ecuaciones $y_0 > 0$, $y_0^2 - y_1^2 - \ldots - y_{n-1}^2 > 0$; T es el tubo $R^n + iW^+$; $K_\mu(\lambda)$ es la función modificada de Bessel de tercera especie.

Sea F(t) una función retardada (o sea tal que sup $F \in \overline{V}^+$), que sea además función de la distancia hiperbólica: $F(t) = \rho(S^2)$, tal que $e^{-\langle t,y \rangle} F(t) \in L(R^n)$ si $y \in W^+$. Sea f(y) la transformada de Laplace de F(t):

$$f(z) = L\{F(t)\} = \int_{R^n} e^{i\langle t,z \rangle} F(t)\, dt$$

TEOREMA 1. Si $z \in T$ vale la fórmula

$$f(z) = (2\pi)^{\frac{1}{2}(n-2)} \int_0^\infty \frac{\rho(\lambda)\lambda^{\frac{1}{2}(n-2)} K_{1/2\,(n-2)}\left(\sqrt{\lambda(z_1^2+z_2^2+\ldots+z_{n-1}^2-z_0^2)}\right)}{\sqrt{\lambda(z_1^2+z_2^2+\ldots+z_{n-1}^2-z_0^2)}}\, d\lambda .$$

El teorema 1 es un análogo hiperbólico del clásico teorema de Bochner que expresa la transformada de Fourier n-dimensional de una función radial por medio de una integral unidimensional de Hankel. El teorema 1 tiene muchas aplicaciones. En esta comunicación nos referimos a dos de ellas: a) obtención de la solución elemental retardada del operador de Klein-Gordon iterado k veces; b) obtención de una fórmula de representación de V.Vladimirov (Translations de la A.M.S., serie 2, 48 (1965) fórmula (4.7), p. 24).

GRATTON, F. y VARGAS, M. (U.N.B.A., M. de Defensa): *Soluciones autosimilares de segunda especie de una ecuación en derivadas parciales de la teoría del plasma focus.*

La evolución temporal de la lámina de corriente en una descarga del tipo "plasma focus" puede ser descripta por la ecuación no lineal

$$4\, x^2\, J_\tau^2 = 1 + J_x^2 - F\, x\, J_{xx}\, (1 + J_x^2)^{-1/2}$$

donde J es una función de x, τ y F es un parámetro real constante [1]. La ecuación admite soluciones del tipo $J = c\tau + \alpha(x)$ que representan láminas que viajan sin deformarse con velocidad incógnita c. Este tipo de soluciones están íntimamente ligadas con la existencia de soluciones autosimilares de segunda especie como ha sido señalado en [2]. La determinación de c depende de la solución de un problema de autovalores con condiciones de contorno para $p = d\alpha/dx$. En nuestro caso: $p = 0$ para $x = 0$, $p \to -\infty$ para $x \to \infty$.

Las siguientes transformaciones de variables

$$z = 4\, c^2 x^2/F\ , \quad p(1+p)^{-1/2} = -(1 + 2\, d \ln v/dz)$$

reducen el problema a una ecuación hipergeométrica confluente para v. Un análisis de pequeñas perturbaciones muestra que las soluciones con $F > 0$ son inestables. Para $h = -1/4F$ entero positivo, las soluciones son

$$v = \sum_{r=0}^{h-1} \binom{h-1}{r} \frac{z^{h-r}}{(h-r)!}$$

Se obtienen luego las posibles formas y velocidades de la lámina de corriente, en función de F.

[1] Gratton, F. y Vargas, M. - Magnetic tension and thickness in the current sheath - Proc. VI Int. Conf. on Controlled Fusion and Plasma Physics, IAEA, Berchtesgaden, 1976. En prensa.

[2] Baremblatt, G. I. y Zeldovich, Ya. B. - Self-similar solutions as intermediate asymptotics - Ann. Rev. Fluid Mech. 4 (1972), 285.

