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SOME PROXIMITY RELATIONS IN A PROBABILISTIC METRIC SPACE(*)

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Dedicated to Professor Luis A. Santaló

0. INTRODUCTION.

Proximities in a probabilistic metric space have been studied previously by R. Fritsche [3], Gh. Constantin and V. Radu [2] and A.Leonte [4]. In this paper we introduce, using some results concerning order and weak convergences [1], a family of semi-proximities $\{\delta_\varphi;\ \varphi \in \Delta^+\}$ analyzing when they are Efremovič-proximities and relating the induced closure operators $\{C_{\delta_\varphi};\ \varphi \in \Delta^+\}$ to those of R. Tardiff [8] and B. Schweizer [7]. In the last section we exhibit a uniform topology where the neighborhood of a point p is precisely the closure of {p} in the topology generated by C_{δ_φ}.

1. PRELIMINARIES.

Let Δ^+ be the set of all one-dimensional positive distribution functions, i.e., let

$\Delta^+ = \{F: R \to [0,1];\ F(0) = 0$, F is non-decreasing and left-continuous$\}$.

Δ^+ has a partial order, namely, $F \geqslant G$ iff $F(x) \geqslant G(x)$, for every x.

$(\Delta^+,\leqslant)$ is a complete lattice with minimum element $\varepsilon_\infty(x) = 0$, for every x, and maximum element the step function given by

$$\varepsilon_0(x) = \begin{cases} 0\ , & \text{for } x \leqslant 0\ , \\ 1\ , & \text{for } x > 0\ . \end{cases} \tag{1.1}$$

It is well known that weak convergence $(\text{w-}\lim_{n\to\infty} F_n)$ in Δ^+ is metrizable by the modified Lévy metric $\mathcal{L}$ introduced by Sibley [6].

(*) **Presented at the INTERNATIONAL CONGRESS OF MATHEMATICIANS, Helsinki, Finland 1978.**

DEFINITION 1.1. A *triangle function* is a two-place function τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G and H in Δ^+,

i) $\tau(F,\varepsilon_0) = F$,

ii) $\tau(F,G) \geqslant \tau(F,H)$ whenever $G \geqslant H$,

iii) $\tau(F,\tau(G,H)) = \tau(\tau(F,G),H)$,

iv) $\tau(F,G) = \tau(G,F)$.

A triangle function τ is *continuous* if it is a continuous function from $\Delta^+ \times \Delta^+$ into Δ^+, where Δ^+ is indowed with the $\mathcal{L}$-metric topology and $\Delta^+ \times \Delta^+$ with the product topology. For a complete study of the fundamental topological semigroups (Δ^+,τ) see [6].

DEFINITION 1.2. A *probabilistic metric space* (briefly, a PM-space) is an ordered pair (S,F), where S is a set, and F is a mapping from $S \times S$ into Δ^+ such that for all $p,q,r \in S$:

I) $F(p,q) = \varepsilon_0$ iff $p=q$,

II) $F(p,q) = F(q,p)$,

III) $\tau(F(p,q),F(q,r)) \leqslant F(p,r)$.

If F satisfies just (I) and (II) we say that (S,F) is a *semi*-PM *space*. The function $F(p,q)$ is denoted by F_{pq}, and $F_{pq}(x)$, for $x > 0$, is interpreted as the probability that the distance between p and q is less than x.

We collect some definitions about proximities which will be used in the sequel. For a complete survey of proximities see [5].

DEFINITION 1.3. Let X be a set and δ a binary relation on P(X), the power set of X. δ is a *semi-proximity* if satisfies, for A, B and C subsets of X, the following conditions:

1) $\emptyset \not\delta A$,

2) If $A \cap B \neq \emptyset$ then $A \delta B$,

3) $A \delta B$ implies $B \delta A$,

4) $A \delta (B \cup C)$, if and only if $A \delta B$ or $A \delta C$.

A semi-proximity δ is called an *Efremovič proximity* if verifies the additional axiom:

5) $A \not\delta B$ implies there exists $E \subset X$ such that $E \not\delta B$ and $(X-E) \not\delta A$.

A semi-proximity δ is said to be *separated* if

6) $a \delta b$ implies $a=b$.

Any semi-proximity δ induces a mapping C_δ from P(X) into itself defi-

a) $C_\delta(\emptyset) = \emptyset$,

b) $C_\delta(A) \supset A$, for every $A \in P(X)$,

c) $C_\delta(A \cup B) = C_\delta(A) \cup C_\delta(B)$ for all $A,B \in P(X)$,

i.e., C_δ is a Cech closure operator which is a Kuratowski closure ($C_\delta(C_\delta(A)) = C_\delta(A)$ for every $A \in P(X)$) whenever δ is an Efremovič proximity. So δ provides a topology on X called the *topology induced* by δ. The topological spaces whose topologies can be derived in this way from proximities are called *proximizable*.

Finally, we summarize some definitions and theorems about order and weak convergences (see [1]).

The supremums and infimums of two functions $F,G \in \Delta^+$, in the lattice $(\Delta^+,\leqslant)$ will be denoted, respectively, by $F \vee G$ and $F \wedge G$.

DEFINITION 1.4. (a) A non-decreasing (resp., non-increasing) sequence (G_n) in Δ^+ is *order* convergent to $G \in \Delta^+$, if and only if $G = \bigvee_{n=1}^{\infty} G_n$ (resp., $G = \bigwedge_{n=1}^{\infty} G_n$). (b) A sequence (F_n) in Δ^+ is *order* convergent to $F \in \Delta^+$ ($F = \text{o-}\lim_{n\to\infty} F_n$), if and only if there exist two sequences (G_n) and (H_n) such that (G_n) is non-decreasing with $\bigvee_{n=1}^{\infty} G_n = F$, (H_n) is non-increasing with $\bigwedge_{n=1}^{\infty} H_n = F$, and for all $n \in N$ is $G_n \leqslant F_n \leqslant H_n$. The order limit is unique.

THEOREM 1.1. *Let* (F_n) *be a sequence in* Δ^+ *and* $F \in \Delta^+$. *Then we have:*

i) $F = \text{o-}\lim_{n\to\infty} F_n$ *iff* $\lim_{n\to\infty} F_n(x) = F(x)$, *for all* $x \in R^+$ (*pointwise convergence*);

ii) *If* $F = \text{o-}\lim_{n\to\infty} F_n$ *then* $F = \text{w-}\lim_{n\to\infty} F_n = \mathcal{L}\text{-}\lim_{n\to\infty} F_n$, *but the reciprocal does not hold in general*;

iii) *If* $F = \text{w-}\lim_{n\to\infty} F_n$ *and* F *is continuous or* (F_n) *is non-decreasing then* $F = \text{o-}\lim_{n\to\infty} F_n$.

THEOREM 1.2. (Weak version of Everett diagonal condition in Δ^+). *Let* $(F_k^n)_{(n,k)\in N\times N}$ *be a collection of sequences in* Δ^+, *let* (F_n) *be a sequence in* Δ^+ *and* $F \in \Delta^+$. *If* F *has at most a finite set of discontinuities,* $F = \text{o-}\lim_{n\to\infty} F_n$, *and for each* $n \in N$, $F_n = \text{o-}\lim_{k\to\infty} F_k^n$, *then there exists a strictly increasing sequence of integers* $k_1 < k_2 < \dots < k_n < \dots$ *in* N, *such that* $F = \text{o-}\lim_{n\to\infty} F_{k_n}^n$.

2. A FAMILY OF PROXIMITIES IN A PM-SPACE.

Let (S,F) be a semi-PM space. For each $\varphi \in \Delta^+$ we define a binary relation δ_φ on $P(S)$ in the following way, for $A,B \in P(S)$,
"$A\ \delta_\varphi\ B$ iff there exists a sequence $((a_n,b_n))_{n\in N}$ in $A \times B$ such that $\varphi = \text{o-}\lim\limits_{n\to\infty} (\varphi \wedge F_{a_n b_n})$".

When $A\ \delta_\varphi\ B$ we will say that A and B have a φ-proximity.

THEOREM 2.1. *δ_φ is a semi-proximity.*

The Čech closure induced by δ_φ will be:

$$C_{\delta_\varphi}(A) = \{x \in S;\ \exists (a_n) \subset A:\ \varphi = \text{o-}\lim_{n\to\infty} (\varphi \wedge F_{xa_n})\}.$$

THEOREM 2.2. *If τ is continuous, $\tau(\varphi,\varphi) = \varphi$ and φ is continuous in R^+, then C_{δ_φ} is a Kuratowski closure.*

Proof. If $x \in C_{\delta_\varphi}(C_{\delta_\varphi}(A))$ there is $(x_n) \subset C_{\delta_\varphi}(A)$ such that $\text{o-}\lim\limits_{n\to\infty} (\varphi \wedge F_{xx_n}) = \varphi$. For each $n \in N$, $x_n \in C_{\delta_\varphi}(A)$, i.e., there exists a sequence $(a_k^n)_{k\in N} \subset A$ such that $\text{o-}\lim\limits_{k\to\infty} (\varphi \wedge F_{x_n a_k^n}) = \varphi$. By Theorem 1.2 there exists an increasing sequence of integers $(k_n)_{n\in N}$ such that $\text{o-}\lim\limits_{n\to\infty} (\varphi \wedge F_{x_n a_{k_n}^n}) = \varphi$. Let $H_n = \tau(\varphi \wedge F_{xx_n}, \varphi \wedge F_{x_n a_{k_n}^n})$, for every $n \in N$. Using the continuity of τ and φ, we have $\text{o-}\lim\limits_{n\to\infty} H_n = \tau(\varphi,\varphi) = \varphi$, and by the triangle inequality $H_n \leqslant F_{xa_{k_n}^n}$ and $H_n \leqslant \tau(\varphi,\varphi) = \varphi$, we will obtain $H_n \leqslant \varphi \wedge F_{xa_{k_n}^n} \leqslant \varphi$ which in turn implies $\text{o-}\lim\limits_{n\to\infty} (\varphi \wedge F_{xa_{k_n}^n}) = \varphi$, i.e., $x \in C_{\delta_\varphi}(A)$.

The following example shows that the strong hypothesis $\varphi = \tau(\varphi,\varphi)$ assumed above, is really necessary.

EXAMPLE 2.1. Consider the PM-space $(R^+, \varepsilon_{|x-y|}, *)$ and $\varphi = U_{\frac{1}{2},1}$. The convolution $*$ is continuous [6] and has no idempotents different from ε_0 and ε_∞. It is easy to see that

$C_{\delta_{U_{\frac{1}{2},1}}}(0) = [0,1/2] \subsetneq C_{\delta_{U_{\frac{1}{2},1}}}(C_{\delta_{U_{\frac{1}{2},1}}}(0))$ because $[0,1] \subset C_{\delta_{U_{\frac{1}{2},1}}}(C_{\delta_{U_{\frac{1}{2},1}}}(0))$.

In order to analyse the special case $\varphi = \varepsilon_0$ we recall the following lemma.

LEMMA 2.1. *Let* I *be any set of indices and let* $\{F_i;\ i \in I\}$ *be in* Δ^+. *The following statements are equivalent:*

i) $\bigvee_{i\in I} F_i = \varepsilon_0$;

ii) *For any* $\varepsilon \in (0,1)$ *there exists* $i \in I$ *such that* $F_i(\varepsilon) > 1-\varepsilon$;

iii) *For any* $\varepsilon \in (0,1)$ *there exists* $i \in I$ *such that* $\mathcal{L}(F_i,\varepsilon_0) < \varepsilon$;

iv) *There is a countable subset* J *of* I *such that* $\bigvee_{i\in J} F_i = \varepsilon_0$.

Then the Efremovič proximity δ_{ε_0} can be presented in the following ways:

"A δ_{ε_0} B iff $\bigvee_{(a,b)\in A\times B} F_{ab} = \varepsilon_0$ iff for every $\varepsilon,\lambda > 0$ there is $(a,b) \in A\times B$ such that $F_{ab}(\varepsilon) > 1-\lambda$"

and

$$C_{\delta_{\varepsilon_0}}(A) = \{x \in S;\ N_x(\varepsilon,\lambda) \cap A \neq \emptyset,\ \varepsilon,\lambda > 0\}\ ,$$

where $N_x(\varepsilon,\lambda) = \{y \in S;\ F_{xy}(\varepsilon) > 1-\lambda\}$ are the neigborhood of the classical ε,λ-topology for these spaces, i.e., the ε,λ-topology is proximizable by δ_{ε_0}.

THEOREM 2.3. *Under the hypothesis of Theorem 2.2, the topological space* (S,C_{δ_φ}) *is completely regular.*

In a PM-space (S,F,τ) and for a fixed $\varphi \in \Delta^+$, Schweizer [7] has introduced the next relation in P(S):

"A I_φ B iff there exists $(a,b) \in A\times B$ such that $F_{ab} \geq \varphi$", and when A I_φ B, A and B are said to be *indistinguishable* (*mod.*φ).

We note that I_φ is a semi-proximity weaker than δ_φ, in the sense that A I_φ B implies A δ_φ B, i.e., indistinguishability (mod.φ) yields φ-proximity. The reciprocal does not hold, in general.

EXAMPLE 2.2. Consider the PM-space $(R^+,\varepsilon,*)$, where $\varepsilon_{pq} = \varepsilon_{|p-q|}$ for all $p,q \in R^+$. Let $k > 0$ and $\varphi = \varepsilon_k$. Take $A = [0,1)$ and $B = (1+k,+\infty)$. Taking for each $n \in N$, $a_n = 1-1/n \in A$ and $b_n = 1+k+1/n \in B$, we have

$$\text{o-}\lim_{n\to\infty} (\varepsilon_k \wedge \varepsilon_{|a_n-b_n|}) = \text{o-}\lim_{n\to\infty} \varepsilon_{k+\frac{2}{n}} = \varepsilon_k,$$

i.e., A δ_{ε_k} B but A $\not{I}_{\varepsilon_k}$ B because for all $(a,b) \in A\times B$ we have $\varepsilon_{|a-b|} < \varepsilon_k$.

Recently, Tardiff has introduced [8] for $\varphi \in \Delta^+$ a closure operator de fined by

$$C_\varphi(A) = \{x \in S;\ (\forall\, h \in (0,1])(\exists\, a = a(h) \in A) \text{ such that } F^h_{xa} \geqslant \varphi\}\ ,$$

being

$$F^h_{xa}(t) = \begin{cases} 0 \quad , & \text{if } \ t \leqslant 0, \\ \min(F_{xa}(t+h)+h,1), & \text{if } \ t \in (0,1/h], \\ 1 \quad , & \text{if } \ t > 1/h\ . \end{cases}$$

The semi-proximity T_φ defined by

$$\text{"}A\ T_\varphi\ B \quad \text{iff} \quad C_\varphi(A) \cap C_\varphi(B) \neq \emptyset\text{"},$$

is stronger than I_φ because if $A\ I_\varphi\ B$ then there is $(a,b) \in A \times B$ such that $F_{ab} \geqslant \varphi$ and for all $h > 0$, $F^h_{ab} \geqslant F_{ab} \geqslant \varphi$, i.e., $A\ T_\varphi\ B$. The reciprocal does not hold, in general.

EXAMPLE 2.3. Consider the PM-space of example 2.1, and the same $\varphi = \varepsilon_k$, $k > 0$. Let $A = [0,1)$. Then $C_{I_{\varepsilon_k}}(A) = [0,1+k) \subset C_{\varepsilon_k}(A)$. But $1+k \in C_{\varepsilon_k}(A)$ because, for any $h \in (0,1]$, taking $1-h \in A$ we have $\varepsilon^h_{1+k-1+h} = \varepsilon^h_{k+h} \geqslant$ $\geqslant \varepsilon_k$, so $\{1+k\}\ T_{\varepsilon_k}\ A$ but $\{1+k\} \not{I}_{\varepsilon_k}\ A$.

Finally we remark that for $\varphi = \varepsilon_0$, $T_{\varepsilon_0} = \delta_{\varepsilon_0}$ is the ε,λ-proximity and for any φ and $p \in S$: $C_{\delta_\varphi}(\{p\}) = C_{I_\varphi}(\{p\}) = C_\varphi(\{p\}) = \{q \in S;\ F_{pq} \geqslant \varphi\}$, and this set is exactly the class of p in the partition of S induced by the equivalence relation of indistinguishability (mod.φ) introduced in [7].

3. A PROXIMITY INDUCED BY AN UNIFORMITY.

DEFINITION 3.1. A triangular function τ is said to be *radical* if for any $F \in \Delta^+ - \{\varepsilon_0\}$ there exists $G \in \Delta^+ - \{\varepsilon_0\}$ such that $F < \tau(G,G) < \varepsilon_0$.

THEOREM 3.1. *If* $\tau \geqslant *$ *then* τ *is radical.*

Proof. We need to show that for any $F < \varepsilon_0$ there is $G < \varepsilon_0$ such that $G*G > F$. In effect, if $F = \varepsilon_k$ for some $k > 0$ then taking $G = \varepsilon_{k/4}$ we have $G*G = \varepsilon_{k/2} > \varepsilon_k$. If $F < \varepsilon_k$ for some $k > 0$ then the same G yields the same conclusion. So we can suppose that there is an interval $(0,k)$ such that $F(x) > 0$ for $x \in (0,k)$. Let

$$H(x) = \begin{cases} 0 \quad , & \text{if } x \leqslant 0, \\ +\sqrt{F(2x)}, & \text{if } 0 < x \end{cases}$$

Obviously $F(x) \leqslant +\sqrt{F(x)} \leqslant +\sqrt{F(2x)}$ for $x > 0$, and consequently $F \leqslant H$. If $F(x) = H(x)$ for all $x > 0$ then $F(x) = \sqrt{F(x)}$ and $F(x)(F(x)-1) = 0$, i.e., there would exist $k' > 0$ such that $F(x) = \varepsilon_{k'}$, which is a contradiction. So $F < H$ and there is a $t > 0$ such that $0 < F(t) < H(t) < < 1$. Let

$$G(x) = \begin{cases} H(x) & , \text{ if } x \leqslant t, \\ 1 & , \text{ if } x > t. \end{cases}$$

$G > H$ and a straightforward computation shows that $F \leqslant H*H$. By the strict isotony of $*$, $F \leqslant H*H < G*G < \varepsilon_0$.

Let (S,F,τ) be a PM-space. For any $F \in \Delta^+-\{\varepsilon_0\}$, let
$U(F) = \{(p,q) \in S \times S;\ F_{pq} > F\}$.

THEOREM 3.2. *If* τ *is radical then the collection* $\{U(F);\ F \in \Delta^+-\{\varepsilon_0\}\}$ *is a basis for a diagonal separated uniformity* $\mathcal{U}$ *on* S.

Proof. Obviously $\Delta_S = \{(p,p);\ p \in S\} \subset U(F)$ and $U(F) = U(F)^{-1}$, for any $F < \varepsilon_0$. If $F,G < \varepsilon_0$ and being τ radical there is $G < \varepsilon_0$ such that $F < \tau(G,G) < \varepsilon_0$. Then $U(G) \circ U(G) \subset U(F)$ because if $(p,r) \in U(G) \circ U(G)$, there is $q \in S$ such that $(p,q) \in U(G)$ and $(q,r) \in U(G)$. By the triangle inequality $F_{pr} \geqslant \tau(F_{pq},F_{qr}) \geqslant \tau(G,G) > F$, so $(p,q) \in U(F)$. Finally note that $\mathcal{U}$ is separated because $\bigcap_{F\in\Delta^+-\{\varepsilon_0\}} U(F) = \Delta_S$.

COROLLARY 3.1. *The topology generated by* $\mathcal{U}$ *is metrizable.*

Proof. Consider the countable family $\{\alpha_{t,t'};\ t,t' \in (0,1) \cap Q\} \subset \Delta^+$, where

$$\alpha_{t,t'}(x) = \begin{cases} 0 & , \text{ if } x \leqslant 0, \\ t' & , \text{ if } 0 < x \leqslant t, \\ 1 & , \text{ if } x > t. \end{cases}$$

If $U \in \mathcal{U}$, there is $F < \varepsilon_0$ such that $U(F) \subset U$. Being $F < \varepsilon_0$ there exists $t \in (0,1) \cap Q$ such that $F(t) < 1$. Let $t' \in (F(t),1)$. Then $F < \alpha_{t,t'}$ and $U(F) \supset U(\alpha_{t,t'})$, i.e., $\{U(\alpha_{t,t'});\ t,t' \in (0,1) \cap Q\}$ is a countable basis for $\mathcal{U}$. We apply then Weyl theorem.

The topology generated by $\mathcal{U}$ can be described by the family $N(\mathcal{U}) = = \{N_p(F);\ F \in \Delta^+-\{\varepsilon_0\},\ p \in S\}$, where each neighborhood $N_p(F)$ is given by

$$N_p(F) = \{q \in S;\ F_{pq} > F\} = C_{I_F}(\{p\}),$$

i.e., $N_p(F)$ is precisely the closure of $\{p\}$ by C_{I_F}, C_{δ_F} or C_F, in other words, if $q \in N_p(F)$ then q is indistinguishable (mod.F) of p.

The uniformity $\mathcal{U}$ induces a proximity $\delta_{\mathcal{U}}$ defined on $P(S)$ by:

"A $\not\delta_{\mathcal{U}}$ B iff for some $F < \varepsilon_0$, $N_p(F) \cap B = \emptyset$, for all $p \in A$".

Applying a well known result of proximity theory we obtain that the topology induced by δ_U is the uniform topology. We remark that this topology is exactly the topology T_F obtained when considering the PM-space as generalized metric space [9].

REFERENCES

[1] C.ALSINA and E.TRILLAS, *Order and weak convergences of distribution functions*, (to appear).

[2] Gh.CONSTANTIN and V.RADU, *Random-proximal spaces*, Pub. Univer. Din Ţimişoara (1974).

[3] R.FRITSCHE, *Proximities in a probabilistic metric space*, (unpub.).

[4] A.LEONTE, *Proximitate stochastica*, Stud. Cerc. Mat.,26 (1974) 1095-1100.

[5] S.A.NAIMPALLY and B.D.WARRAK, *Proximity spaces*, Cambridge Univ. Press (1970).

[6] B.SCHWEIZER, *Multiplications on the space of probability distribution functions*, Aeq. Math., 12 (1975), 151-183.

[7] B.SCHWEIZER, *Sur la possibilité de distinguer les points dans un espace métrique aléatoire*, C.R. Acad. Sc. Paris, 280 (1975) 459-461.

[8] R.TARDIFF, *Topologies for probabilistic metric spaces*, Pacific Jour. of Math., 65 (1976), 233-251.

[9] E.TRILLAS and C.ALSINA, *Introducción a los espacios métricos generalizados*, Serie Univ., n°49 (Fundación Juan March). Madrid 1978.

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PREFERENCIAS SUBDIFERENCIABLES

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Dedicado al Profesor Luis A. Santaló

SUMMARY. Under the assumption of monotone and convex preferences, compensated demand correspondences are singled-valued on a dense, full subset of the price domain, and the Slutsky equation holds for a dense, full subset of price variations.

La presente nota, que dedico afectuosa y respetuosamente al profesor doctor Luis A. Santaló, se propone aclarar un aspecto de las correspondencias de demanda que no ha sido dilucidado en la literatura sobre el tema.

Partimos de supuestos normales en la teoría económica del consumo:

HIPOTESIS. *El conjunto de consumo es* $\overline{R}^n_+$. *Las preferencias del consumidor están representadas por un preorden completo y son continuas, convexas y estrictamente monótonas. El ingreso del consumidor y los precios de los* n *bienes son positivos.*

Procedemos del siguiente modo. Tomamos cualquier variedad de indiferencia

$$\{x \in \overline{R}^n_+ \mid U(x) = c\}$$

donde c es un número real. En virtud de la Hipótesis expresada obtenemos una función explícita

$$x_1 = F(x_2,\dots,x_n) ,$$

que extendemos a todo R^{n-1} atribuyéndole el valor $+\infty$ en los demás puntos.

Elegimos el bien 1 como numerario ($p_1 \equiv 1$) y llamamos q al vector $(p_2,\dots,p_n)$, donde p_i es el precio del bien i en unidades del bien 1. Las propiedades de F, de su conjugada F* y de sus respectivos mapas subdiferenciales ∂F y ∂F^* se describen en la siguiente proposición.

LEMA. (a) F *es convexa propia y cerrada*;

(b) $-q \in \partial F(x_2,\dots,x_n)$ *si y solo si* $(x_2,\dots,x_n) \in \partial F^*(-q)$;

(c) $(x_2,\dots,x_n) = \partial F^*(-q)$ *si y solo si* F* *es diferenciable en* -q.

Demostración. Resulta inmediatamente de los hechos postulados en la Hipótesis por aplicación de proposiciones de análisis convexo (cf. Rockafellar, [5], Teorema 12.2, Corolario 23.5.1 y Teorema 25.1).

Pasamos ahora a las correspondencias de demanda. Agregamos los siguientes símbolos: I, ingreso del consumidor; φ, correspondencia de demanda individual no compensada; σ, correspondencia de demanda individual compensada.

TEOREMA 1. *Con* x *fija, los vectores de precios en los cuales* $\sigma(p;x)$ *contiene un solo punto forman un subconjunto denso del conjunto de todos los vectores de precios. El complemento de dicho subconjunto es de medida nula.*

Demostración. $\sigma(p;x)$ consta de un único elemento si y solo si la respectiva F^* es diferenciable en $-q$, de acuerdo con el Lema anterior.

De este hecho se desprende lo afirmado por el Teorema, teniendo en cuenta las propiedades generales de diferenciabilidad de funciones convexas propias (v. Rockafellar, [5], Teorema 25.5).

Una manera alternativa de probar el Teorema consiste en deducirlo de la semicontinuidad superior del mapa subgradiente, propiedad estudiada por Moreau [2].

TEOREMA 2. *Los pares* $(p,p+\Delta p)$ *que satisfacen la ecuación de Slutsky constituyen un subconjunto denso del conjunto de todos los pares de vectores de precios. El complemento de dicho subconjunto es de medida nula* (cf. [4]).

Demostración. Dado el ingreso I y los precios iniciales p, sea un incremento Δp. Introducimos un selector arbitrario

$$s\colon \varphi(p,I) \longmapsto s\varphi(p,I) \in \varphi(p,I) \ ,$$

y consideramos la descomposición:

$$\varphi(p+\Delta p,I) - \varphi(p,I) = \varphi(p+\Delta p,I) - \sigma(p+\Delta p,s\varphi(p,I)) + \\ + \sigma(p+\Delta p,s\varphi(p,I)) - \varphi(p,I) \ ,$$

que equivale a la ecuación de Slutsky en la forma de incrementos finitos (Nikaido, [3], capítulo VI; Ellis, [1]).

La descomposición indicada es posible si y solo si $\sigma(p+\Delta p,s\varphi(p,I))$ contiene un único elemento. Basta entonces aplicar el Teorema 1 para concluir la demostración.

BIBLIOGRAFIA

[1] ELLIS, D.F., *A Slutsky Equation for Demand Correspondences*, Econometrica, 44 (1976), 825-828.

[2] MOREAU, J.J., *Semi-continuité du sous-gradient d'une fonctionelle*, Comptes rendus hebdomadaires des séances de l'Académie des Sciences (Paris), t.260 (25 de enero de 1965), 1067-1070.

[3] NIKAIDO, H., *Convex Structures and Economic Theory*, Nueva York, Academic Press, 1968.

[4] OLIVERA, J.H.G., *Ecuación de Slutsky para correspondencias de demanda*, Seminario Interno del Centro de Investigaciones Económicas, Instituto Torcuato Di Tella, 13 de julio de 1979.

[5] ROCKAFELLAR, R.T., *Convex Analysis*, Princeton, N.J., Princeton University Press, 1970.

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PROJECTORS ON CONVEX SETS IN REFLEXIVE BANACH SPACES

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Dedicated to Professor Luis A. Santaló

Selfadjoint operators in Hilbert space can be synthetized out of orthogonal projectors by the process of forming the integrals of numerical functions with respect to an increasing one-parameter family of projectors. To be viable such a mechanism - known as spectral synthesis - requires from projectors a certain number of algebraic properties. Not long ago I have shown [7,8,9] that these properties subsist if the class of linear projectors is enlarged so as to include projectors on closed convex cones, conceived as nearest point mappings, and thus I was able to synthetize a new class of operators, mostly nonlinear. But then, having freed the spectral theory from its original confinement I was faced with the question of how far one can go on extending it. For instance, would it be valid in spaces other than Hilbert space?. It is precisely to this question that I am addressing myself in this paper, beginning with the study of projectors in reflexive Banach spaces. A first basic question is to decide what projectors on convex sets should be. Nearest point mappings certainly do not qualify, as they form an unruly class devoid of any algebraic structure, nor does any class of operators mapping the space into itself, since for these many of the required properties do not even make sense. This realized, one is led to the view that projectors must be mappings, perhaps multivalued, acting from the dual into the space, view which in Hilbert space is thoroughly concealed by the standard identification of the space with its dual. At this stage a choice offers itself in a most natural way: The projector on a closed convex set K in a real reflexive Banach space X is the mapping $P_K\colon X^* \longrightarrow 2^X$ assigning to each $x^* \in X^*$ the set of points minimizing $\frac{1}{2}\,\|x^*\|^2 + \frac{1}{2}\,\|x\|^2 - \langle x^*,x\rangle$ over K. A series of familiar looking results soon brings out the certainty of being on the right track. So reassured, I have proceeded to investigate these new mathematical objects, not so much on their own right but rather as possible instruments for the spectral theory. My results are inconclu-

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sive as they failed to prove or disprove a couple of essential points. It is however apparent that the very existence of an increasing family of projectors requires from the space a good deal of Hilbert space structure, and therefore that there is not much occasion for the spectral theory to take place in a reflexive space chosen at random.

§1. PROJECTORS ON CONVEX SETS.

All throughout this article we shall be working in a real reflexive Banach space X, whose dual we shall denote X*. As usual the double bar indicates the norm in either space, and the angular brackets the bilinear form effecting the pairing of X and X*. We shall let $J: X \longrightarrow 2^{X^*}$ denote the duality mapping:

$$Jx = \{x^* \mid \langle x^*,x\rangle = \|x\|^2 = \|x^*\|^2\}$$

of X onto X*, and $J^{-1}: X^* \longrightarrow 2^X$,

$$J^{-1}x^* = \{x \mid \langle x^*,x\rangle = \|x\|^2 = \|x^*\|^2\},$$

the duality mapping of X* onto X. Let us recall that $Jx = \partial\frac{1}{2}\|x\|^2$, and $J^{-1}x^* = \partial\frac{1}{2}\|x^*\|^2$, and that the relation

$$\frac{1}{2}\|x^*\|^2 + \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle = 0$$

is equivalent to $x^* \in Jx$ and to $x \in J^{-1}x^*$. Mappings, even when singlevalued, are considered here in the context of multivalued mappings, and so the inverses always exist. The conjugate of a proper lower semicontinuous function $f: X \longrightarrow (-\infty,+\infty]$ is denoted f^*. We shall often use the letter Q for the function $x \longrightarrow \frac{1}{2}\|x\|^2$, and Q* for its adjoint $x^* \longrightarrow \frac{1}{2}\|x^*\|^2$. If K is a closed convex set ψ_K denotes its indicator function. The infraconvolution of convex functions is indicated by the symbol $\square$.

To bring out the analogy with projectors in Hilbert space we shall follow closely our discussion of the Hilbert space theory expounded in [9]; the reader is invited to compare the results step by step.

DEFINITION 1. The projector on a closed convex set K in X is the mapping $P_K: X^* \longrightarrow 2^X$ assigning to each x* the set of points minimizing the function

$$\frac{1}{2}\|x^*\|^2 + \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle$$

over K, that is

$$(1)\qquad P_Kx^* = \{x \in K \mid \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle \leq \frac{1}{2}\|y\|^2 - \langle x^*,y\rangle,\ \forall\, y \in K\}$$

Since $\|x\|^2 - \langle x^*,x\rangle$ is l.s.c. convex function of x tending to $+\infty$ with

$\|x\|$ the infimum is always attained and $P_K x^*$ is never empty. In Hilbert space P_K is simply the nearest point mapping on K. If $K = X$ then $P_K = J^{-1}$, whereas if $K = \{tz\}_{t\geq 0}$ then $P_K x^* = \langle x^*, \frac{z}{\|z\|}\rangle^+ \frac{z}{\|z\|}$. In the latter case we recognize $P_K x^*$ as the ordinary projection of x^* on a half-line.

THEOREM 1.

(2) $$P_K x^* = \{x \mid (Q+\psi_K)x + (Q+\psi_K)^* x^* = \langle x^*,x\rangle\} = (J+\partial\psi_K)^{-1}x^*.$$

Proof. From (1) we obtain

$$\{x \in P_K x^*\} \iff \{\langle x^*,x\rangle - (\tfrac{1}{2}\|x\|^2 + \psi_K(x)) = \sup_y [\langle x^*,y\rangle - \tfrac{1}{2}(\|y\|^2+\psi_K(y))]\}$$

$$\iff \{(Q+\psi_K)(x) + (Q+\psi_K)^*(x)^* = \langle x^*,x\rangle\} \iff \{x \in \partial(Q+\psi_K)^* = (J+\partial\psi_K)^{-1}x^*\}$$

COROLLARY 1. P_K is a subdifferential.

COROLLARY 2. The function $\frac{1}{2}\|x\|^2 - \langle x^*,x\rangle$ remains constant over $P_K x^*$.

This corollary justifies the notation $\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2$ for the common value of $\langle x^*,x\rangle - \frac{1}{2}\|x\|^2$ on $P_K x^*$.

COROLLARY 3.

(3) $$\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2 = (Q+\psi_K)^* x^*.$$

Proof. The left hand side coincides with the supremum of $\langle x^*,y\rangle - (\frac{\|y\|^2}{2} + \psi_K(y))$, which is $(Q+\psi_K)^* x^*$.

COROLLARY 4. P_K satisfies the subdifferential equation

(4) $$P_K x^* = \partial[\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2].$$

COROLLARY 5.

(5) $$P_K x^* \cap P_K y^* \subset P_K(tx^* + (1-t)y^*).$$

Proof. This is just another way of saying that $P_K^{-1}x = Jx + \partial\psi_K x$ is convex. On the other hand convexity follows from the maximal monotonicity of $J + \partial\psi_K$.

COROLLARY 6.

(6) $$\{x \in P_K x^*\} \iff \{\exists\, \bar{x}^* \in Jx \mid \langle x^*-\bar{x}^*,x-y\rangle \geqslant 0,\ \forall\ y \in K\}.$$

Proof. $\langle x \in P_K x^*\rangle \iff \{x^* \in Jx + \partial\psi_K(x)\} \iff \{\exists\, \bar{x}^* \in Jx \mid x^*-\bar{x}^* \in \partial\psi_K x\}$

Let us recall a few basic notions. A vector $u^* \in X^*$ is said to be normal to a closed convex set K at a point $x \in K$ if

$$\langle u^*, x-y\rangle \geqslant 0 \ , \quad y \in K \ ;$$

such vectors are called *normals*. It is evident that $\partial\psi_K(x)$ is the set of all normals to K at x.

A hyperplane is said to support a convex set K if it bounds a minimal halfspace containing K. If K is closed the intersections of a supporting hyperplane with K is called a face of K; if the face is not empty the hyperplane is said to support K at any point of this face, otherwise it supports K at infinity. As intersections of closed convex sets faces are closed convex sets. The equation of any hyperplane supporting K at finite distance can be written in the form: $\langle u^*, x\rangle - r = 0$, with u^* normal to K, and $r = \sup\limits_{y\in K} \langle u^*, y\rangle$. It follows that a K-face is the set of points having a common nonvanishing normal. To also include the case $u^* = 0$, K itself is considered to be a face, if only an improper one. In this context it is important to bear in mind that Jx is the set of normals at x to the ball of radius $\|x\|$ centered at the origin with norms all equal to $\|x\|$, and also the face of the ball of radius $\|x\|$ in X^* having x as normal.

THEOREM 2. *Any* $P_K x^*$ *is the intersection of a* K*-face with a face of a ball centered at the origin, and conversely. The* K*-face is proper if* $x^* \notin JK$.

Proof. For fixed u^* and v^* we have

$$\{x \mid u^* \in Jx\} \cap \{x \mid v^* \in \partial\psi_K(x)\} \subset \{x \mid u^*+v^* \in Jx+\partial\psi_K x\} = P_K(u^*+v^*).$$

Moreover, by definition of P_K,

$$\{x_1 \in P_K(u^*+v^*)\} \Longleftrightarrow \{u^*+v^* = u_1^* + v_1^*,\ u_1^* \in Jx_1,\ v_1^* \in \partial\psi_K(x_1)\}$$

and if x belongs to the intersection set on the left in the previous equation,

$$\{x_1 \in P_K(u^*+v^*)\} \Rightarrow \{0 = \langle u^*-u_1^*, x-x_1\rangle + \langle v^*-v_1^*, x-x_1\rangle\}$$

and by the monotonicity of J and $\partial\psi_K$,

$$0 = \langle u^*-u_1^*, x-x_1\rangle = \langle v^*-v_1^*, x-x_1\rangle$$

But $0 = \langle u^*-u_1^*, x-x_1\rangle = \langle u^*, x\rangle + \langle u_1^*, x_1\rangle - \langle u^*, x_1\rangle - \langle u_1^*, x\rangle =$

$$= [\tfrac{1}{2}\|u^*\|^2 + \tfrac{1}{2}\|x_1\|^2 - \langle u^*, x_1\rangle] + [\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|u_1^*\|^2 - \langle u_1^*, x\rangle],$$

and since both terms on the right are nonnegative, they vanish, implying that $u^* \in Jx_1$, $u_1^* \in Jx$. Furthermore, from $0 = \langle v^*-v_1^*, x-x_1\rangle$ we deduce for any $z \in K$,

$$\langle v^*, x_1-z\rangle = \langle v^*, x-z\rangle + \langle v^*, x_1-x\rangle = \langle v^*, x-z\rangle + \langle v_1^*, x_1-x\rangle + \langle v^*-v_1^*, x-x_1\rangle$$
$$= \langle v^*, x-z\rangle + \langle v_1^*, x_1-x\rangle \geqslant 0.$$

whence $v^* \in \partial\psi_K(x_1)$. In conclusion,

$$\{x_1 \in P_K(u^*+v^*)\} \Rightarrow \{u^* \in Jx_1,\ v^* \in \partial\psi_K(x_1)\},$$

and therefore

$$P_K(u^*+v^*) = \{x \mid u^* \in Jx\} \cap \{x \mid v^* \in \partial\psi_K(x)\}.$$

Of these two last sets the former is the face of the ball through x having u* as normal and the latter the K-face perpendicular to v*. This concludes the proof because any x* can be written in the form $x^* = u^*+v^*$, with v* normal to K at a point x, and u* normal at x to the ball through x. It is clear that if $x^* \notin JK$ then $u^* \neq 0$, and the corresponding K-face is proper.

COROLLARY 1. If J^{-1} is single valued so is P_K for any K.

This corollary can also be stated by saying that if the unit ball in X* is smooth then P_K is singlevalued.

COROLLARY 2. The functions $\frac{1}{2}\|x\|^2$ and $\langle x^*,x\rangle$ take constant values for $x \in P_K x^*$.

We can now use the notation $\frac{1}{2}\|P_K x^*\|^2$, $\langle x^*,P_K x^*\rangle$ without any ambiguity, because the results do not depend on the representative point in $P_K x^*$ used to calculate them.

COROLLARY 3. $P_K x^*$ is a bounded closed convex set for every $x^* \in X^*$.

THEOREM 3. $x^* \in JK$ *if and only if* $P_K x^* = J^{-1}x^* \cap K$.

Proof. It is obvious that if $P_K x^* = J^{-1}x^* \cap K$ then $x^* \in JK$. Conversely, if $x \in K$ and $x^* \in Jx$, then for each $y \in P_K x^*$ there is a $y^* \in Jy$ and a $u^* \in \partial\psi_K(y)$ such that $x^* = y^*+u^*$, and so

$$\langle x^*-y^*,x-y\rangle + \langle u^*,y-x\rangle = 0.$$

The two terms on the left are nonnegative, the first by monotonicity, and the second because u* is normal to K at y. Hence both vanish. From $\langle x^*-y^*,x-y\rangle = 0$ it follows that $y \in J^{-1}x^*$, and hence, since this holds for every y in $P_K x^*$, that $P_K x^* \subset J^{-1}x^* \cap K$. The opposite inclusion being obvious, the theorem is proved.

COROLLARY 1. $R(P_K) = K$.

Proof. From the definition of projector $R(P_K) \subset K$, and from the above theorem $P_K(JK) \supset K$, so $R(P_K) = K$.

COROLLARY 2.

COROLLARY 3.