HERRERA, M. (U.N.B.A.): *Cálculo de la traza mediante corrientes residuales.*

Sea X una variedad analítica compleja de dimensión n, compacta, y Y una intersección completa en X. El objeto de la exposición es describir una construcción explícita del diagrama

$$\begin{array}{ccc} H_y^{2n}(X;C) & \longrightarrow & H^{2n}(X;C) \\ & {\scriptstyle t_n} \searrow \quad \swarrow {\scriptstyle \int} & \\ & C & \end{array}$$

de cohomología local mediante corrientes residuales.

MARCHI, E. y MILLAN, L. (U.N. San Luis): *Soluciones de ecuaciones diferenciales del tipo descripto en el modelo de axon de Hodgkin-Huxley, usando desarrollos en serie.*

Se reduce el problema del sistema de ecuaciones diferenciales a derivadas parciales a un sistema de ecuaciones diferenciales ordinarias, haciendo $V(x,t) = V(x - \theta t)$ donde θ es la velocidad de conducción; obteniendo así un sistema de tres ecuaciones de 1er. orden para n, m, h y de 2do. orden para $V(x - \theta t)$. Este es el caso de una onda uniforme que se propaga a velocidad constante y que fue resuelto en forma numérica por Hodgkin-Huxley en 1952. Linealizando los coeficientes que aparecen en las ecuaciones diferenciales de n, m, h se ensayan soluciones del tipo $V(\xi) = \sum_j V_j \xi^j$, $n(\xi) = \sum_j n_j \xi^j$, etc. siendo $\xi = x - \theta t$. Se obtienen expresiones para $n(\xi)$, $m(\xi)$, $h(\xi)$ en función de n_0, m_0, h_0 y los coeficientes V_j. Se hace una simplificación adicional en la corriente iónica, al linealizarla; eliminándose por lo tanto el h. Se obtiene una relación de recurrencia para los V_k, en función de los V_j para $j < k$ y se ensaya una solución del tipo

$$V_k = \sum_i^{\frac{k-1}{2}} \sum_j^{k-(2i+1)} S_{ij}^{(k)} V_0^j V_1^i$$

obteniéndose una nueva relación de recurrencia, muy complicada, para los $S_{ij}^{(k)}$. Actualmente se busca resolver esa relación para ciertas condiciones iniciales.

MARCHI, E. y TARAZAGA, P. (U.N. San Luis): *The minimax theorem for continuous games using an elimination procedure.*

Tiempo atrás fue introducido un proceso de eliminación de estrategias superfluas el cual ha sido usado en forma elemental para probar el teorema del minimax para la extensión mixta de un juego finito de dos personas a suma cero. Posteriormente, uno de los autores de este trabajo extendió el método para algunas clases de juegos infinitos. En este trabajo se generaliza un proceso de eliminación de estrategias superfluas aplicándolo a juegos continuos, y se demuestra de esta manera el teorema de Bohnenblust-Karlin y Shapley.

MIGUEL, O. (U.N. San Luis): *La partición de un gráfico de conocimientos en unidades pedagógicas.*

Basándose en las estructuras del proceso interactivo de enseñar-aprender desarrolladas previamente (Marchi - Miguel, 1974) se estudia en este artículo la partición de un curso o gráfico de conocimientos en celdas que son llamadas unidades pedagógicas. Se describe un algoritmo para realizar dicha partición y se estudia la relación temporal para el dictado consecutivo de las unidades; esta relación está determinada por las relaciones de prerequisitos entre los puntos pertenecientes a celdas distintas. Para las interacciones entre los profesores que dictan cursos simultáneos se describe un juego en el que las estrategias consisten principalmente en el orden de dictado de las unidades pedagógicas. Este juego tiene una estrategia conjunta óptima o punto de equilibrio. Por último, se estudia el proceso interactivo de enseñar-aprender en sus dos aspectos: el de las trayectorias posibles dentro de una unidad pedagógica y el del orden secuencial de dictado de las mismas. Se estudia la existencia de un punto de equilibrio para este proceso.

[1] Marchi, E. y Miguel, O. - On the structure of the teaching-learning interactive process - Int. Jour. of Game Theory, 3,2 (1974), 83.