(9) $$P_K x^* \subset P_K(tx^* + (1-t)\, J\, P_K x^*)\ , \quad 0 \leqslant t \leqslant 1.$$

Proof. From Theorem 1, Corollary 5 and Corollary 2 above.

THEOREM 4. *A subdifferential operator* $P\colon X^* \longrightarrow 2^X$ *is a projector if and only if it satisfies*

(10) $$Px^* = \partial[\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2],$$

where the notation is construed to mean that $\langle x^*, x\rangle - \frac{1}{2}\,\|x\|^2$ *takes a constant value for* $x \in Px^*$, *and that the resulting function, assumed equal to* $+\infty$ *when* Px^* *is empty, is a proper l.s.c. convex function of* x^*.

Proof. Necessity is the content of Theorem 1, Corollary 4. As for sufficiency start out by remarking that $\mathcal{D}(P)$ is convex because by hypothesis it coincides with the domain of a l.s.c. convex function. We claim that P is locally bounded about each point in space. Indeed, if it were not there would be a point x^* and a sequence $\{x_n^*\}_1^\infty \subset \mathcal{D}(P)$ such that $x_n^* \longrightarrow x^*$, $\|Px_n^*\| \uparrow +\infty$, and then $\langle x_n^*, Px_n^*\rangle - \frac{1}{2}\,\|Px_n^*\|^2 \longrightarrow -\infty$, implying, by lower semicontinuity, that $\langle x^*, Px^*\rangle - \frac{1}{2}\,\|Px^*\|^2 = -\infty$, which is impossible. Then, local boundedness coupled with demicontinuity (itself a consequence of maximal monotonicity) require that $\mathcal{D}(P)$ be closed. Now, if u is normal to $\mathcal{D}(P)$ at x^* then, by maximal monotonicity again, $Px^* + tu \in Px^*$, $t \geqslant 0$, and $u = 0$, since Px^* is a bounded set. Having no nonvanishing normal $\mathcal{D}(P)$ is the whole space. (The foregoing argument is a particular case of the theorem that says that a maximal monotone operator is surjective if and only if its inverse is locally bounded [4]).

Next we observe that (10) amounts to

$$[\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2] - [\langle y^*, Py^*\rangle - \tfrac{1}{2}\,\|Py^*\|^2] \geqslant \langle x^*-y^*, y\rangle\ ,$$

$\forall\ x^*, y^* \in X^*$, $\forall\ y \in Py^*$, that is, to

$$\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2 \geqslant \langle x^*, y\rangle - \tfrac{1}{2}\,\|y\|^2,\ \forall\ x^*, y^* \in X^*\ ,\ \forall\ y \in Py^*.$$

Hence, since for $y \in Px^*$ the right hand member of this inequality coincides with the one on the left,

$$\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2 = \sup_{y \in \overline{R(P)}} \{\langle x^*, y\rangle - \tfrac{1}{2}\,\|y\|^2\}.$$

As the closure of the range of a maximal monotone operator $\overline{R(P)}$ is convex [cf.5], and the supremum above is $(Q + \psi_{\overline{R(P)}})^*(x^*) =$
$= \langle x^*, P_{\overline{R(P)}}\, x^*\rangle - \frac{1}{2}\,\|P_{\overline{R(P)}}\, x^*\|^2$. Finally,

$$Px^* = \partial[\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2] = \partial[\langle x^*, P_{\overline{R(P)}}\, x^*\rangle - \tfrac{1}{2}\,\|P_{\overline{R(P)}}\, x^*\|^2] =$$
$$= P_{\overline{R(P)}}\, x^*. \quad \text{Q.E.D.}$$

THEOREM 6. $\sum_1^n P_{K_i}$ *is a projector if and only if*

$$\sum_1^n \|P_{K_i}x^*\|^2 - \|\sum_1^n P_{K_i}x^*\|^2 = \text{const.} \tag{11}$$

In such a case $\sum_1^n P_{K_i} = P_{\sum_1^n K_i}$.

Proof. If $\sum_1^n P_{K_i}$ is a projector then the subdifferential of $\langle x^*,(\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2$, namely $\sum_1^n P_{K_i}x^*$, is contained in that of $\sum_1^n [\langle x^*,P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2]$, and in consequence both convex functions coincide up to an additive constant, that is, (11) holds. Conversely, if (11) holds, then

$$\langle x^*,(\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2 = \sum_1^n [\langle x^*,P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2] + \text{const},$$

and

$$\partial[\langle x^*,(\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2] \supset \sum_1^n \partial[\langle x^*,P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2] =$$
$$= (\sum_1^n P_{K_i})x^*.$$

Since the subdifferential of a convex function is monotone, and $\sum_1^n P_{K_i}$ maximal monotone [6], the above inclusion is in fact an equality, and $\sum_1^n P_{K_i}$ is a projector because it satisfies relation (10). Thus the first part of the theorem is proved. As to the last, note first that if $f_i(x) = \frac{1}{2}\|x\|^2 + \psi_{K_i}(x)$, $i = 1,2,\dots,n$ then $\sum_1^n P_{K_i} = \sum_1^n \partial f_i^* = \partial \sum_1^n f_i^*$ because the f_i^*'s are continuous [6]. Hence

$\overline{\mathcal{R}(\sum_1^n P_{K_i})} = \overline{\mathcal{R}(\partial \sum_1^n f_i^*)} = \overline{\mathcal{D}(\partial(\sum_1^n f_i^*)^*} = \overline{\mathcal{D}(\partial(f_1\square f_2\square\dots\square f_n))}$, and,as the domain of the subdifferential of a l.s.c. convex function is dense in the domain of the function [1],

$$\overline{\mathcal{D}(\partial(f_1\square f_2\square\dots\square f_n)} = \overline{\mathcal{D}(f_1\square f_2\square\dots\square f_n)} = \overline{\mathcal{D}(f_1)+\mathcal{D}(f_2)+\dots+\mathcal{D}(f_n)} =$$
$$= \overline{K_1+K_2+\dots+K_n}.$$

Therefore, $\overline{\mathcal{R}(\sum_1^n P_{K_i})} = \overline{\sum_1^n K_i}$. Now, if $\sum_1^n P_{K_i}$ is a projector its range is closed and $\overline{\sum_1^n K_i} = \mathcal{R}(\sum_1^n P_{K_i}) \subset \sum_1^n K_i$, whence $\mathcal{R}(\sum_1^n P_{K_i}) = \sum_1^n K_i$. The its range.

§2. CONICAL PROJECTORS.

Projectors on closed convex cones with vertex at the origin are called conical projectors. It is clear that a projector on a convex set is positive homogeneous when the set is a cone with vertex at 0, and only then, so that the class of conical projectors coincides with that of positive homogeneous projectors. The letter C will be reserved to designate the above type of cones, so that P_C will always indicate a conical projector.

The dual of a cone $C \subset X$ in the cone in X^*

(12) $$C^{\perp} = \{x^* \in X^* \mid \langle x^*,x\rangle \leqslant 0,\ x \in C\}.$$

$C^{\perp}$ is nonempty, closed and convex. The operation of taking duals has the following properties:

(13) $$C^{\perp\perp} = C\ ,\quad \{C_1 \subset C_2\} \Leftrightarrow \{C_1^{\perp} \supset C_2^{\perp}\}\ ,\quad (\bigcap_i C_2)^{\perp} = \overline{\mathrm{co} \cup C_i^{\perp}}.$$

For linear spaces $\perp$ coincides with the operation of taking anihilators. The indicator functions of dual cones are conjugate of each other. We leave to the reader the verification of these facts.

The original definition (1) acquires a special form in the case of projectors on cones:

THEOREM 6.

(14) $$P_C x^* = \{x \in C \mid \langle x^*,x\rangle = \|x\|^2 = [\sup_{u\in C,\|u\|\le 1} \langle x^*,u\rangle]^2\}$$

Proof. If x minimizes $\frac{1}{2}\|y\|^2 - \langle x^*,y\rangle$ over C, then, for any $x \in C$, $\frac{1}{2}t^2\|x\|^2 - t\langle x^*,x\rangle$ as a function of t attains its minimum on the positive real axis at $t = 1$, and hence $\|x\|^2 = \langle x^*,x\rangle$. Therefore $x \in P_C x^*$ if and only if $\|x\|^2 = \langle x^*,x\rangle$ and

$$-\frac{\|x\|^2}{2} = \frac{\|x\|^2}{2} - \langle x^*,x\rangle = \inf_{y\in C} \frac{1}{2}\{\|y\|^2 - \langle x^*,y\rangle\} =$$

$$= \inf_{y\in C}\inf_{t\ge 0}\{\frac{1}{2}t^2\|y\|^2 - t\langle x^*,y\rangle\} =$$

$$= \inf_{y\in C}\begin{cases} 0, & \text{if } \langle x^*,y\rangle \leqslant 0 \\ -\frac{1}{2}\langle x^*,\frac{y}{\|y\|}\rangle^2, & \text{if } \langle x^*,y\rangle > 0 \end{cases} = -\frac{1}{2}[\sup_{u\in C,\|u\|\le 1}\langle x^*,u\rangle]^2 .$$

Q.E.D.

It is worth remarking that any $x \neq 0$ in $P_C x^*$ is of the form $\langle x^*,u\rangle^+ u$, where u is a vector in C maximizing $\langle x^*,v\rangle^+$, so that $P_C x^*$ is simply obtained by looking for the directions in C making the smallest angle with x* and projecting on them in the ordinary sense. This geometrical definition may very well be taken as the point of departure for the theory of conical projectors. It is indeed the idea of "least angle mapping" what lies at the roots of projectors. J.P. Aubin has used this idea to define projectors on linear spaces [1].

THEOREM 7.

$$\|P_C x^*\|^2 = \langle x^*, P_C x^*\rangle = [\sup_{u\in C,\, \|u\|\le 1} \langle x^*, u\rangle]^2 = \delta^2_{C^\perp}(x^*), \tag{15}$$

where $\delta_{C^\perp}(x^*)$ *denotes the distance from* x^* *to* $C^\perp$.

Proof. Only the last equality requires a proof. By Theorem 1, Corollary 3,

$$\langle x^*, P_C x^*\rangle - \frac{1}{2}\|P_C x^*\|^2 = (Q+\psi_C)^*(x^*) = (Q^* \,\square\, \psi_C^*)(x^*) = (Q^* \,\square\, \psi_{C^\perp})(x^*) =$$

$$= \inf_{y^*\varepsilon C^\perp} \frac{1}{2}\|x^*-y^*\|^2 = \frac{1}{2}\delta^2_{C^\perp}(x^*).$$

Since $\langle x^*, P_C x^*\rangle - \frac{1}{2}\|P_C x^*\|^2$ is equal to both $\frac{1}{2}\|P_C x^*\|^2$ and $\frac{1}{2}\langle x^*, P_C x^*\rangle$, the theorem is proved.

COROLLARY 1. $\eta(P_C) = C^\perp$.

COROLLARY 2.

$$P_C x^* = \partial \frac{1}{2}\|P_C x^*\|^2 = \partial \frac{1}{2}\delta^2_{C^\perp}(x^*). \tag{16}$$

Next theorem establishes a relation between projectors and nearest point mappings.

THEOREM 8. $(I^*-JP_C)x^* \cap C^\perp$ *is the set of points in* $C^\perp$ *closest to* x^*. (I^* *denotes the identity map in* X^*).

Proof. If $z^* \in (I^*-JP_C)x^* \cap C^\perp$ then $x^*-z^* \in JP_C x^*$ and $\|x^*-z^*\| = \|JP_C x^*\| = \|P_C x^*\| = \delta_{C^\perp}(x^*)$, which shows that z^* minimizes the distance from x^* to points in $C^\perp$.

Conversely, if $z^* \in C^\perp$ realizes the distance from x^* to $C^\perp$, then $\frac{1}{2}\delta^2_{C^\perp}(x^*) = \frac{1}{2}\|x^*-z^*\|^2$. Since on the other hand $\frac{1}{2}\delta^2_{C^\perp}(y^*) \leqslant \frac{1}{2}\|y^*-z^*\|^2$ for all $y^* \in X^*$, and since $\partial\frac{1}{2}\delta^2_{C^\perp}(x^*) = P_C x^*$ (Corollary above),

$$\frac{1}{2}\|y^*-z^*\|^2 - \frac{1}{2}\|x^*-z^*\|^2 \geqslant \frac{1}{2}\delta^2_{C^\perp}(y^*) - \frac{1}{2}\delta^2_{C^\perp}(x^*) \geqslant \langle y^*-x^*, P_C x^*\rangle,\ y^*\in X^*,$$

whence by definition of subgradient,

$$P_C x^* \subset \partial\frac{1}{2}\|x^*-z^*\|^2 = J^{-1}(x^*-z^*)\ ,$$

If we let $\Pi_{C^\perp}: X^* \to 2^{X^*}$ denote the nearest point mapping on $C^\perp$ we can give this theorem a form suggestive of Moreau's decomposition of a vector in Hilbert space along orthogonal directions in dual cones [3].

COROLLARY. For any $x^* \in X^*$ there are vectors u and v^* such that

$$x^* \in Ju+v^*,\ u \in C,\ v^* \in C^\perp,\ \langle v^*,u\rangle = 0. \tag{17}$$

Moreover, if (17) holds then $u \in P_C x^*$ and $v^* \in \Pi_{C^\perp} x^*$.

Proof. The possibility of decomposition (17) follows from Theorem 1, Corollary 6 and the theorem above. As to the last part notice that if $v^* \in C^\perp$ and $\langle v^*,u\rangle = 0$ then $v^* \in \partial\psi_C(u)$, and apply Theorems 1 and 10. Projectors and nearest point mappings are the same objects in Hilbert space. If the identification of the space with its dual is made explicit this coincidence can be expressed by the equation

$$\Pi_C = P_C J. \tag{18}$$

Now, is this relation characteristic of Hilbert space? We don't know, we only conjecture that it is. The following theorem gives some support to our contention.

THEOREM 9. *Let* X *and* X* *be dual reflexive Banach spaces. Then if the duality mapping* $J\colon X \to 2^{X^*}$ *is bijective, and*

$$\Pi_C = P_C J \text{ for all straight lines and hyperplanes } C \subset X, \tag{19}$$

$$\Pi_{C^*} = P_{C^*} J^{-1} \text{ for all straight lines and hyperplanes } C^* \subset X^*, \tag{20}$$

X *is a Hilbert space.*

Proof. By Theorem 2, Corollary 1 all projectors are single valued, and on use of Theorem 8, (19) and (20) can be written in the form

$$(I-J^{-1}P_{C^\perp})x = P_C Jx \quad , \quad (I^*-J\,P_C)x^* = P_{C^\perp} J^{-1}x^*.$$

If in the first of these equations $P_{C^\perp}x$ is replaced by its expression derived from the last one obtains

$$(I-P_C J)x = J^{-1}(J-JP_C J)x$$

that is,

$$J(x-P_C Jx) = Jx - JP_C Jx.$$

In a similar manner

$$J^{-1}(x^*-P_{C^*}J^{-1}x^*) = J^{-1}x^* - J^{-1}P_{C^*}J^{-1}x^*.$$

Making in the above equations the following identifications

$$C = \{tu\}_{-\infty<t<+\infty}\ ,\ C^* = \{tJu\}_{-\infty<t<+\infty}\ ,\ x = v,\ x^* = Jv$$

where u and v are any two unit vectors, one gets

$$J(v - \langle Jv,u\rangle\, u) = Jv - \langle Jv,u\rangle\, Ju$$

$$J(v - \langle Ju,v\rangle\, u) = Jv - \langle Ju,v\rangle\, Ju\ .$$

Set $r = v-\beta u$, $s = v-\alpha u$, $\alpha = \langle Ju,v\rangle$, $\beta = \langle Jv,u\rangle$, and on use of these identities proceed to the following calculations:

$$\|r\|^2 = \langle Jr,r\rangle = \langle Jv-\beta Ju, v-\beta u\rangle = 1+\beta^2-\beta^2-\beta\alpha = 1-\alpha\beta$$

$$\|s\|^2 = \langle Js,s\rangle = \langle Jv-\alpha Ju, v-\alpha u\rangle = 1+\beta^2-\alpha\beta-\alpha^2 = 1-\alpha\beta$$

$$\langle Jr,s\rangle = \langle Jv-\beta Ju, v-\alpha u\rangle = 1+\alpha\beta-\alpha\beta-\alpha\beta = 1-\alpha\beta.$$

Therefore, $\langle Jr,s\rangle = \|Jr\|^2 = \|s\|^2$ and by definition of J, $Jr = Js$. This implies $r=s$, which in turn yields $\alpha=\beta$, that is, $\langle Ju,v\rangle = \langle Jv,u\rangle$. This equation, valid for unitary u and v, is at once extended to all u's and v's in X by use of the homogeneity of J. But then J is a self adjoint mapping of X onto X*, and as such linear. It follows that $\|x\|^2 = \langle Jx,x\rangle$ is a quadratic form, and the theorem is proved.

Theorem 4 takes a simpler form in the case of conical projectors:

THEOREM 10. *A positive homogeneous-subdifferential operator* $P\colon X^* \to 2^X$ *is a conical projector if and only if it satisfies*

$$Px^* = \partial \tfrac{1}{2}\ \|Px^*\|^2. \tag{21}$$

Proof. It follows from Theorem 4, and equation (15) that a conical projector satisfies (21). Conversely, if a positive homogeneous subdifferential P satisfies (21), then, since it also satisfies $Px^* = \partial\frac{1}{2}\langle x^*,Px^*\rangle$, [9], $\|Px^*\|^2 = \langle x^*,Px^*\rangle$ (use the fact that $P0^*=0$), that is $\frac{1}{2}\|Px^*\|^2 = \langle x^*,Px^*\rangle - \frac{1}{2}\|Px^*\|$. Hence, (10) holds for P, and P is a projector.

COROLLARY. A positive homogeneous subdifferential operator $P\colon X^* \to 2^X$ is a conical projector if and only if

$$\|Px^*\|^2 = \langle x^*,Px^*\rangle\ ,\ \forall\ x^* \in \mathcal{D}(P). \tag{22}$$

Proof. Necessity is contained in Theorem 7. If, on the other hand, P is a subdifferential operator satisfying (22), then $Px^* = \partial\frac{1}{2}\langle x^*,Px^*\rangle = \partial\frac{1}{2}\|Px^*\|^2$, and I is a projector by the above theorem.

Now we turn our attention to the important question of when a sum of projectors is a projector.

n

(23) $$\Big\| \sum_1^n P_{C_i} x^* \Big\|^2 = \sum_1^n \| P_{C_i} x^* \|^2 .$$

In such a case

$$\sum_1^n P_{C_i} = P_{\sum_1^n C_i} .$$

Proof. This is a particular case of Theorem 5. The constant in equation (11) is zero because all P_{C_i}'s vanish at $x^* = 0$.

It may be checked that if all C_i's are rays: $\{tu_i\}_{t \geq 0}$, $\|u_i\| = 1$, (23) simply says that $\|x\|^2$ is quadratic over the n-hedron $\{\sum_1^n t_i u_i\}_{t_i \geq 0}$, and that the u_i's are orthogonal with regard to the induced scalar product, or more briefly, that $\{\sum_1^n C_i, \| \ \|\}$ is a 2^n-tant of an n-dimensional Hilbert space. Based on this remark the system of n cones satisfying the Pythagorean relation (23) may be conceived as a generalization of an orthogonal n-tuple of vectors where the vectors are replaced by cones. Accordingly we shall say that such cones form an orthogonal n-tuple, and shall use the notation $C_1 \perp C_2 \perp \ldots \perp C_n$ or $P_{C_1} \perp P_{C_2} \perp \ldots \perp P_{C_n}$ to denote this fact. It is remarkable how much of the Hilbert space structure is brought into the space by the requirement that a projector should split into the sum of others.

THEOREM 12. $C_1 \perp C_2 \perp \ldots \perp C_n$ *if and only if*

(24) $$\inf_{\sum_1^n x_i = x,\, x_i \in C_i} \sum_1^n \|x_i\|^2 = \|x\|^2 , \ \forall \ x \in C_1 + C_2 + \ldots + C_n .$$

In such a case the infimum is always attainable.

Proof. $C_1 \perp C_2 \perp \ldots \perp C_n$ is equivalent to

$$\sum_1^n \frac{1}{2} \|P_{C_i} x^*\|^2 = \frac{1}{2} \|P_{\sum_1^n C_i} x^*\|^2 ,$$

which by taking conjugates and recalling that the conjugate of $\frac{1}{2} \|P_C x^*\|^2$ is $\frac{1}{2} \|x\|^2 \perp \psi_C(x)$ (Theorem 1, Corollary 3, and (15)) becomes (24). To see that the infimum is attained take n sequences $\{x_i^{(k)}\}_1^\infty \subset C_i$, such that $\sum_{i=1}^n \|x_i^{(k)}\|^2 \downarrow \|x\|^2$, $\sum_{i=1}^n x_i^{(k)} = x$. Since the sequences are obviously bounded they can be assumed to be weakly convergent to limits x_i in C_i respectively. Then, the limit inferior of the norms being larger than the norm of the weak limit, we must have

$\sum_{i=1}^{n} \|x_i\|^2 \leqslant \|x\|^2$, $\sum_1^n x_i = x$, that is $\sum_{i=1}^{n} \|x_i\|^2 = \|x\|^2$, $\sum_{i=1}^{n} x_i = x$.
(Briefer but less direct: $R(\sum_1^n P_{C_i}) = R(P_{\sum_1^n C_i}) = \sum_1^n C_i$).

For the inversion of the statement: If $C_1 \perp C_2 \perp \ldots \perp C_n$, then $\{x_i \in P_{C_i} x^*, i = 1,2,\ldots n\} \Rightarrow \{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2\}$, we need a couple of lemmas.

LEMMA 1. *Let* $C_1 \perp C_2 \perp \ldots \perp C_n$. *Then*
$\{J(\sum_1^n x_i) \cap J(\sum_1^n x_i') \neq \emptyset$, $\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2$, $\|\sum_1^n x_i'\|^2 = \sum_1^n \|x_i'\|^2$,
$x_i, x_i' \in C_i$, $i = 1,2,\ldots,n\}$ *implies* $\{Jx_i \cap Jx_i' \neq \emptyset$, $i = 1,2,\ldots,n\}$.

Proof. From $J(\sum_1^n x_i) \cap J(\sum_1^n x_i') \neq \emptyset$ it follows $\|t(\sum_1^n x_i)+(1-t)(\sum_1^n x_i')\|^2 =$ $=$ const., for $0 \leqslant t \leqslant 1$. Then,

$$\|t(\sum_1^n x_i)+(1-t)(\sum_1^n x_i')\|^2 = t\|\sum_1^n x_i\|^2+(1-t)\|\sum_1^n x_i'\|^2 = \sum_1^n (t\|x_i\|^2+(1-t)\|x_i'\|^2) \geqslant$$
$$\geqslant \sum_1^n \|tx_i+(1-t)x_i'\|^2, \quad 0 \leqslant t \leqslant 1,$$

and by Theorem 12, since $tx_i + (1-t)x_i' \in C_i$,

$$\sum_1^n \|tx_i+(1-t)x_i'\|^2 \geqslant \|\sum_1^n (tx_i+(1-t)x_i')\|^2 = \|t\sum_1^n x_i+(1-t)\sum_1^n x_i'\|^2,$$

so

$$\sum_1^n \|tx_i+(1-t)x_i'\|^2 = \|t\sum_1^n x_i+(1-t)\sum_1^n x_i'\|^2 = \text{const.}$$

Now, the sum of the squares of convex functions being constant if and only if the individual terms are constant, we must have, $\|tx_i+(1-t)x_i'\|^2 =$ const., $0 \leqslant t \leqslant 1$, from which it follows $Jx_i \cap Jx_i' \neq \emptyset$, $i = 1,2,\ldots,n$. Q.E.D.

LEMMA 2. *If* $C_1 \perp C_2 \perp \ldots \perp C_n$, *then*
$\{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2$, $x_i \in C_i\} \Rightarrow \{P_{C_i} x^* \in J^{-1}Jx_i$, $i = 1,2,\ldots,n$, $\forall\ x^* \in J(\sum_1^n x_i)\}$.

Proof. By Theorems 3 and 11,

$$P_{\sum_1^n C_i} x^* = J^{-1}x^* \cap (\sum_1^n C_i) = \sum_1^n P_{C_i} x^*, \qquad \forall\ x^* \in J(\sum_1^n x_i).$$

Hence, if $x_1', x_2', \ldots, x_n'$ are any n points in $P_{C_1}x^*, P_{C_2}x^*, \ldots, P_{C_n}x^*$ respectively, $\sum_1^n x_i' \in J^{-1}x^*$, and $x^* \in J(\sum_1^n x_i) \cap J(\sum_1^n x_i')$.

The lemma above then yields $Jx_i \cap Jx_i^! \neq \emptyset$, that is, $x_i^! \in J^{-1}Jx_i$, $i = 1,2,\ldots,n$, and since $x_i^!$ is any point in $P_{C_i}x^*$, $P_{C_i}x^* \subset J^{-1}Jx_i$, $i = 1,2,\ldots,n$.

THEOREM 13. *If* $C_1 \perp C_2 \perp \ldots \perp C_n$, *then*

$$(25)\ \{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2,\ x_i \in C_i,\ i = 1,2,\ldots,n\} \iff$$
$$\iff \{x_i \in P_{C_i}x^*,\ i = 1,2,\ldots,n,\ \forall\ x^* \in J(\sum_1^n x_i)\}$$

Proof. Assume that the proposition on the left holds. Then, by last lemma, $\|P_{C_i}x^*\| = \|x_i\|$, $x^* \in J(\sum_1^n x_i)$, and so, since $\|x_i\| = \|P_{C_i}x^*\| = \sup_{u_i \varepsilon C_i, \|u_i\| \leq 1} \langle x^*,u_i\rangle$, $\langle x^*,x_i\rangle - \|x_i\|^2 \leqslant 0$, $i = 1,2,\ldots,n$, and adding up these inequalities,

$$\sum_1^n [\langle x^*,x_i\rangle - \|x_i\|^2] = \langle x^*,\sum_1^n x_i\rangle - \sum_1^n \|x_i\|^2 = \|\sum_1^n x_i\|^2 - \|\sum_1^n x_i\|^2 = 0$$

Therefore, $\langle x^*,x_i\rangle = \|x_i\|^2 = \|P_{C_i}x^*\|^2 = [\sup_{u_i \varepsilon C_i, \|u_i\| \leq 1} \langle x^*,u_i\rangle]^2$, and by (14) $x_i \in P_{C_i}x^*$, proving the implication from left to right. The opposite implication is but a quantification of (23).

COROLLARY.

$$(26)\ \{\sum_1^n P_{C_i} = P_{\sum_1^n C_i}\} \Rightarrow \{P_{C_i}x^* \subset P_{C_i}JP_{\sum_1^n C_j}x^* \subset J^{-1}JP_{C_i}x^* \cap C_i\ ,$$
$$i = 1,2,\ldots,n,\ \ \forall\ \ x^* \in X^*\}.$$

Proof. Let $x_1,x_2,\ldots,x_n$ be points in $P_{C_1}x^*,P_{C_2}x^*,\ldots,P_{C_n}x^*$ respectively. Then by Lemma 2 and the theorem above $x_i \in P_{C_i}J(\sum_1^n x_i) \subset \subset J^{-1}Jx_i \cap C_i$, $i = 1,2,\ldots,n$, whence (26) follows from the fact that when the x_i's range over the sets $P_{C_i}x^*$, $\sum_1^n x_i$ ranges over $P_{\sum_1^n C_i}x^*$.

REMARK. By (8) $J^{-1}JP_{C_i}x^* \cap C_i = P_{C_i}JP_{C_i}x^*$, so that the right member of (26) can be written in the form $P_{C_i}x^* \subset P_{C_i}JP_{\sum_1^n C_j}x^* \subset P_{C_i}JP_{C_i}x^*$.

Comparison with (8) prompts the conjecture that the last inclusion is not proper, that is, that $P_{C_i}JP_{\sum_1^n C_k} = P_{C_i}JP_{C_i}$. However, this is not true in general. Consider the following example:

Let X and X* be the dual two dimensional Banach space with norms:

$$\|x\| = \begin{cases} (|\xi_1|^2 + |\xi_2|^2)^{1/2}, & \xi_1\xi_2 \geqslant 0 \\ |\xi_1| + |\xi_2| , & \xi_1\xi_2 \leqslant 0 \end{cases} , \quad \|x^*\| = \begin{cases} (|\xi_1^*| + |\xi_2^*|)^{1/2}, & \xi_1^*\xi_2^* \geqslant 0 \\ \max(|\xi_1^*|,|\xi_2^*|) , & \xi_1^*\xi_2^* \leqslant 0. \end{cases}$$

The second and fourth quadrants in X, wich we call C_1 and C_2 respectively, form an orthogonal couple, and $J^{-1} = P_{C_1}+P_{C_2}$. For any $x^* \in X^*$ in the first quadrant and away from the axes $JP_{C_1+C_2}x^* = x^*$, and $JP_{C_1}x^* = x_i^*$, where x_i is the Euclidean projection of x^* on the i-axis. Moreover, $P_{C_1}J\, P_{C_1+C_2}x^* = P_{C_1}x^*$ is a singleton x_1 on the 2-axis, whereas $P_{C_1}JP_{C_1}x^* = J^{-1}x^* \cap C_1$ is a straight line segment through x_1 across C_1 parallel to the first quadrant bisector. Obviously $P_{C_1}\, J\, P_{C_1+C_2}x^* \neq P_{C_1}\, J\, P_{C_1}x^*$.

All that has been said of conical projections from Theorem 11 on applies also to projections on general convex sets, the only difference being the presence of an additive constant all throughout.

THEOREM 14. *If* $C_1 \perp C_2 \perp \ldots \perp C_n$ *then*

$$(27) \qquad P_{C_i}(tI^*+(1-t)\, J\, P_{\sum_1^n C_j}) = P_{C_i} , \quad 0 < t \leqslant 1,\ i=1,2,\ldots,n$$

Proof. For $x^* \in X^*$ and $y^* \in J\, P_{\sum_1^n C_j} x^*$ set $z^*(t) = tx^* + (1-t)y^*$, $0 < t \leqslant 1$. Now

$$\sup_{u_i\varepsilon C_i,\|u_i\|\leq 1} \langle z^*(t),u_i\rangle \leqslant t \sup_{u_i\varepsilon C_i,\|u_i\|\leq 1} \langle x^*,u_i\rangle + (1-t) \sup_{u_i\varepsilon C_i,\|u_i\|\leq 1} \langle y^*,u_i\rangle =$$

$$= t\|P_{C_i}x^*\| + (1-t)\|P_{C_i}y^*\|.$$

By (26) $\|P_{C_i}y^*\| = \|P_{C_i}x^*\|$ so,

$$\sup_{u_i\varepsilon C_i,\|u_i\|\leq 1} \langle z^*,u_i\rangle \leqslant \|P_{C_i}x^*\|.$$

Moreover, by hypothesis and choice of y^* there are points $x_i \in P_{C_i}x^*$, $i = 1,2,\ldots,n$ such that $y^* \in J\sum_1^n x_i$. Since $\langle x^*,x_i\rangle = \|x_i\|^2$ by (14), and $\langle y^*,x_i\rangle = \|x_i\|^2$ by (25), we have $\langle z^*(t),x_i\rangle = \|x_i\|^2 = \|P_{C_i}x^*\|^2$, $t = 1,2,\ldots,n$. In view of what has already been proved these equations mean that the suprema of $\langle z^*(t),u_i\rangle$, $\langle x^*,u_i\rangle$, $\langle y^*,u_i\rangle$ over the u_i's in C_i with $\|u_i\| \leqslant 1$ are attained simultaneously and are equal to $\|P_{C_i}x^*\|$. Then,

$$\{v_i \in P_{C_i}z^*(t)\} \Longleftrightarrow \{\|v_i\| = \sup_{u_i\varepsilon C_i,\|u_i\|\leq 1} \langle z^*,u_i\rangle = \|P_{C_i}x^*\| , \langle z^*(t),v_i\rangle = \|P_{C_i}x^*\|\}$$

$$\Longleftrightarrow \{\|v_i\| = \sup_{u_i \in C_i, \|u_i\| \le 1} \langle x^*, u_i \rangle = \sup_{u_i \in C_i, \|u_i\| \le 1} \langle y^*, u_i \rangle = \|P_{C_i} x^*\|, \ \langle x^*, v_i \rangle = \|P_{C_i} x^*\|^2 =$$

$$= \langle y^*, v_i \rangle = \|v_i\|^2\} \Longleftrightarrow \{v_i \in P_{C_i} x^*, \ v_i \in P_{C_i} y^*\}$$

and hence

$$P_{C_i}(tx^* + (1-t)y^*) = P_{C_i} x^* \cap P_{C_i} y^*, \quad 0 < t \leqslant 1 \ , \ i = 1,2,\ldots,n.$$

Since y^* is any point in $J\, P_{\sum_1^n C_j} x^*$,

$$P_{C_i}(t\, x^* + (1-t) J\, P_{\sum_1^n C_j} x^*) = P_{C_i} x^* \cap P_{C_i}\, J\, P_{\sum_1^n C_j} x^* \ ,$$

and an appeal to the previous theorem concludes the proof.

COROLLARY. For any conical projector,

(28) $$P_C(tI^* + (1-t) J\, P_C) = P_C \ , \quad 0 < t \leqslant 1.$$

Proof. Set in (27) $C_1 = C$, $C_2 = C_3 = \ldots = C_n = \{0\}$.

The geometrical meaning of the relation $C_1 \perp C_2 \perp \ldots \perp C_n$ is not sufficiently clear from defining Pythagorean relation (23), nor from (24). In Hilbert space each cone is the dual of the sum of the others relatively to the total sum [9, Equation 2.10]. A similar result holds in reflexive Banach spaces.

LEMMA 3.

(29) $$C_1 \perp C_2 \perp \ldots \perp C_n \Rightarrow \{JC_j \subset (\sum_{i \neq j} C_i)^\perp \ , \ j = 1,2,\ldots,n\}.$$

Proof. Let $x_j \in C_j$, $y_j^* \in Jx_j$. Then, since by (8) $x_j \in P_{C_j} y_j^* \subset J^{-1} J x_j$,

$$\|x_j + \sum_{i \neq j} P_{C_i} y_j^*\|^2 = \sum_{i=1}^n \|P_{C_i} y_j^*\|^2 = \|x_j\|^2 + \sum_{i \neq j} \|P_{C_i} y_j^*\|^2$$

and by definition of J,

$$\sum_{i \neq j} \|P_{C_i} y_j^*\|^2 = \|x_j + \sum_{i \neq j} P_{C_i} y_j^*\|^2 - \|x_j\|^2 \geqslant 2 \langle y_j^*, \sum_{i \neq j} P_{C_i} y_j^* \rangle = 2 \sum_{i \neq j} \|P_{C_i} y_j^*\|^2.$$

Hence, $P_{C_i} y_j^* = 0$, that is, $y_j^* \subset C_i^\perp$, $i \neq j$, and $J\, C_j \subset \bigcap_{i \neq j} C_i^\perp = (\sum_{i \neq j} C_i)^\perp$.

THEOREM 15.

(30) $$C_1 \perp C_2 \perp \ldots \perp C_n \Rightarrow C_j = J^{-1}[(\sum_{i \neq j} C_i)^\perp] \cap (\sum_{k=1}^n C_k) \ , \ j = 1,2,\ldots,n.$$

Proof. By Lemma 3, $C_j \subset J^{-1}[(\sum_{i \neq j} C_i)^\perp]$, and since $C_j \subset \sum_1^n C_k$,

$C_j \subset J^{-1}[(\sum_{i \neq j} C_i)^\perp] \cap (\sum_1^n C_k)$.

This is half of (30). To prove the other half start with an x_j in $J^{-1}[(\sum_{i\neq j} C_i)^{\perp}] \cap (\sum_1^n C_k)$, and then observe that
$x_j \in C_1 + C_2 + \ldots + C_n$, $x_j \in J^{-1}x_j^*$, for some $x_j^* \in (\sum_{i\neq j} C_i)^{\perp} = \bigcap_{i\neq j} C_i^{\perp}$.
So

$$x_j^* \in Jx_j \subset J(C_1 + C_2 + \ldots + C_n)$$

and by Theorem 3,

$$x_j \in J^{-1}x_j^* \cap (C_1 + C_2 + \ldots + C_n) = P_{\sum_1^n C_k} x_j^* = \sum_1^n P_{C_k} x_j^* = P_{C_j} x_j^* \subset C_j,$$

and since x_j was any point in $J^{-1}[(\sum_{i\neq j} C_i)] \cap (\sum_1^n C_k)$,

$$J^{-1}[(\sum_{i\neq j} C_i)] \cap (\sum_1^n C_k) \subset C_j,$$

concluding the proof.

COROLLARY. In the relation $P_C = \sum_1^n P_{C_k}$ any n projectors determine the remaining one.

We do not know if the arrow in (30) can be reversed. The most that we can say is that this is the case in Hilbert spaces of dimension not larger than three.

§3. CONCLUSION AND COMMENTS.

The material set forth in the preceeding pages is essentially all we know about projectors in reflexive Banach spaces. No doubt the discussion can be carried further still, and we hope that it will be, for, as it stands the extent of our knowledge is insufficient for the proper development of a spectral theory. Let us point out here to some of the most visible shortcomings.

In the first place it is not known if the relation $P_{C_1} > P_{C_2}$, defined as meaning that $P_{C_1} - P_{C_2}$ is a projector, is a partial ordering for projectors. Indeed, there is no proof of it being transitive.

Important as transitivity is, spectral theory requires something stronger still, namely that any sub k-tuple of an orthogonal n-tuple of cones be again orthogonal. This is necessary if the spectral measure built out of a spectral resolution is to be projector-valued. In Hilbert space this is a consequence of $P_{C_1} > P_{C_2}$ being equivalent to $P_{C_2}JP_{C_1} = P_{C_2}$. No such equivalence has been established in reflexive Banach spaces, we only know that if J^{-1} is single valued $P_{C_1} > P_{C_2}$ implies $P_{C_2}JP_{C_1} = P_{C_2}$ (Corollary, Theorem 13).

the homogenity of orthogonality, is the following:

If $C_1 \perp C_2 \perp \ldots \perp C_n$, and $x_i \in C_i$, $i = 1,2,\ldots,n$, then

$$(\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2) \Rightarrow \{\|\sum_1^n \alpha_i x_i\|^2 = \sum_1^n \alpha_i^2 \|x_i\|^2 , \alpha_i \geqslant 0\}.$$

The whole of functional calculus is based on it. Needless to say that we have no evidence that it holds in reflexive Banach spaces.

These examples should suffice to show the need of further research. Maybe some of the sought properties are not valid in general. If so, we anticipate serious difficulties in bringing such facts to light, for the construction of counterexamples is a hard task in this field.

REFERENCES

[1] AUBIN,J.P., *Optimal approximation and characterization of the error and stability function in Banach spaces*, J.Approx. Th.3 (1970) 430-444.

[2] BROENDSTED,A. and ROCKAFELLAR,R.T., *On the subdifferentiability of convex functions*, Proc.Am.Math.Soc., 16(1966), 605-611.

[3] MOREAU,J.J., *Décomposition orthogonale dans un espace hilbertien selon deux cônes mutuellement polaires*, C.R.Acad.Sci.Paris, 255 (1962), 233-240.

[4] ROCKAFELLAR,R.T., *Local boundedness of monotone operators*, Mich. Math.J., 16(1969), 397-407.

[5] ROCKAFELLAR,R.T., *On the virtual convexity of the domain and range of nonlinear maximal monotone operators*, Math.Ann. 185 (1970), 81-90.

[6] ROCKAFELLAR,R.T., *Extension of Fenchel's duality theorem for convex functions*, Duke Math.J., 33(1966), 81-90.

[7] ZARANTONELLO,E.H., *Projections on convex sets and spectral theory*, in Contributions to Nonlinear Functional Analysis, Acad. Press, New York, 1971, pp. 237-424.

[8] ZARANTONELLO,E.H., *The product of commuting projections is a projection*, Proc.Am.Math.Soc., 38(1973), 591-593.

[9] ZARANTONELLO,E.H., *L'algèbre des projecteurs coniques*, in Analyse convexe et ses applications, pp. 232-243. Lecture notes in Economics and Mathematical Systems, No.102, Springer Verlag, 1974.

[10] ZARANTONELLO,E.H., *The meaning of the Cauchy-Schwarz-Buniakovsky inequality*, Proc.Am.Math.Soc., 58(1976), 133-138.

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ELEMENTARY GEOMETRY OF THE UNSYMMETRIC MINKOWSKI PLANE

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Dedicated to Luis A. Santaló

1.