PATETTA, N. (U.N. Mar del Plata): *Estudio cualitativo del movimiento en el problema plano de tres cuerpos.*

El objeto de este trabajo es determinar los intervalos de variación admisibles para las distancias mutuas entre tres cuerpos, cuando estos evolucionan a lo largo de una solución plana. Se determinan cinco comportamientos en base a las constantes c_1 y c_2 de energía y cantidad de movimiento respectivamente, estableciéndose una condición necesaria para la existencia de soluciones estacionarias. Para el caso de soluciones estacionarias equiláteras esta condición necesaria se reduce a que las constantes c_1 y c_2 verifiquen la condición

$$9 M^2 [m_0^{-1} + m_1^{-1} + m_2^{-1}]^3 c_1^{-1} c_2^{-2} h^{-2} g^{-2} = 1$$

donde m_0, m_1 y m_2 son las masas de los cuerpos, y $M = m_0 + m_1 + m_2$, $h = (m_0+m_1)m_0m_1$, $g = Mm_2^{-1}(m_0+m_1)^{-1}$.

PEJSACHOWICZ, J. (U.N. San Luis): *Homología y homotopía de p-aplicaciones.*

Se determina en esta nota la categoría de homotopía de aplicaciones multivaluadas con pesos en un anillo Λ (p-aplicaciones), introducida por G. Darbo en Teoria dell'omologia in una categoria de mappe plurivalente ponderate, Rend. Sem. Mat., Padova XXVIII, 1959. Este tipo de aplicaciones fue recientemente redescubierto por R.Jerrard, Homology with multivalued functions applied to fixed points, Trans. of the Am. Math. Soc., 213, 1975; aunque su definición difiere ligeramente de la de Darbo esta diferencia desaparece al nivel de homotopía y, por lo tanto, lo dicho aquí vale para las m-aplicaciones de su trabajo.

ROFMAN, E. (I.M. "Beppo Levi"): *Métodos constructivos en control óptimo.*

Se presentan algoritmos de optimización tales de poder ser utilizados en computadoras de memoria reducida. Tras caracterizar a los operadores integrales para los cuales el método tiene validez se estudia la convergencia para el problema libre, el problema con restricciones como también la extensión al caso en que el conjunto de controles admisibles es no convexo. Se presentan, finalmente, problemas resueltos con estos nuevos métodos.

MENALDI, J. L. y ROFMAN, E. (I.M. "Beppo Levi"): *Regularización y penalización en problemas elípticos no coercitivos.*

Para el siguiente problema: "Sean $V \subset H$ dos espacios de Banach convexos con inyección continua e imagen densa. Sea K un convexo cerrado de V y sea $A: K \longrightarrow H'$ un operador acotado, monótono y hemicontinuo. Hallar $u \in K$ tal que $\langle Au, v-u \rangle \geq 0 \quad \forall\, v \in K$"; presentamos en esta comunicación una combinación de las técnicas de aproximación de espacios, regularización y penalización de la que demostramos su convergencia a la solución, supuesta existente. Agregamos, además, distintas acotaciones para el error del método.

SANCHEZ, C. (U.N. San Luis): *Operadores de Dirac para variedades Spin(p,q).*

Se han definido en [2] las variedades Spin(p,q) y establecido algunas

Sea k un cuerpo de característica cero. R una k-álgebra local con radical M tal que $R/M \simeq k \subset R$. Supóngase que el módulo diferencial D(R/k) es de tipo finito, como R-módulo. Entonces R es regular si y sólo si D(R/k) es libre. Este resultado se debe a Nakai. La nueva demostración se basa en conectar la condición de regularidad de $\hat{R}$ (completado de R) con las extensiones jacobianas del ideal I definido por la sucesión exacta $0 \to I \to k[[x_1,\ldots,x_n]] \xrightarrow{\lambda} \hat{R} \to 0$. Asimismo, tales extensiones resultan ser imágenes inversas por λ de invariantes de Fitting del módulo diferencial M-ádico $D_c(\hat{R}/k)$. Finalmente se usa una caracterización de los módulos libres finamente generados sobre un anillo local, en término de los invariantes de Fitting asociados al módulo.