Plane unsymmetric Minkowski geometry is given by a proper convex body Z in the affine plane and a point $O \in \operatorname{int} Z$. Z is called the *indicatrix* of the geometry; it defines a pseudonorm for any vector x: Write $x = OX$, then

$$\|x\| = \inf \lambda \mid X \subset \lambda Z. \qquad (1)$$

Then
$$\|x\| \geqslant 0, \qquad \|x\| = 0 \text{ only if } x = 0 \ ,$$
$$\|\alpha x\| = \alpha \|x\| \quad \text{if} \quad \alpha \geqslant 0 \ ,$$
$$\|x + x'\| \leqslant \|x\| + \|x'\| \ .$$

The pseudonorm is a norm, $\|\alpha x\| = |\alpha| \, \|x\|$ for all x and all $\alpha \in R$, if and only if Z is symmetric of center O, $Z = -Z$. (All operations of vector algebra will be taken for the origin at O). The elementary geometry of the symmetric Minkowski plane was studied in detail by C.M.Petty [9], we are interested in the unsymmetric case. Although we are going to use trigonometry and analytic geometry, no smoothness conditions will be imposed on Z.

We shall parametrize the convex curve ∂Z by t, two times its polar area function in the sense of polar coordinates, relative to a fixed polar axis. (For an arbitrary monotone and continuous parameter τ on ∂Z, t is a Stieltjes integral $t = \int_{\tau_0}^{\tau} \det(z(\sigma), dz(\sigma))$). Let $\hat{Y}$ be the dual of Z [7]. The curve $\partial\hat{Y}$ will be referred to s, two times the area function in the sense of polar coordinates computed from the polar axis. In a homothety of ratio c, the area of Z is multiplied by c^2 and that of $\hat{Y}$ by c^{-2}. Since (1) is affine and the dual is an affine covariant of Z, we may *normalize* the geometry by requiring

$$\text{Area } (Z) = \text{Area } (\hat{Y}) \ .$$

We shall assume from now on that we work with a normalized geometry (or, that we have defined the unit area so as to make the original bo-in general, where Π denotes the area common to Z and $\hat{Y}$ [5]. For symmetric geometries, $2\sqrt{2} \leqslant \Pi \leqslant \pi$.

To any vector z(t) from 0 to ∂Z there correspond all vectors y(s) from 0 to $\partial\hat{Y}$ for which $z(t).y(s) = 1$. Therefore, if Y is the image of $\hat{Y}$ in the rotation of angle $-\frac{\pi}{2}$ and center 0, the relation between the *isoperimetrix* Y [2] and the indicatrix is given by

$$\det(y(s),z(t)) = 1 \ . \tag{2}$$

Since Y also is a proper convex body, it defines a norm $\| x \|$. The relation (2) defines a map Φ of the circle $S = R/2\Pi$ onto arcs of S:

$$\Phi(t) = \{a(t) \leqslant s \leqslant b(t)\}$$

by: $-y[\varphi(t)]$ is the oriented direction of a support line of Z at z(t),

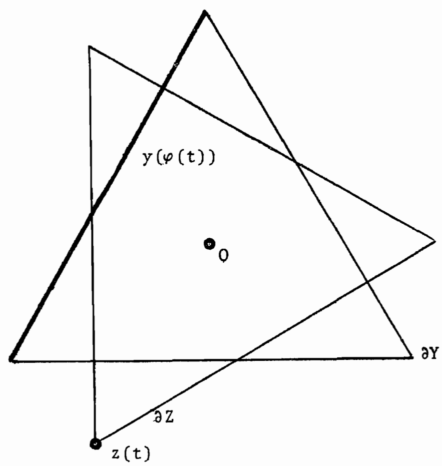

Fig. 1

where we denote by $\varphi(t)$ any $s \in \Phi(t)$. The map Φ satisfies

a) $a(t) = b(t)$ if Z has a unique line of support at z(t)

b) $\mathrm{int}\Phi(t_1) \cap \mathrm{in}\Phi(t_2) \neq \emptyset$ implies $\Phi(t_1) = \Phi(t_2)$

c) $\bigcup_{t\in S} \Phi(t) = S$.

Since the dual of the dual is the original convex body, (2) defines in the same way a map $\Psi(s) = \{A(s) \leqslant t \leqslant B(s)\}$, with properties a)-c) for s. Clearly, $s \in \Phi \circ \Psi(s)$, $t \in \Psi \circ \Phi(t)$. For the endpoints of the intervals we shall write $a(t) = \varphi_-(t)$, $b(t) = \varphi_+(t)$. Similarly, the interval $\Psi(s)$ is written $\Psi_-(s) \leqslant \Psi(s) \leqslant \Psi_+(s)$.

We shall pair vectors in a frame

$$\begin{bmatrix} y(s) \\ z(t) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -z(t) \\ y(s) \end{bmatrix}$$

only if $s = \varphi(t)$, $t = \Psi(s)$. Since the frames are unimodular, so is the matrix in

$$\begin{bmatrix} y(s) \\ z(t) \end{bmatrix} = \begin{bmatrix} cm_s(s,s_o) & st(t_o,s) \\ -sm(s_o,t) & cm_t(t_o,t) \end{bmatrix} \begin{bmatrix} y(s_o) \\ z(t_o) \end{bmatrix} \qquad (3)$$

whose elements are the *trigonometric functions* of the geometry. It follows that the "cosine" function cm is "even",

$$cm_s(s,s_o) = cm_t(t,t_o)$$

and the "sine" functions sm and st are odd:

$$sm(s_o,t) = -sm(s,t_o)$$

$$st(t_o,s) = -st(t,s_o) \ .$$

As a consequence, we drop the indices of the cm-functions since the arguments alone identify the functions. These functions have been studied in detail in [5] for smooth indicatrices; here we note only that

$$s,s_o \in \Phi(t) \text{ implies } cm(s,s_o) = 1 \ , \ sm(s_o,t) = sm(s,t) = 0$$

$$t,t_o \in \Psi(s) \text{ implies } cm(t_o,t) = 1 \ , \ st(t,s) = st(t_o,s) = 0 \ .$$

Let t^*,s^* be the values of t and s, respectively, for which $z(t^*)$ has the direction of $-z(t)$, $y(s^*)$ the direction of $-y(s)$. Then

$$sm(s_o,t) = 0 \quad \text{implies} \quad s_o = \varphi(t) \quad \text{or} \quad s_o = \varphi(t^*)$$

$$st(t_o,s) = 0 \quad \text{implies} \quad t_o = \Psi(s) \quad \text{or} \quad t_o = \Psi(s^*) \ .$$

2.

All triangles will be *oriented*. For a triangle ABC, the leg a is the vector $a = \overrightarrow{BC}$. All notations allow for cyclic permutations. We write $a = \|a\| Z(t_a) = \pmb{|}a\pmb{|} Y(s_a)$; this defines the angle variables. We have to distinguish several notions of orthogonality (really, transversality in the sense of the Calculus of Variations).

A vector v is *orthogonal to* a vector $x = \|x\| Z(t)$ if $v = \pmb{|}v\pmb{|} Y(s)$, $s = \varphi(t)$. The orthogonal direction is unique only if $\Phi(t)$ is a single point. A vector v is *orthogonal from* $x = \pmb{|}x\pmb{|} Y(s)$ if $v = \|v\| Z(t)$, $t = \Psi(s)$. An *altitude* h_a is a vector orthogonal *from* A *to* a. The height is $\pmb{|}h_a\pmb{|}$. The area Δ of the triangle is

$$\Delta = \frac{1}{2}\det(h_a,a) = \frac{1}{2}\|a\|\pmb{|}h_a\pmb{|}\det(Y(s_{h_a}),Z(t_a)) = \frac{1}{2}\|a\|\pmb{|}h_a\pmb{|} \ . \qquad (4)$$

This is the Minkowski form of the area formula; it allows for cyclic

$= \|c\|\,\|h_c\|$ holds only if Z and Y are homothetic, i.e., if they are Radon curves; this is a theorem of Tamássy [10].

From a+b+c = 0 we get the cosine theorem

$$\|a\| + \|b\|\,\mathrm{cm}(t_a,t_b) + \|c\|\,\mathrm{cm}(t_a,t_c) = 0 \tag{5}$$

for the components in the direction of $Z(t_a)$ and the sine theorem

$$\frac{\|b\|}{\mathrm{sm}(\varphi(t_c),t_a)} = \frac{\|c\|}{\mathrm{sm}(\varphi(t_a),t_b)} = \frac{\|a\|}{\mathrm{sm}(\varphi(t_b),t_c)} \tag{6}$$

for the transversal components. The sm-function was first defined by Busemann [2] who also found the sine theorem (for symmetric metric) from the area formula

$$\begin{aligned}\Delta = \frac{1}{2}\det(a,b) &= \frac{1}{2}\|a\|\,\|b\|\ \det(Z(t_a),Z(t_b)) = \\ &= \frac{1}{2}\|a\|\,\|b\|\ \mathrm{sm}(\varphi(t_a),t_b)\ .\end{aligned}$$

In the norm of Y, the formula $\Delta = \frac{1}{2}\pmb{|}a\pmb{|}\,\pmb{|}b\pmb{|}\ \mathrm{st}(\Psi(s_a),s_b)$ which yields

$$\frac{\pmb{|}a\pmb{|}}{\mathrm{st}(\Psi(s_b),s_c)} = \frac{\pmb{|}b\pmb{|}}{\mathrm{st}(\Psi(s_c),s_a)} = \frac{\pmb{|}c\pmb{|}}{\mathrm{st}(\Psi(s_a),s_b)} \tag{7}$$

We say that ABC is a *right triangle* if $b = h_a$. Then

$$\begin{aligned}-c &= \|a\|\,Z(t_a) + \pmb{|}b\pmb{|}\,Y(\varphi(t_a))\\ &= \|-c\|\,[\mathrm{cm}(t_a,t_c^*)Z(t_a) - \mathrm{sm}(\varphi(t_a),t_c^*)Y(\varphi(t_a))]\end{aligned}$$

or
$$\begin{aligned}\|a\| &= \|-c\|\,\mathrm{cm}(t_a,t_c^*) = \|c\|\,|\mathrm{cm}(t_c,t_c^*)|\,\mathrm{cm}(t_a,t_c^*)\\ \pmb{|}b\pmb{|} &= \|-c\|\,|\mathrm{sm}(\varphi(t_a),t_c^*)| = \|c\|\,|\mathrm{cm}(t_c,t_c^*)\ \mathrm{sm}(\varphi(s_a),t_c^*)|\ .\end{aligned}$$

Since the determinant of the matrix in (3) is 1,

$$\|-c\|^2 = \|-c\|\,\mathrm{cm}(t_a,t_c^*)\|-c\|\,\mathrm{cm}(t_c^*,t_a) + \|-c\|\,\mathrm{sm}(\varphi(t_a),t_c^*)\|-c\|\,\mathrm{st}(t_c^*,\varphi(t_a))\ ;$$

we obtain the generalization of the theorem of Pythagoras

$$\|-c\|^2 = \|a\|^2\,\frac{\mathrm{cm}(t_c^*,t_a)}{\mathrm{cm}(t_a,t_c^*)} + \pmb{|}b\pmb{|}^2\,\frac{\mathrm{st}(t_a,\varphi(t_c^*))}{\mathrm{sm}(\varphi(t_a),t_c^*)}\ . \tag{8}$$

This leads to a characterization of euclidean geometry. We start with

LEMMA 1. *If* $\mathrm{cm}(t_o,t) = \mathrm{cm}(t,t_o)$ *for all* t,t_o *then the geometry is euclidean.*

Denote the matrix in (3) by $M(t,t_o;s,s_o)$. From the composition formula $M(t_1,t;s_1,s)M(t,t_o;s,s_o) = M(t_1,t_o;s_1,s_o)$ and the fact that the diagonal elements in M are equal it follows that M is of the form

$$M = \begin{bmatrix} a & b \\ qb & a \end{bmatrix}\ .$$

A coordinate transformation of matrix $\mathrm{diag}(1,|q|^{-\frac{1}{2}})$ then transforms all M into the form $\begin{bmatrix} a & b \\ \pm b & a \end{bmatrix}$. Since Z is convex, Z is in the halfplane of the line of direction $y(s_o)$ through $z(t_o)$ that contains 0, therefore

$$cm(t_o,t) \leqslant 1 \quad \text{for all} \quad t_o,t . \tag{9}$$

(The inequality in absolute value holds only in symmetric geometries). For t sufficiently close to t_o, the diagonal elements in M are positive and < 1 in absolute value. Since $\det M = 1 = a^2 \mp b^2$ and $a^2 < 1$, it follows that q is negative and M is a rotation matrix. Since q is the same for all M, all M are rotation matrices and both Z and Y are the unit circle.

If $cm(t_c^*,t_a) = \lambda cm(t_a,t_c^*)$ for all directions a and c with λ independent of a and c, it follows by a change of names that $\lambda = 1$ and the geometry is euclidean by the lemma.

If $st(t_a,\varphi(t_c^*)) = \mu sm(\varphi(t_a),t_c^*)$ for all directions with constant μ it follows that the linear dependence of $Y(\varphi(t_o))$ and $Y(\varphi(t))$ implies the linear dependence of $Z(t_o)$ and $Z(t)$: $t_o = \Psi\circ\varphi(t_o)$, $t = \Psi\circ\varphi(t)$. Therefore φ (and Ψ) are point-valued functions, Z and Y are rotund ovals. Also, the map by oppositely oriented parallel lines of support is the map which commutes with φ (and Ψ). Hence, both Z and Y are symmetric ovals. From the composition formula of the matrices M we get the addition formulas

$$\begin{aligned} sm(s_o,t_2) &= sm(s_1,t_2)cm(s_1,s_o)+cm(t_1,t_2)sm(s_o,t_1) \\ st(t_o,s_2) &= cm(s_2,s_1)st(t_o,s_1)+st(t_1,s_2)cm(t_o,t_1). \end{aligned} \tag{10}$$

Under our hypothesis, we either have $\mu = 0$ which is impossible or simultaneously

$$st(t_o,s_2) = st(t_1,s_2)cm(t_1,t_o)+cm(t_1,t_2)st(t_o,s_1)$$

$$st(t_o,s_2) = st(t_1,s_2)cm(t_o,t_1)+cm(t_2,t_1)st(t_o,s_1)$$

i.e., $st(t_1,t_2)[cm(t_1,t_o)-cm(t_o,t_1)] = st(t_o,s_1)[cm(t_2,t_1)-cm(t_1,t_2)]$.

Since the directions t_o,t_1,t_2 are arbitrary, they can be chosen so that $st(t_o,s_1) = 0$, $st(t_1,s_2) \neq 0$. Therefore, the cm-function is symmetric and the geometry is euclidean by Lemma 1. We have proved:

PROPOSITION 1. *If there exists a constant* λ *or a constant* μ *such that*

either $\quad \|-c\|^2 = \lambda\|a\|^2 + f(a,b)\|b\|^2$

or $\quad \|-c\|^2 = g(a,b)\|a\|^2 + \mu\|b\|^2$

holds for all right triangles, then the geometry is euclidean.

3.

One of the definitions of an angle bisector in euclidean geometry is the set of centers of the circles that touch both legs of an angle. For the isoperimetric inequality, the Minkowski analog of a euclidean circle is the isoperimetrix [3,5] and its homothetic images. The center of the circle is the image of 0 in the homothety. The radius of the circle is the ratio of homothety, or the $\| \ \|$-norm of any vector from the center to the circumference.

DEFINITION. A Y-*bisector* of ABC is the set of centers of the circles that touch two of the lines that carry the legs of the triangle in one fixed angle domain. An *interior* bisector is a bisector that contains interior points of the triangle and a vertex as endpoint. A bisector that is not interior is exterior. All bisectors are continua that have a vertex as only relative boundary point.

Since all circles that touch two concurrent rays are homothetic images of one another in homotheties centered at the vertex, each bisector is a straight ray through that vertex. Two concurrent straight lines define four concurrent-bisectors, they form two straight lines only in symmetric geometry. By construction, the intersection of two Y-bisectors of a triangle is the center of a tritangent circle :

PROPOSITION 2. *The interior* Y-*bisectors of a triangle are concurrent. There are three triples consisting of two exterior and one interior* T-*bisector each.*

The existence of the points of intersection is an easy consequence of Pasch's axiom.

The point of concurrence of the three interior bisectors is the center I of the *incircle*, the homothetic image of Y tangent to the three sides of the triangle. The contact is oriented if both ABC and ∂Y are positively oriented. Let I_a be a point of contact of the incircle and a. Then a is orthogonal *to* II_a and the area of IBC is $\frac{1}{2} r \|a\|$. Therefore, for $p = \frac{1}{2} [\|a\| + \|b\| + \|c\|]$, we have

$$\Delta = pr \tag{11}$$

just as in euclidean geometry.

Another definition of the bisector of an angle is as axis of symmetry or as line making equal but opposite angles with the legs:

DEFINITION. An sm-*bisector* of a and b at C is a line of direction parameter t directed towards C for which $sm(\varphi(t_a),t) = sm(\varphi(t_b),t)$.

PROPOSITION 3. *The* sm-*bisectors are the* Y-*bisectors.*

If ∂Y is not strictly convex then the incircle and a will have a segment in common and for some $\varphi(t_a)$ and some $\varphi(t_b)$ the condition is satisfied. Let I_a be the point on $\partial Y \cap a$ for which $s(II_a) = \varphi(t_a)$. Then

$$\boldsymbol{|} II \boldsymbol{|}_a = \det(II_a, Z(t_a)) = \det(IC, Z(t_a)) = \|IC\| \det(Z(t), Z(t_a)) =$$
$$= -\|IC\| \, \mathrm{sm}(\varphi(t_a), t) \; .$$

Therefore, there exists I_b such that $II_a = II_b$ and I is the center of a circle of radius II_a which touches both legs (and this holds for every point on the bisector, not just the incenter).

A similar theory holds for Z - and st-bisectors.

4.

DEFINITION. The *perpendicular Z-bisector* of a segment AB is the set of all points P for which $\|PA\| = \|PB\|$. The *perpendicular Y-bisector* is defined by $\boldsymbol{|} PA \boldsymbol{|} = \boldsymbol{|} PB \boldsymbol{|}$. A Z-(Y-) *midpoint* of AB is a point of the intersection of AB and its Z-(Y-) perpendicular bisector.

The Z-bisector of a segment may have nonzero twodimensional measure. For example, in the normalized geometry for which Z is the square of side length $2^{3/4}$ parallel the axes with center at O, let AB be a segment parallel the y-axis of length $2a < 2^{3/4}$. From the endpoints A,B we draw the lines parallel the diagonals of Z and get the diamond ACBD. Then for any P in one of the exterior vertical angles at C and D (the shaded domains in fig.2), $\|PA\| = \|PB\|$. All bisectors are zerodimensional if Z and Y are rotund.

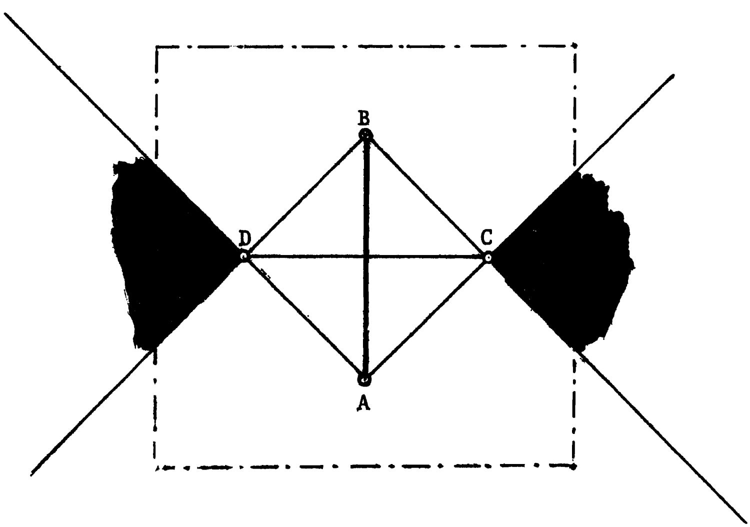

Fig.2

PROPOSITION 4. *The* Z-(*or* Y-) *midpoint of any segment is unique.*

For $P \in AB$, the ratio of division $\lambda = \|PA\| / \|PB\|$ is strictly monotone increasing and continuous, by the first and third properties of the pseudonorm. It increases from 0 to ∞, therefore it is = 1 at exactly one point.