VILLAMAYOR, O. E. (I.A.M.): *Geometría algebraica en característica $p \neq 0$.*

Sobre una definición de morfismos de pre-esquemas de k-álgebras en característica $p \neq 0$, de sus singularidades y de la construcción de un espacio de Jets en analogía con la construcción de Boardman Thom en el caso de variedades diferenciables. Finalmente desearía nombrar algunos resultados y problemas propios de la característica $p > 0$.

VILLAMAYOR, O. E. (I.A.M.): *Polarización de polinomios homogéneos.*

Se extiende a cuerpos de característica $p \neq 0$ la construcción de la matriz de polarización para un polinomio homogéneo y se determina, usando una sucesión de invariantes que resultan de dicha matriz, un sistema de coordenadas donde el polinomio resulta escrito en forma minimal. Los núcleos de la matriz de polarización definen, además, los vértices del cono determinado por dicho polinomio homogéneo.

YOHAI, V. J. (U.N.B.A.) y MARONNA, R. A. (Fundación Bariloche): *Comportamiento asintótico de M-estimadores para el modelo lineal.*

Este trabajo trata con M-estimadores para el modelo lineal $y_i = \underset{\sim}{x}_i' \underset{\sim}{\theta} + u_i$ $i = 1,\ldots,n$, donde los $\underset{\sim}{x}_i$ son vectores p-dimensionales fijos y los u_i variables aleatorias i.i.d con función de distribución F. Los estimadores considerados son soluciones de la ecuación $\sum_{i=1}^{n} \Psi(y_i - \underset{\sim}{x}_i' \hat{\underset{\sim}{\theta}}) \underset{\sim}{x}_i = 0$. Sea $\underset{\sim}{X}$ la matriz cuya i-ésima fila es $\underset{\sim}{x}_i'$, luego se prueba que $(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})' \underset{\sim}{X}'\underset{\sim}{X} (\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})$ es acotado en probabilidad, suponiendo que Ψ satisface un conjunto de condiciones que incluyen monotonicidad. Esto implica que una condición suficiente para la consistencia de $\hat{\underset{\sim}{\theta}}$ es que el menor autovalor de $\underset{\sim}{X}'\underset{\sim}{X}$ tienda a infinito. Para el caso en que $p = p_n \to \infty$, se prueba que $p(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})' \underset{\sim}{X}'\underset{\sim}{X} (\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})$ está acotada en probabilidad, suponiendo que $p\xi \to 0$ donde $\xi = \max_{1 \le i \le n} \underset{\sim}{x}_i' \underset{\sim}{X}'\underset{\sim}{X} \underset{\sim}{x}_i$.

ZUPPA, C. (U.N. San Luis): *Campos vectoriales de Anosov.*

Sea ϕ_t: M $\longrightarrow$ M (t $\in$ R) un flujo de Anosov de clase C^r ($r \geqslant 1$) sobre una variedad compacta M. Esto quiere decir que existe una descomposición ϕ_t-invariante TM = $E^U \oplus E^S + E^T$, donde E^T es el fibrado lineal tangente al flujo no singular ϕ_t, y el flujo $T\phi_t$ es expansivo en E^U y contractivo en E^S. Es ya bien conocida la conjetura siguiente:
C. El conjunto de puntos no errantes (nonwandering) de ϕ_t, $\Omega(\phi_t)$, es M.
Si el fibrado $E^U \oplus E^S$ es integrable, entonces ϕ_t es topológicamente equivalente a la suspensión de un difeomorfismo de Anosov (J.F. Plante). En este caso (C) se reduce a una conjetura similar para difeomorfismos. En esta nota nos ocupamos de estudiar (C) cuando $E^U \oplus E^S$ no es integrable y, bajo esta hipótesis, se obtiene el resultado parcial:
TEOREMA. Si $r \geqslant 2$ y $E^U \oplus E^S$ es de clase C^1, entonces $\Omega(\phi_t)$ = M.

J.F. Plante ha obtenido este resultado cuando dim M = 3 (Amer. Journal of Math., 1972).

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VILLARINO 739 - B. B.