In unsymmetric geometry there is no direct connection between the bisector sets in the two halfplanes defined by the line AB. For example, in the geometry defined by a triangle Z and a point $O \in \text{int } Z$, the Z-bisectors p, p' of the segment PB, $P = BO \cap b$, are the rays complementary to the segments OC,OA. For any other segment P_1Q_1, $P_1 \in b$, $Q_1 \in a$, $O \in P_1Q_1$, the bisector p_1 in the halfplane opposite C is p but the bisector p_1' in the halfplane of C is the union of a segment OS on OC and a ray parallel p'.

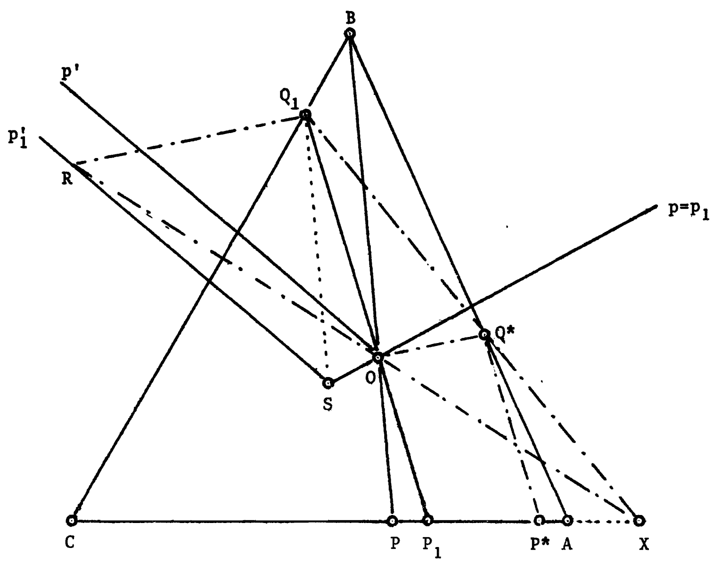

Fig. 3

A point R on that ray can be found as follows: Since $\|RQ_1\| = \|RP_1\|$, RQ_1P_1 is homothetic to a triangle OQ^*P^*; there is a one-to-one correspondence between $Q^* \in BA$ and R. The center X of the homothety is $b \cap Q_1Q^*$; R is the intersection of XO and the line through Q_1 parallel Q^*O. The locus of R is a conic as intersection of two projectively related pencils of lines (through O and Q_1). The line OQ_1 corresponds to itself in the projectivity. Hence, the projectivity is a perspectivity and the conic is a double line. Since $Q^* \to A$ implies $X \to A$, AO is an asymptote, i.e. $SR \parallel AO$. S is found by $Q_1S \parallel BO$.

Busemann [3] has shown that in any symmetric G-space the perpendicular bisectors are flat only if the geometry is Klein. The bisector of a symmetric Minkowski geometry are flat only if the geometry is euclidean.

The theorem can be extended to unsymmetric Minkowski geometry using an argument of Blaschke.

PROPOSITION 5. *A geometry in which all* Z-(*or* Y-) *perpendicular bisectors are straight lines is euclidean. The same conclusion holds if every* Z- (*or* Y-) *perpendicular bisector is the union of two straight rays and if, in addition, the* Z- (Y-) *perpendicular bisectors of two segments* AB, CD *intersecting at their common midpoint have only that midpoint in common.*

Let M be the midpoint of AB. (We use only Z, the argument for Y is identical). We prove first that the bisectors are straight lines if for any other segment CD with midpoint M the bisectors of AB and CD have only M in common. We may assume without loss of generality that $M=O$, $\|OA\| = \|OB\| = 1$. Let r_1, r_2 be the two rays that form the bisector. For $P \in r_1$, let OA'B' be the homothetic image of PAB in the map that brings P onto O and $A', B' \in \partial Z$. Clearly, $A'B' \parallel AB$. $\|PO\| \longrightarrow \infty$ implies $\|A'B'\| \longrightarrow 0$. Therefore, the line $\bar{r}_1$ defined by r_1 intersects ∂Z at a point P" where a support line is parallel AB. The other point on ∂Z with support line parallel AB is $\bar{r}_2 \cap \partial Z$ by the same argument. By hypothesis, the couples of points of parallel support are in 1-1 order preserving correspondence with the directions through O: no point of ∂Z has more than one support line and no line more than one support point; ∂Z is rotund (strictly convex and smooth). For AB fixed, $A' \longrightarrow B'$ defines an affine relation of axis r_1 in the terminology of Veblen and Young [11] in one halfplane of AB. If $\bar{r}_2 \neq \bar{r}_1$ then at least one of A or B would admit two distinct support lines, since the support line at A cannot be the image of the support line at B in two elations with different axes. Hence, r_1 and r_2 are collinear.

Now let σ_{AB} be the affine extension of the map $A' \longrightarrow B'$. The σ_{AB} generate a group of affine maps that admit O as fixed point and Z as invariant convex body. Therefore, the group is linear and bounded, it is conjugate to an orthogonal group ([7], prop.14-10), Z is an ellipse and the geometry is euclidean.

The Z-midpoint M_a of a = BC is defined by $\|M_aB\| = \|M_aC\|$. Since $\|M_aB\| Z(t_a^*) = \|M_aC\| cm(t_a, t_a^*) Z(t_a)$, we have

$$\frac{|M_aB|}{|M_aC|} = |cm(t_a, t_a^*)| \quad . \tag{12}$$

The Z-*medians* of ABC are the lines AM_a, BM_b, CM_c. Then we have from Ceva's theorem [6]: The Z-medians of a triangle are concurrent if and only if

$$cm(t_a, t_a^*) cm(t_b, t_b^*) cm(t_c, t_c^*) = -1 \quad . \tag{13}$$

The Z-medians are concurrent for all triangles if (13) holds for all triples of directions. For a degenerate triangle we have, for example, $t_b \longrightarrow t_a$, $t_c \longrightarrow t_a^*$ and therefore $cm(t_a^*, t_a) = -1$ for all directions t_a. That means for $s^* \in \Phi(t^*)$, $s \in \Phi(t)$ that

$$\begin{bmatrix} Y(s^*) \\ Z(t^*) \end{bmatrix} = \begin{bmatrix} -1 & st(t,s^*) \\ -sm(s,t^*) & -1 \end{bmatrix} \begin{bmatrix} Y(s) \\ Z(t) \end{bmatrix} .$$

Since the determinant is 1, either $st(t,s^*) = 0$ or $sm(s,t^*) = 0$. It follows from the convexity of Z that st and sm are continuous functions of s and t. Therefore, $0 \leqslant s,t \leqslant 2\Pi$ is the union of a countable set of intervals on which either $Z(t^*) = -Z(t)$ or $Y(s^*) = -Y(s)$. The construction of the dual can be given a local version: If ∂Z is $r = r(\theta)$ in euclidean polar coordinates and $\underline{n}(\theta) = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}$ then the local dual is the envelope of the lines $\underline{n}(\theta).\underline{y} = 1/r$, [7]. Therefore, the local symmetry of ∂Z implies that of ∂Y, $sm(s,t^*) = st(t,s^*) = 0$:

PROPOSITION 6. *The Z-medians of a geometry are concurrent for all triangles if and only if the geometry is symmetric. In that case, the Z-medians are the affine medians and the Y-medians.*

5.

In euclidean geometry, the altitudes are concurrent at the orthocenter. A definition of the orthocenter derived from the euclidean theory of circles was studied by Asplund and Grünbaum [1], their results are valid for unsymmetric metrics and lead to a characterization of the geometries defined by strictly convex, symmetric ovals. Golab and Tamássy [4] proved that the altitudes are concurrent in Radon geometries. The only symmetric Radon curve is the circle, this is a characterization of euclidean geometry.

A triangle is *isosceles* in the Z-norm if $\|a\| = \|b\|$, it is *equilateral* if $\|a\| = \|b\| = \|c\|$. By the sine theorem, a triangle is isosceles if and only if $sm(\varphi(t_c), t_a) = sm(\varphi(t_b), t_c)$. It is not obvious that equilateral triangles exist for all directions of the legs. Without loss of generality, we assume $\|a\| = \|b\| = \|c\| = 1$. For $a = OA$, an equilateral exists if $A \in 2Z^* = -2Z$. For a symmetric metric, $Z = -Z$ and the condition is always satisfied:

PROPOSITION 7. *In symmetric metric, equilateral triangles exist for every direction of the leg* a.

The proposition does not hold for all unsymmetric metrics. Since $Z \cap -Z \neq \emptyset$, we can only say that equilateral triangles exist for a set of directions with positive linear measure. An example is the geometry given by Z the triangle $(0,1)$, $(\pm 1,-3)$, 0 at the origin. The admissible directions OA are those for which the y-coordinate of A is $\geqslant -\frac{1}{2}$.

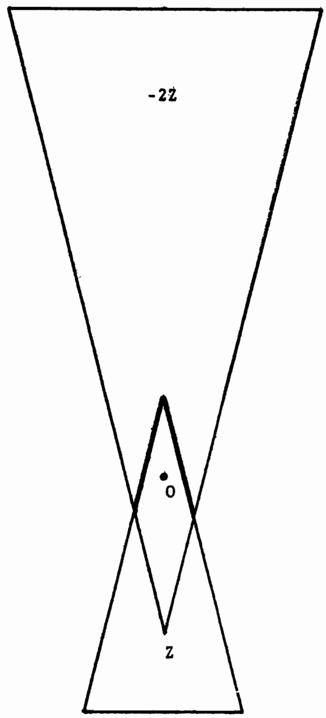

Fig.4

The theory of equilateral triangles can be expected to be simple only for symmetric metrics. A few sample theorems:

PROPOSITION 8. *In symmetric metric, an exterior* Y*-bisector of the equal legs of a* Z*-isosceles triangle is parallel to the basis.*

$\|a\| = \|-b\|$ implies $sm(\varphi(t_a),t_c) = sm(\varphi(t_b^*),t_c)$; the direction of the bisector is that of c. By a similar argument, we get:

PROPOSITION 9. *In symmetric metric, an interior* Z*-bisector of the equal legs of a* Y*-isosceles triangle is the altitude* <u>*from*</u> *base to vertex.*

The interior Z-bisector is the st-bisector and satisfies $st(\Psi(s_a),s)=$ $= st(\Psi(s_b^*),s)$. For an isosceles triangle,

$$st(\Psi(s_b),s_c) = st(\Psi(s_c),s_a) = -st(\Psi(s_a),s_c) = -st(\Psi(s_b^*),s_c) .$$

Hence, $s_c = s$ and $\Psi(s_c)$ is the direction of the normal *from* the basis.

In this way, many theorems of elementary geometry become valid in an appropriate interplay of the two norms; for theorems of differential and integral geometry see [8].

REFERENCES

[1] E.Asplund and B.Grünbaum, *On the geometry of Minkowski planes.* Ens. Math. 6(1960)299-306.

[2] H.Busemann, *The foundations of Minkowskian geometry* , Comment. Math.Helv. 24(1950)156-187.

[3] H.Busemann, *The Geometry of Geodesics*, Academic Press, New York 1955.

[4] S.Golab and L.Tamâssy, *Eine Kennzeichnung der euklidischen Ebene unter den Minkowskischen Ebenen*, Publ. Math. Debrecen 7(1960) 187-193.

[5] H.Guggenheimer, *Pseudo-Minkowski differential geometry*, Ann. di mat. pur. appl. (4)70(1965)305-370.

[6] H.Guggenheimer, *Plane Geometry and its Groups*, Holden-Day, San Francisco 1967.

[7] H.Guggenheimer, *Applicable geometry*, Krieger, Malabar Fla.1977.

[8] H.Guggenheimer, *On plane Minkowski geometry*, Geom. Dedic. 12 (1982)371-381.

[9] C.M.Petty, *On the geometry of the Minkowski plane*, Riv.Mat.Univ. Parma 6(1955)269-292.

[10] L.Tamâssy, *Ein Problem der zweidimensionalen Minkowskischen Geometrie*, Ann. Polon. Math. 9(1960)39-48.

[11] O.Veblen and J.W.Young, *Projective Geometry*, Reprint Blaisdell, New York.

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THE NUMBER OF DIAMETERS THROUGH A POINT INSIDE AN OVAL

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Dedicated with greatest admiration and respect to Professor L. A. Santaló

1. INTRODUCTION.

In [6], Professor Santaló raised the question of determining bounds on the expected number of normals that can be drawn from a random point inside a convex body to its boundary. If the body has constant width this is equivalent to determining bounds on the expected number of diameters passing through a random point inside the body, since in this case the expected number of normals is just twice the expected number of diameters.

Let K be a plane convex body. Then a *diameter* is a chord of K whose endpoints lie on parallel supporting lines of K. For each $(x,y) \in K$, let $n(x,y)$ be the number of diameters of K passing through (x,y) (note that $n(x,y)$ might take the value $+\infty$). We are interested in the functional $I(K)$ given by

$$I(K) = \iint_K n(x,y)\, dx\, dy .$$

If we denote by $n(K)$ the expected number of diameters passing through a random point of K, then we have

$$n(K) = I(K)/A(K) ,$$

where $A(K)$ is the area of K.

Let $DK = K + (-K)$ be the *difference body* of K. In case the boundary of K is sufficiently regular, we shall prove that

$$\frac{1}{4} A(DK) \leqslant I(K) \leqslant \frac{1}{2} A(DK) . \tag{1.1}$$

For any plane convex body K, the difference body satisfies the inequalities (see Bonnesen and Fenchel [1])

$$(1.2) \qquad 4A(K) \leqslant A(DK) \leqslant 6A(K) .$$

Combining this with (1.1) gives

$$A(K) \leqslant I(K) \leqslant 3A(K) .$$

As a consequence we have

$$(1.3) \qquad 1 \leqslant n(K) \leqslant 3 .$$

The lower bound is not surprising, since a theorem of Hammer [3] guarantees that $n(x,y) \geqslant 1$ for all $(x,y) \in K$.

In Section 3 we shall prove (1.1), which leads to (1.3). We shall also show that the given bounds are sharp, in that $n(K) = 1$ iff K is centrally symmetric, and there exist K satisfying the regularity conditions we shall impose for which n(K) is as close to 3 as we please.

Our proofs will depend on transforming I(K) to an integral involving the length of a variable diameter and the instantaneous radius of rotation of that diameter. Indeed, let $D(\theta)$ be the length of a diameter making angle θ with the horizontal and $\rho(\theta)$ the distance from the instantaneous center of rotation to one endpoint. Then we shall show in Section 2 that

$$(1.4) \qquad I(K) = \frac{1}{2}\int_0^{2\pi}[\rho^2(\theta)-\rho(\theta)D(\theta) + \frac{1}{2}D^2(\theta)]\,d\theta .$$

It will follow from this that

$$(1.5) \qquad I(K) = \frac{1}{2}\int_0^{2\pi} \rho^2(\theta)\,d\theta .$$

The latter expression is geometrically plausible when we think of K as covered by the infinitesimal sectors of area swept out by diameters rotating through an angle $d\theta$ about their instantaneous centers of rotation (see Fig. 2).

Let $R(\varphi)$ be the radius of curvature at a boundary point of K where the supporting line makes angle φ with the horizontal, and let $w(\varphi)$ be the *width* of K in direction φ, that is, the distance between the parallel supporting lines making angle φ with the horizontal. In section 4 we shall derive from (1.5) the expression

$$(1.6) \qquad I(K) = \frac{1}{2}\int_0^{2\pi} \frac{R^2(\varphi)w(\varphi)}{R(\varphi) + R(\varphi+\pi)}\,d\varphi .$$

In case K has constant width $w(\varphi) \equiv b$ we have in addition $R(\varphi) + R(\varphi+\pi) \equiv b$, so (1.6) gives

$$(1.7) \qquad I(K) = \frac{1}{2}\int_0^{2\pi} R^2(\varphi)\,d\varphi .$$

This latter expression also follows from (1.5), since for sets of constant width we have $\rho(\theta) = R(\varphi)$ and $d\theta = d\varphi$ (where θ and φ are as in Fig. 1).

Since K has constant width b iff DK is a circular disk of radius b, we obtain from (1.1)

$$\frac{\pi}{4} b^2 \leqslant I(K) \leqslant \frac{\pi}{2} b^2 . \tag{1.8}$$

The area of a plane set K of constant width b satisfies

$$\frac{\pi-\sqrt{3}}{2} b^2 \leqslant A(K) \leqslant \frac{\pi}{4} b^2 , \tag{1.9}$$

with equality on the lefthand side for a Reuleaux triangle and on the righthand side for a circular disk. Using this in (1.8) yields

$$1 \leqslant n(K) \leqslant \frac{\pi}{\pi-\sqrt{3}} , \tag{1.10}$$

for plane sets of constant width. The upper bound corresponds to that given in [6] for the expected number of normals that can be drawn to the boundary from a random point inside a set of constant width. The lower bound is achieved precisely when K is a circular disk, and the upper bound when K is a Reuleaux triangle. Our methods give (1.10) only for sets of constant width satisfying our regularity assumptions, and among such K there are those (approximating Reuleaux triangles) for which n(K) is arbitrarily close to the upper bound in (1.10).

Section 5 contains a discussion of how (1.6) may be viewed as the analogue of (1.7) for a plane convex set K of constant relative width 1 in the relative geometry whose unit disk is DK.

We introduce in Section 2 the background necessary for our development and proceed to the proofs of the formulas (1.4) and (1.5).

2. PROOFS OF (1.4) AND (1.5).

We shall restrict our considerations to plane convex bodies having a certain degree of regularity. In the following, K will be a plane convex body whose boundary, to be denoted C, is a convex curve of class C^3 with nowhere vanishing curvature. We shall refer to such a K as an *oval*. In this case C admits the parametric representation

$$x = x(\varphi) \quad , \quad y = y(\varphi) \quad , \quad 0 \leqslant \varphi \leqslant 2\pi , \tag{2.1}$$

where φ is the angle the tangent line at $P(\varphi) = (x(\varphi), y(\varphi))$ makes with the x-axis (Fig.1).

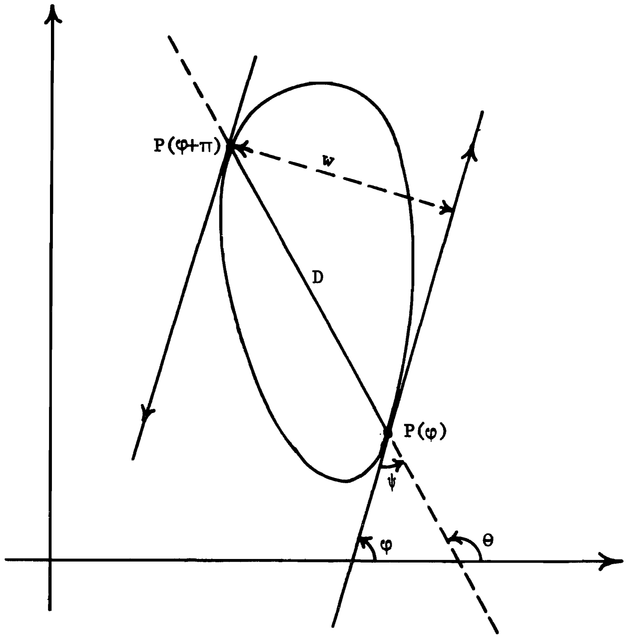

Figure 1

The chord $\overline{P(\varphi)P(\varphi+\pi)}$ is a diameter of K making angle $\theta = \theta(\varphi)$ with the x-axis (as indicated in Fig. 1). Since K is an oval, it is easy to see that θ is a strictly monotonic function of φ, so it is also in fact possible to express $\varphi = \varphi(\theta)$ as a smooth function of θ.

Let $\bar{D}(\varphi)$ denote the length of the diameter $\overline{P(\varphi)P(\varphi+\pi)}$. Then any point (x,y) on this diameter has coordinates of the form

$$\begin{aligned} x &= x(\varphi) + \lambda \cos\theta(\varphi) \\ y &= y(\varphi) + \lambda \sin\theta(\varphi) \end{aligned} \qquad 0 \leqslant \lambda \leqslant \bar{D}(\varphi). \tag{2.2}$$

If S is the region in the (φ,λ)-plane defined by $S = \{(\varphi,\lambda): 0 \leqslant \lambda \leqslant \bar{D}(\varphi)\ ,\ 0 \leqslant \varphi \leqslant 2\pi\}$, then the equations (2.2) define a smooth mapping of S into K. The theorem of Hammer [3] mentioned in the introduction tells us that in fact this mapping sends S *onto* K. Since (φ,λ) and $(\varphi+\pi,\bar{D}(\varphi)-\lambda)$ always have the same image under this mapping, we see that each $(x,y) \in K$ is the image of $2n(x,y)$ points of S, where $n(x,y)$ is the number of diameters through (x,y). Thus, if $J = J(\varphi,\lambda)$ is the Jacobian determinant of the mapping, we have (see Federer [2, p. 243])

$$2I(K) = 2\iint_K n(x,y)\,dx\,dy = \iint_S |J(\varphi,\lambda)|\,d\varphi\, d\lambda\ . \tag{2.3}$$

Direct calculation from (2.2) gives

$$(2.4) \qquad J(\varphi,\lambda) = x'(\varphi)\sin\theta - y'(\varphi)\cos\theta - \lambda\theta' ,$$

where $\theta = \theta(\varphi)$, and the prime represents differentiation with respect to φ. But

$$(2.5) \qquad x'(\varphi) = R(\varphi)\cos\varphi \quad , \quad y'(\varphi) = R(\varphi)\sin\varphi \quad , \quad 0 \leqslant \varphi \leqslant 2\pi ,$$

where $R(\varphi)$ is the radius of curvature of C at $P(\varphi)$. Denoting by $\psi = \psi(\varphi)$ the angle between the tangent line ard the diameter, as in Fig. 1, we obtain by substitution of (2.5) into (2.4),

$$(2.6) \qquad J = R\sin(\theta-\varphi) - \lambda\theta' = R\sin\psi - \lambda\theta' .$$

Let $\bar\rho(\varphi)$ be the instantaneous radius of rotation of the diameter $\overline{P(\varphi)P(\varphi+\pi)}$, that is, the distance from the instantaneous center of rotation to the point $P(\varphi)$. Let ds be the element of arclength of C at $P(\varphi)$. Then we have (see Fig. 2)

$$(2.7) \qquad \bar\rho(\varphi)d\theta = \sin\psi\, ds = R(\varphi)\sin\psi\, d\varphi .$$

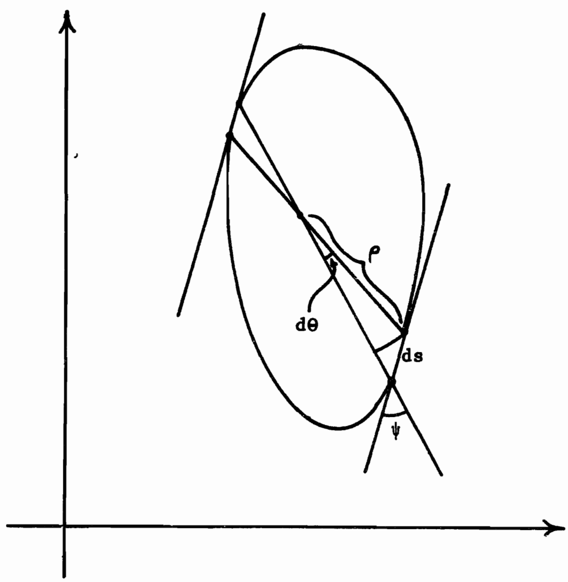

Figure 2

These relations can be derived from the results given in Hammer and Smith [4]. We have from (2.7) that $R(\varphi)\sin\psi = \bar\rho(\varphi)\theta'$. Substitution of this into (2.6) gives

$$(2.8) \qquad J(\varphi,\lambda) = (\bar\rho(\varphi)-\lambda)\theta'(\varphi) .$$

Iteration of the rightmost integral in (2.3) then gives

$$I(K) = \frac{1}{2}\int_0^{2\pi}\{\int_0^{\bar{D}(\varphi)} |\bar{\rho}(\varphi)-\lambda|d\lambda\}\theta'(\varphi)d\varphi \quad . \tag{2.9}$$

We let $\rho(\theta) = \bar{\rho}(\varphi(\theta))$ and $D(\theta) = \bar{D}(\varphi(\theta))$. Changing variables from φ to θ in (2.9) leads to

$$I(K) = \frac{1}{2}\int_0^{2\pi}\{\int_0^{D(\theta)} |\rho(\theta)-\lambda|d\lambda\}d\theta \quad . \tag{2.10}$$

Since any two diameters of an oval K intersect inside K, the centers of rotation all belong to K. Consequently $\rho(\theta) \leqslant D(\theta)$, and the inner integral in (2.10) takes the form

$$\int_0^{D(\theta)} |\rho(\theta)-\lambda|d\lambda = \int_0^{\rho(\theta)} (\rho(\theta)-\lambda)d\lambda + \int_{\rho(\theta)}^{D(\theta)} (\lambda-\rho(\theta))d\lambda \quad . \tag{2.11}$$

Evaluation of these integrals then gives, with (2.10), the required formula (1.4).

To obtain (1.5), we rewrite (1.4) in the form

$$I(K) = \frac{1}{4}\int_0^{2\pi} [\rho^2(\theta) + (D(\theta)-\rho(\theta))^2]d\theta \quad . \tag{2.12}$$

Since $\rho(\theta) + \rho(\theta+\pi) = D(\theta)$, this becomes

$$I(K) = \frac{1}{4}\int_0^{2\pi} [\rho^2(\theta) + \rho^2(\theta+\pi)]d\theta \quad , \tag{2.13}$$

from which (1.5) follows by the periodicity of ρ.

3. THE BOUNDS ON I(K).

Write equation (1.4) in the form

$$I(K) = \frac{1}{4}\int_0^{2\pi} D^2(\theta)d\theta - \frac{1}{2}\int_0^{2\pi} \rho(\theta)(D(\theta)-\rho(\theta))d\theta \quad . \tag{3.1}$$

Applying to (3.1) the fact that $0 \leqslant \rho(D-\rho) \leqslant D^2/4$, we obtain

$$\frac{1}{8}\int_0^{2\pi} D^2(\theta)d\theta \leqslant I(K) \leqslant \frac{1}{4}\int_0^{2\pi} D^2(\theta)d\theta \quad . \tag{3.2}$$

The boundary of the difference body DK has the polar coordinate representation $r = D(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, so

$$A(DK) = \frac{1}{2}\int_0^{2\pi} D^2(\theta)d\theta \quad . \tag{3.3}$$

The required bounds in (1.1) now follow from (3.2) and (3.3).

Equality holds on the lefthand side of (3.2), and so of (1.1), iff

$\rho(\theta)(D(\theta)-\rho(\theta)) \equiv D^2(\theta)/4$, which happens precisely when $\rho(\theta) \equiv D(\theta)/2$. In this case each diameter of K is an area bisector, and it follows that K is centrally symmetric (see Hammer and Smith [4]). As a further consequence, since A(DK) = 4A(K) iff K is centrally symmetric, we see that n(K) = 1 iff K is centrally symmetric.

The theorems of Hammer and Sobczyk [5] imply that when K is not centrally symmetric there exist three diameters surrounding a triangle Δ such that $n(x,y) \geqslant 3$ for $(x,y) \in \Delta$. In this case, since $n(x,y) \geqslant 1$ for all $(x,y) \in K$, one must have that $n(K) > 1$. This shows in another way that n(K) = 1 only if K is centrally symmetric.

Equality can hold on the righthand side of (3.2) and (1.1) iff $\rho(\theta)(D(\theta)-\rho(\theta)) \equiv 0$. This is not possible for our class of ovals; however we can find ovals K for which I(K) is arbitrary close to A(DK)/2. For example, appropriate approximations of triangles will have this property, and we can find such K with n(K) as close to 3 as we please. In that sense the bounds in (1.3) are sharp.

4. PROOF OF (1.6).

If $w(\varphi)$ is the width of K, then we have $w(\varphi) = D(\theta)\sin\psi$ (see Fig.1). Thus from (2.7) we obtain

$$\rho(\theta)d\theta = \sin\psi\, ds = \frac{w(\varphi)}{D(\theta)} R(\varphi)d\varphi \, . \tag{4.1}$$

Since $w(\varphi+\pi) = w(\varphi)$ and $D(\theta+\pi) = D(\theta)$, we also have

$$\rho(\theta+\pi)d\theta = \frac{w(\varphi)}{D(\theta)} R(\varphi+\pi)d\varphi \, . \tag{4.2}$$

Comparison of (4.1) and (4.2) yields

$$\frac{\rho(\theta+\pi)}{\rho(\theta)} = \frac{R(\varphi+\pi)}{R(\varphi)} \, ,$$

from which it follows that

$$\frac{D(\theta)}{\rho(\theta)} = \frac{\rho(\theta)+\rho(\theta+\pi)}{\rho(\theta)} = \frac{R(\varphi)+R(\varphi+\pi)}{R(\varphi)} \, .$$

Thus we have

$$\rho(\theta) = \frac{D(\theta)R(\varphi)}{R(\varphi)+R(\varphi+\pi)} \, . \tag{4.3}$$

Then (4.1) and (4.3) yield

$$\rho^2(\theta)d\theta = \rho(\theta)\rho(\theta)d\theta = \frac{R^2(\varphi)w(\varphi)}{R(\varphi)+R(\varphi+\pi)}\, d\varphi \quad , \tag{4.4}$$

5. INTERPRETATION OF (1.6) IN RELATIVE GEOMETRY.

In relative differential geometry in the plane (see, for example, Bonnesen and Fenchel [1]), one replaces the ordinary Euclidean unit disk by an arbitrary centrally symmetric convex body E centered at the origin. The *relative width* of a convex set K is the Euclidean width divided by half the width of E in the same direction. Then K has *constant relative width* b iff $DK = K + (-K) = bE$.

Given an oval K, we take $E = DK$ as our unit disk for a relative geometry. Then K has contant relative width 1, relative to E. Let $ds(\varphi)$ be the Euclidean element of arclength of K at $P(\varphi)$, and $dS(\varphi)$ the Euclidean element of arclength of E at the boundary point with outward normal parallel to the outward normal of K at $P(\varphi)$. The *relative radius of curvature* of K at $P(\varphi)$, denoted by $\tilde{R}(\varphi)$, is

$$\tilde{R}(\varphi) = \frac{ds(\varphi)}{dS(\varphi)} .$$

But we have $ds(\varphi) = R(\varphi)d\varphi$ and, since $E = DK$, $dS(\varphi) = (R(\varphi)+R(\varphi+\pi))d\varphi$. Hence

$$\tilde{R}(\varphi) = \frac{R(\varphi)}{R(\varphi)+R(\varphi+\pi)} . \tag{5.1}$$

The *relative arclength element* of E, at a boundary point where the supporting line makes angle φ with the horizontal, is $d\tilde{S}(\varphi) = h(E,\varphi)dS(\varphi)$, where $h(E,\varphi)$ is the supporting function of E. Since $E = DK$ we have $h(E,\varphi) = w(\varphi)$ = the width of K. This gives

$$d\tilde{S}(\varphi) = w(\varphi)dS(\varphi) = w(\varphi)(R(\varphi)+R(\varphi+\pi))d\varphi . \tag{5.2}$$

From (5.1) and (5.2) we obtain then for (1.6) the form,

$$I(K) = \frac{1}{2} \int \tilde{R}^2 \, d\tilde{S} , \tag{5.3}$$

where the integration is over the boundary of $E = DK$ with respect to the relative arclength induced by E. Thus (1.6) may be viewed as the generalization of (1.7) to sets of constant relative width.

REFERENCES

[1] T.Bonnesen and W.Fenchel, *Theorie der konvexen Körper*, Sprin-Verlag, Berlin 1934.

[2] H.Federer, *Geometric measure theory*, Springer-Verlag, Berlin-Heidelberg-New York 1969.

[3] P.C.Hammer, *Convex bodies associated with a convex body*, Proc. Amer. Math. Soc. 2(1951), 781-793.

[4] P.C.Hammer and T.J.Smith, *Conditions equivalent to central symmetry of convex curves*, Proc. Cambridge Philos. Soc. 60(1964), 779-785.

[5] P.C.Hammer and A.Sobczyk, *Planar line families II*, Proc. Amer. Math. Soc. 4(1953), 341-349.

[6] L.A.Santaló, *Note on convex spherical curves*, Bull. Amer. Math. Soc. 50(1944), 528-534.

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SOME NEW CHARACTERIZATIONS OF VERONESE SURFACE AND STANDARD FLAT TORI

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Dedicated to Professor Luis A. Santaló

1. INTRODUCTION.

Let M be a (connected) surface in a Euclidean m-space E^m. For any point p in M and any unit vector t at p tangent to M, the vector t and and the normal space $T_p^{\perp}M$ of M at p determine an (m-1)-dimensional vector subspace E(p,t) of E^m through p. The intersection of M and E(p,t) gives rise a curve γ in a neighborhood of p which is called the *normal section* of M at p in the direction t. The surface M is said to have *planar normal sections* if normal sections of M are planar curves. In this case, for any normal section γ, we have $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ identically. A surface M is said to have *pointwise planar normal sections* if, for each point p in M, normal sections at p satisfy $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ at p (i.e., normal sections at p have "zero torsion" at p). It is clear that if a surface M lies in a linear 3-subspace E^3 of E^m, then M has planar normal sections and has pointwise planar normal sections.

We shall now define the Veronese surface. Let (x,y,z) be the natural coordinate system in E^3 and (u^1,u^2,u^3,u^4,u^5) the natural coordinate system in E^5. We consider the mapping defined by

$$u^1 = \frac{1}{\sqrt{3}}\,yz \quad , \quad u^2 = \frac{1}{\sqrt{3}}\,zx \quad , \quad u^3 = \frac{1}{\sqrt{3}}\,xy \quad ,$$

$$u^4 = \frac{1}{2\sqrt{3}}\,(x^2 - y^2) \quad , \quad u^5 = \frac{1}{6}\,(x^2 + y^2 - 2z^2).$$

This defines an isometric immersion of $S^2(\sqrt{3})$ into the unit hypersphere $S^4(1)$ of E^5. Two points (x,y,z) and (-x,-y,-z) of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane imbedded in E^5 is called the *Veronese surface* (see, for instance, [4].)

In [2], we have proved the following.

THEOREM A. *Let* M *be a surface in* E^m. *If* M *has pointwise planar normal sections, then, locally,* M *lies in a linear* 5*-subspace* E^5 *of* E^m.

The classification of surfaces in E^m with planar normal sections was obtained in [3].

THEOREM B. *Let* M *be a surface in* E^m. *If* M *has planar normal sections, then, either, locally,* M *lies in a linear* 3*-subspace* E^3 *or, up to similarity transformations of* E^m, M *is an open portion of the Veronese surface in a* E^5.

In view of Theorems A and B, it is an interesting problem to classify surfaces in E^5 with pointwise planar normal sections. As we already mentioned, every surface in E^3 has pointwise planar normal sections. A surface M in E^m is said to *lie essentially in* E^m if, locally, M does not lie in any hyperplane E^{m-1} of E^m. According to Theorem A, the classification problem of surfaces in E^m with pointwise planar normal sections remains open only for surfaces which lie essentially either in E^5 or in E^4.

In this paper, we will solve this problem completely for surfaces which lie essentially in E^5. Furthermore, we will obtain three classification theorems for surfaces in E^4. As biproducts some new geometric characterizations of the Veronese surface and standard flat tori are then obtained.

2. PRELIMINARIES.

Let M be a surface in E^m. We choose a local field of orthonormal frame $\{e_1,\dots,e_m\}$ in E^m such that, restricted to M, the vectors e_1, e_2 are tangent to M and $e_3,\dots,e_m$ are normal to M. We denote by $\{\omega^1,\dots,\omega^m\}$ the field of dual frames. The structure equations of E^5 are given by

$$(2.1) \qquad d\omega^A = -\sum \omega^A_B \wedge \omega^B \quad , \quad \omega^A_B + \omega^B_A = 0 \ ,$$

$$(2.2) \qquad d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B \ ,$$

$$A,B,C,\dots = 1,2,\dots,m.$$

Restricting these forms on M, we have $\omega^r = 0$, $r,s,t,\dots = 3,\dots,m$. Since

Cartan's Lemma implies

$$(2.4)\qquad \omega_i^r = \sum h_{ij}^r\,\omega^j \quad , \qquad h_{ij}^r = h_{ji}^r \;.$$

From these formulas we obtain

$$(2.5)\qquad d\omega^i = -\sum \omega_j^i \wedge \omega^j \;,$$

$$(2.6)\qquad \omega_j^i + \omega_i^j = 0 \;,$$

$$(2.7)\qquad d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i \quad , \qquad \Omega_j^i = \frac{1}{2}\sum R_{jk\ell}^i\,\omega^k \wedge \omega^\ell \;,$$

$$(2.8)\qquad R_{jk\ell}^i = \sum (h_{ik}^r h_{j\ell}^r - h_{i\ell}^r h_{jk}^r) \;,$$

$$(2.9)\qquad d\omega_s^r = -\sum \omega_t^r \wedge \omega_s^t + \Omega_s^r \quad , \qquad \Omega_s^\ell = \frac{1}{2}\sum R_{sij}^r\,\omega^i \wedge \omega^j \;,$$

$$(2.10)\qquad R_{sij}^r = \sum_k (h_{ki}^r h_{kj}^s - h_{kj}^r h_{ki}^s) \;.$$

The Riemannian connection of M is defined by (ω_j^i). The form (ω_s^r) defines a connection D in the normal bundle of M. We call $h = \sum h_{ij}^r \omega^i \omega^j e_r$ the second fundamental form of the surface M. We call $H = \frac{1}{2}\,\mathrm{tr}\,h$ the mean curvature vector of M. We take exterior differentiation of (2.4) and define h_{ijk}^r by

$$(2.11)\qquad \sum h_{ijk}^r \omega^k = dh_{ij}^r - \sum h_{i\ell}^r \omega_j^\ell - \sum h_{\ell j}^r \omega_i^\ell + \sum h_{ij}^s \omega_s^r \;.$$

Then we have the following equation of Codazzi,

$$(2.12)\qquad h_{ijk}^r = h_{ikj}^r \;.$$

If we denote by ∇ and $\tilde\nabla$ the covariant derivatives of M and E^m, respectively, then, for any two vector fields X, Y tangent to M and any vector field ξ normal to M, we have

$$(2.13)\qquad \tilde\nabla_X Y = \nabla_X Y + h(X,Y) \;,$$

$$(2.14)\qquad \tilde\nabla_X \xi = -A_\xi X + D_X \xi \;,$$

where A_ξ denotes the Weingarten map with respect to ξ. If $\langle\, ,\, \rangle$ denotes the inner product of E^m, then

$$(2.15)\qquad \langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle \;.$$

If we define $\bar\nabla h$ by

$$(2.16)\qquad (\bar\nabla_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \;,$$

then equation (2.12) of Codazzi becomes

$$(2.17) \qquad (\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) .$$

It is well-known that $\bar{\nabla} h$ is a normal-bundle-valued tensor of type (0,3).

We need the following theorems for the proof of Theorem 1.

THEOREM C. (Chen [1]). *A surface* M *of* E^m *has pointwise planar normal sections if and only if* $(\bar{\nabla}_t h)(t,t) \wedge h(t,t) = 0$ *for any* $t \in TM$.

THEOREM D. (Chen [2]). *Let* M *be a surface in* E^m *with pointwise planar normal sections. Then* $\operatorname{Im} h$ *is parallel.*

3. CLASSIFICATION OF SURFACES IN E^5.

In this section we shall prove the following.

THEOREM 1. *Let* M *be a surface which lies essentially in* E^5. *Then, up to similarities of* E^5, M *is an open portion of the Veronese surface in* E^5 *if and only if* M *has pointwise planar normal sections.*

Proof. Let M be a surface in E^5 with pointwise planar normal sections. We choose a local field of orthonormal frame $\{e_1,e_2,e_3,e_4,e_5\}$ such that, restricted to M, e_3 is in the direction of the mean curvature vector H, e_1, e_2 are the principal directions of $A_3 = A_{e_3}$. Then e_3 is perpendicular to $h(e_1,e_2)$. We further choose e_5 so that e_5 is in the direction of $h(e_1,e_2)$. Then, with respect to $\{e_1,e_2,e_3,e_4,e_5\}$, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} , \quad A_4 = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} , \quad A_5 = \begin{pmatrix} \eta & \delta \\ \delta & -\eta \end{pmatrix}$$

Thus, we have

$$(3.1) \qquad h(e_1,e_1) = \alpha e_3 + \gamma e_4 + \eta e_5 , \quad h(e_1,e_2) = \delta e_5 ,$$
$$h(e_2,e_2) = \beta e_3 - \gamma e_4 - \eta e_5 .$$

It is easy to see that $\dim \operatorname{Im} h = 3$ if and only if $h(e_1,e_1) \wedge h(e_1,e_2) \wedge h(e_2,e_2) \neq 0$. Therefore, $\dim \operatorname{Im} h = 3$ if and only if $(\alpha+\beta)\gamma\delta \neq 0$. We put

$$(3.2) \qquad M_3 = \{p \in M \mid \dim \operatorname{Im} h = 3\} .$$

that M does not lie essentially in E^5. From now on, we assume that M lies essentially in E^5. Then M_3 is not empty. We denote by N a component of M_3. On N, we have

$$(\alpha+\beta)\gamma\delta \neq 0 . \tag{3.3}$$

From (2.16) and (3.1) we find

$$(\bar{\nabla}_{e_1} h)(e_1,e_1) = [e_1(\alpha) + \gamma\omega_4^3(e_1) + \eta\omega_5^3(e_1)]e_3 + \\ + [\alpha\omega_3^4(e_1) + e_1(\gamma) + \eta\omega_5^4(e_1)]e_4 + \\ + [\alpha\omega_3^5(e_1) + \gamma\omega_4^5(e_1) + e_1(\eta) - 2\delta\omega_1^2(e_1)]e_5 , \tag{3.4}$$

$$(\bar{\nabla}_{e_2} h)(e_1,e_1) = [e_2(\alpha) + \gamma\omega_4^3(e_2) + \eta\omega_5^3(e_2)]e_3 + \\ + [\alpha\omega_3^4(e_2) + e_2(\gamma) + \eta\omega_5^4(e_2)]e_4 + \\ + [\alpha\omega_3^5(e_2) + \gamma\omega_4^5(e_2) + e_2(\eta) - 2\delta\omega_1^2(e_2)]e_5 , \tag{3.5}$$

$$(\bar{\nabla}_{e_1} h)(e_1,e_2) = [\delta\omega_5^3(e_1) + (\alpha-\beta)\omega_1^2(e_1)]e_3 + \\ + [\delta\omega_5^4(e_1) + 2\gamma\omega_1^2(e_1)]e_4 + \\ + [e_1(\delta) + 2\eta\omega_1^2(e_1)]e_5 , \tag{3.6}$$

$$(\bar{\nabla}_{e_1} h)(e_2,e_2) = [e_1(\beta) - \gamma\omega_4^3(e_1) - \eta\omega_5^3(e_1)]e_3 + \\ + [\beta\omega_3^4(e_1) - e_1(\gamma) - \eta\omega_5^4(e_1)]e_4 + \\ + [\beta\omega_3^5(e_1) - \gamma\omega_4^5(e_1) - e_1(\eta) - 2\delta\omega_2^1(e_1)]e_5 , \tag{3.7}$$

$$(\bar{\nabla}_{e_2} h)(e_1,e_2) = [\delta\omega_5^3(e_1) + (\alpha-\beta)\omega_1^2(e_2)]e_3 + \\ + [\delta\omega_5^4(e_2) + 2\gamma\omega_1^2(e_2)]e_4 + \\ + [e_2(\delta) + 2\eta\omega_1^2(e_2)]e_5 , \tag{3.8}$$

$$(\bar{\nabla}_{e_2} h)(e_2,e_2) = [e_2(\beta) - \gamma\omega_4^3(e_2) - \eta\omega_5^3(e_2)]e_3 + \\ + [\beta\omega_3^4(e_2) - e_2(\gamma) - \eta\omega_5^4(e_2)]e_4 + \\ + [\beta\omega_3^5(e_2) - \gamma\omega_4^5(e_1) - e_2(\eta) - 2\delta\omega_2^1(e_2)]e_5 . \tag{3.9}$$

Because M has pointwise planar normal sections, Theorem C implies

$$(\bar{\nabla}_{e_1} h)(e_1,e_1) = \lambda_1 h(e_1,e_1) \quad , \quad (\bar{\nabla}_{e_2} h)(e_2,e_2) = \lambda_2 h(e_2,e_2) , \tag{3.10}$$

for some local functions λ_1,λ_2. Combining (3.1), (3.4), (3.9) with (3.10) we obtain

$$e_1(\alpha) = \alpha\lambda_1 + \gamma\omega_3^4(e_1) + \eta\omega_3^5(e_1) \quad , \tag{3.11}$$

$$e_1(\gamma) = \gamma\lambda_1 - \alpha\omega_3^4(e_1) + \eta\omega_4^5(e_1) \quad , \tag{3.12}$$

$$e_1(\eta) = \eta\lambda_1 - \alpha\omega_3^5(e_1) - \gamma\omega_4^5(e_1) + 2\delta\omega_1^2(e_1) \quad , \tag{3.13}$$

$$e_2(\beta) = \beta\lambda_2 - \gamma\omega_3^4(e_2) - \eta\omega_3^5(e_2) \quad , \tag{3.14}$$

$$e_2(\gamma) = \gamma\lambda_2 + \beta\omega_3^4(e_2) + \eta\omega_4^5(e_2) \quad , \tag{3.15}$$

$$e_2(\eta) = \eta\lambda_2 + \beta\omega_3^5(e_2) - \gamma\omega_4^5(e_2) + 2\delta\omega_1^2(e_2) \quad . \tag{3.16}$$

Moreover, from (3.5), (3.6), (3.7), (3.8) and equation (2.17) of Codazzi, we also have

$$e_2(\alpha) = \gamma\omega_3^4(e_2) - \delta\omega_3^5(e_1) + \eta\omega_3^5(e_2) + (\alpha-\beta)\omega_1^2(e_1) \quad , \tag{3.17}$$

$$e_1(\beta) = -\gamma\omega_3^4(e_1) - \delta\omega_3^5(e_2) - \eta\omega_3^5(e_1) + (\alpha-\beta)\omega_1^2(e_2) \quad , \tag{3.18}$$

$$e_1(\delta) = \eta\lambda_2 + (\alpha+\beta)\omega_3^5(e_2) - 2\eta\omega_1^2(e_1) \quad , \tag{3.19}$$

$$e_2(\delta) = -\eta\lambda_1 + (\alpha+\beta)\omega_3^5(e_1) - 2\eta\omega_1^2(e_2) \quad , \tag{3.20}$$

$$\lambda_1\gamma - (\alpha+\beta)\omega_3^4(e_1) - \delta\omega_4^5(e_2) + 2\gamma\omega_1^2(e_2) = 0 \quad , \tag{3.21}$$

$$\lambda_2\gamma + (\alpha+\beta)\omega_3^4(e_2) + \delta\omega_4^5(e_1) - 2\gamma\omega_1^2(e_1) = 0 \quad . \tag{3.22}$$

Let $t = e_1 + ke_2$. Then, from Theorem C, we have

$$(\overline{\nabla}_{e_1+ke_2}h)(e_1+ke_2, e_1+ke_2) \wedge h(e_1+ke_2, e_1+ke_2) = 0 \tag{3.23}$$

for any k. Because $e_3 \wedge e_4$, $e_3 \wedge e_5$ and $e_4 \wedge e_5$ are linearly independent, (3.1), (3.3), (3.4) - (3.10), and (3.23) imply

$$-\gamma\delta\omega_3^5(e_1) + \alpha\delta\omega_4^5(e_1) - (\alpha+\beta)\gamma\omega_1^2(e_1) = 0 \quad , \tag{3.24}$$

$$(\alpha+\beta)\gamma\lambda_1 + 3\gamma\delta\omega_3^5(e_2) - 3\alpha\delta\omega_4^5(e_2) + 3(\alpha+\beta)\gamma\omega_1^2(e_2) = 0 \quad , \tag{3.25}$$

$$(\alpha+\beta)\gamma\lambda_2 + 3\gamma\delta\omega_3^5(e_1) + 3\beta\delta\omega_4^5(e_1) - 3(\alpha+\beta)\gamma\omega_1^2(e_1) = 0 \quad , \tag{3.26}$$

$$\gamma\delta\omega_3^5(e_2) + \beta\delta\omega_4^5(e_2) - (\alpha+\beta)\gamma\omega_1^2(e_2) = 0 \quad , \tag{3.27}$$

$$2\gamma\delta\lambda_1 - 3\gamma\eta\lambda_2 - 3(\alpha+\beta)\gamma\omega_3^5(e_2) - 3\delta\eta\omega_4^5(e_1) + 6\gamma\eta\omega_1^2(e_1) = 0 \ , \tag{3.28}$$

$$-3\gamma\eta\lambda_1 - 2\gamma\delta\lambda_2 + 3(\alpha+\beta)\gamma\omega_3^5(e_1) + 3\delta\eta\omega_4^5(e_2) - 6\gamma\eta\omega_1^2(e_2) = 0 \ . \tag{3.29}$$

From (3.25) and (3.27) we find

$$(3.30) \qquad \gamma\lambda_1 - 3\delta\omega_4^5(e_2) + 6\gamma\omega_1^2(e_2) = 0 .$$

From (3.24) and (3.26) we find

$$(3.31) \qquad \gamma\lambda_2 + 3\delta\omega_4^5(e_1) - 6\gamma\omega_1^2(e_1) = 0 .$$

Similarly, from (3.21), (3.22), (3.28) and (3.29), we also have

$$(3.32) \qquad 2\gamma\delta\lambda_1 + 3(\alpha+\beta)\eta\omega_3^4(e_2) - 3(\alpha+\beta)\gamma\omega_3^5(e_2) = 0 ,$$

$$(3.33) \qquad -2\gamma\delta\lambda_2 - 3(\alpha+\beta)\eta\omega_3^4(e_1) + 3(\alpha+\beta)\gamma\omega_3^5(e_1) = 0 .$$

From (3.22) and (3.24) we find

$$(3.34) \qquad -\alpha\gamma\lambda_2 - \alpha(\alpha+\beta)\omega_3^4(e_2) - \gamma\delta\omega_3^5(e_1) + (\alpha-\beta)\gamma\omega_1^2(e_1) = 0 .$$

Similarly, from (3.21) and (3.27) we get

$$(3.35) \qquad \beta\gamma\lambda_1 - \beta(\alpha+\beta)\omega_3^4(e_1) + \gamma\delta\omega_3^5(e_2) - (\alpha-\beta)\gamma\omega_1^2(e_2) = 0 .$$

From (3.21), (3.30) and (3.22) and (3.31), we obtain, respectively,

$$(3.36) \qquad (\alpha+\beta)\omega_3^4(e_1) - 2\delta\omega_4^5(e_2) + 4\gamma\omega_1^2(e_2) = 0 ,$$

$$(3.37) \qquad (\alpha+\beta)\omega_3^4(e_2) - 2\delta\omega_4^5(e_1) + 4\gamma\omega_1^2(e_1) = 0 .$$

From (3.21) and (3.36), we obtain

$$(3.38) \qquad -2\gamma\lambda_1 + 3(\alpha+\beta)\omega_3^4(e_1) = 0 .$$

Similarly, from (3.22) and (3.37), we obtain

$$(3.39) \qquad 2\gamma\lambda_2 + 3(\alpha+\beta)\omega_3^4(e_2) = 0 .$$

Combining (3.21) and (3.38) we have

$$(3.40) \qquad \gamma\lambda_1 - 3\delta\omega_4^5(e_2) + 6\gamma\omega_1^2(e_2) = 0 .$$

Equations (3.22) and (3.39) imply

$$(3.41) \qquad \gamma\lambda_2 + 3\delta\omega_4^5(e_1) - 6\gamma\omega_1^2(e_1) = 0 .$$

From (3.34) and (3.39) we find

$$(3.42) \qquad \alpha\lambda_2 + 3\delta\omega_3^5(e_1) - 3(\alpha-\beta)\omega_1^2(e_1) = 0 .$$

Similarly, we have

$$\beta\lambda_1 + 3\delta\omega_3^5(e_2) - 3(\alpha-\beta)\omega_1^2(e_2) = 0 . \tag{3.43}$$

From (3.32) and (3.39) we find

$$2\delta\lambda_1 - 2\eta\lambda_2 - 3(\alpha+\beta)\omega_3^5(e_2) = 0 . \tag{3.44}$$

Similarly, we also have

$$2\eta\lambda_1 + 2\delta\lambda_2 - 3(\alpha+\beta)\omega_3^5(e_1) = 0 . \tag{3.45}$$

Now, we want to claim that N is pseudo-umbilical in E^5, i.e., $\alpha \equiv \beta$ on N. Assume that $\alpha \neq \beta$ at a point $p \in N$. Then there is an open neighborhood U of p in N such that $\alpha \neq \beta$ everywhere on U. From (3.38) - (3.45), we obtain the following expression of ω_1^2 and ω_r^s on U,

$$\omega_1^2 = \left\{\frac{2\delta\eta\lambda_1 + [\alpha(\alpha+\beta) + 2\delta^2]\lambda_2}{3(\alpha^2-\beta^2)}\right\}\omega^1 + \tag{3.46}$$

$$+ \left\{\frac{[\beta(\alpha+\beta) + 2\delta^2]\lambda_1 - 2\delta\eta\lambda_2}{3(\alpha^2-\beta^2)}\right\}\omega^2 ,$$

$$\omega_3^4 = \left\{\frac{2\gamma\lambda_1}{3(\alpha+\beta)}\right\}\omega^1 - \left\{\frac{2\gamma\lambda_2}{3(\alpha+\beta)}\right\}\omega^2 , \tag{3.47}$$

$$\omega_3^5 = \left\{\frac{2\eta\lambda_1 + 2\delta\lambda_2}{3(\alpha+\beta)}\right\}\omega^1 + \left\{\frac{2\delta\lambda_1 - 2\eta\lambda_2}{3(\alpha+\beta)}\right\}\omega^2 , \tag{3.48}$$

$$\omega_4^5 = \left\{\frac{4\gamma\delta\eta\lambda_1 + \gamma[(\alpha+\beta)^2 + 4\delta^2]\lambda_2}{3\delta(\alpha^2-\beta^2)}\right\}\omega^1 + \tag{3.49}$$

$$+ \left\{\frac{\gamma[(\alpha+\beta)^2 + 4\delta^2]\lambda_1 - 4\gamma\delta\eta\lambda_2}{3\delta(\alpha^2-\beta^2)}\right\}\omega^2 .$$

Now, we shall make a careful study of the integrability condition to obtain a contradiction. In order to do so, we need to compute the exterior derivatives of (ω_r^s).

From (3.47) we have

$$d\omega_3^4 = d\left(\frac{2\gamma}{3(\alpha+\beta)}\right) \wedge (\lambda_1\omega^1 - \lambda_2\omega^2) + \left(\frac{2\gamma}{3(\alpha+\beta)}\right)d(\lambda_1\omega^1 - \lambda_2\omega^2) . \tag{3.50}$$

Thus, by applying (3.11) - (3.18), (3.46) and a direct long computation, we may find

$$d\omega_3^4 = -\frac{2\gamma}{3(\alpha+\beta)}\left\{e_2(\lambda_1) + e_1(\lambda_2) - \frac{\lambda_1\lambda_2}{3} + \right. \tag{3.51}$$

$$+ \frac{\eta[(\alpha+\beta)^2+2\delta^2]}{3\delta(\alpha^2-\beta^2)} (\lambda_1^2+\lambda_2^2)\Big\} \omega^1 \wedge \omega^2 .$$

Similarly, we may also obtain

$$(3.52) \qquad d\omega_3^5 = \frac{1}{9\delta(\alpha+\beta)^2(\alpha-\beta)} \{6(\alpha^2-\beta^2)\delta^2[e_1(\lambda_1)-e_2(\lambda_2)] - 6(\alpha^2-\beta^2)\delta\eta[e_2(\lambda_1)+e_1(\lambda_2)] - 2\{(\delta^2-\gamma^2)[(\alpha+\beta)^2+4\delta^2]+2\delta^2\eta^2\}(\lambda_1^2+\lambda_2^2) + 2\delta^2[\beta(\alpha+\beta)+2\delta^2]\lambda_1^2+2\delta^2[\alpha(\alpha+\beta)+2\delta^2]\lambda_2 + 2\delta(\alpha^2-\beta^2)\eta\lambda_1\lambda_2\} \omega^1 \wedge \omega^2 ,$$

$$(3.53) \qquad d\omega_4^5 = \frac{1}{9\delta(\alpha^2-\beta^2)^2} \cdot \{3(\alpha^2-\beta^2)\gamma[(\alpha+\beta)^2+4\delta^2][e_1(\lambda_1)-e_2(\lambda_2)] - 12\gamma\delta\eta(\alpha^2-\beta^2)[e_2(\lambda_1)+e_1(\lambda_2)] - [(\alpha+\beta)^2(\alpha^2+\alpha\beta+\beta^2)\gamma + 2\gamma\delta^2(5\alpha^2+5\beta^2+4\eta^2+4\delta^2)](\lambda_1^2+\lambda_2^2) + \gamma[(\alpha+\beta)^2+4\delta^2](\beta^2\lambda_1^2+\alpha^2\lambda_2^2) + 4\gamma\delta\eta(\alpha^2-\beta^2)\lambda_1\lambda_2\} \omega^1 \wedge \omega^2 .$$

On the other hand, by using (2.10) and (3.1), we have

$$(3.54) \qquad R^4_{312} = 0 ,$$

$$(3.55) \qquad R^5_{312} = (\beta-\alpha)\delta ,$$

$$(3.56) \qquad R^5_{412} = -2\gamma\delta .$$

Therefore, by equation (2.9) of Ricci, equations (3.47) - (3.49) and (3.54) - (3.56), we also have

$$(3.57) \qquad d\omega_3^4 = -\left(\frac{2\gamma\eta}{9\delta(\alpha-\beta)}\right)(\lambda_1^2+\lambda_2^2)\ \omega^1 \wedge \omega^2$$

$$(3.58) \qquad d\omega_3^5 = \frac{1}{9\delta(\alpha^2-\beta^2)(\alpha+\beta)} \{2\gamma^2[(\alpha+\beta)^2+4\delta^2](\lambda_1^2+\lambda_2^2) - 9\delta^2(\alpha^2-\beta^2)^2\} \omega^1 \wedge \omega^2 ,$$

$$(3.59) \qquad d\omega_4^5 = \frac{-2\gamma\delta}{9(\alpha+\beta)^2} \{2(\lambda_1^2+\lambda_2^2)+9(\alpha+\beta)^2\} \omega^1 \wedge \omega^2 .$$

Comparing (3.51) with (3.57), we find

$$(3.60)\qquad e_2(\lambda_1)+e_1(\lambda_2) = \frac{1}{3}\,\lambda_1\lambda_2 - \frac{2\eta\delta}{3(\alpha^2-\beta^2)}\,(\lambda_1^2+\lambda_2^2)\ .$$

Comparing (3.52) with (3.58), we find

$$(3.61)\qquad \delta[e_1(\lambda_1)-e_2(\lambda_2)]-\eta[e_2(\lambda_1)+e_1(\lambda_2)] =$$

$$= \frac{\delta}{3(\alpha^2-\beta^2)}\{[\alpha(\alpha+\beta)+2\delta^2+2\eta^2]\lambda_1^2+[\beta(\alpha+\beta)+2\delta^2+2\eta^2]\lambda_2^2\} -$$

$$- \frac{1}{3}\,\eta\lambda_1\lambda_2 - \frac{3}{2}(\alpha^2-\beta^2)\delta\ .$$

Combining (3.53) with (3.59), we get

$$(3.62)\qquad [(\alpha+\beta)^2+4\delta^2][e_1(\lambda_1)-e_2(\lambda_2)]-4\delta\eta[e_2(\lambda_1)+e_1(\lambda_2)] =$$

$$= \frac{1}{3(\alpha^2-\beta^2)}\{(\alpha+\beta)^2(\alpha^2+\alpha\beta+\beta^2) +$$

$$+ 2\delta^2(3\alpha^2+4\alpha\beta+3\beta^2+4\eta^2+4\delta^2)](\lambda_1^2+\lambda_2^2) -$$

$$- \frac{1}{3(\alpha^2-\beta^2)}\{[(\alpha+\beta)^2+4\delta^2](\beta^2\lambda_1^2+\alpha^2\lambda_2^2)\} -$$

$$- \frac{4}{3}\,\delta\eta\lambda_1\lambda_2-6\delta^2(\alpha^2-\beta^2)\ .$$

Substituting (3.60) into (3.61), we obtain

$$(3.63)\qquad e_1(\lambda_1)-e_2(\lambda_2) = \frac{1}{3(\alpha^2-\beta^2)}\{[\alpha(\alpha+\beta)+2\delta^2]\lambda_1^2 +$$

$$+ [\beta(\alpha+\beta)+2\delta^2]\lambda_2^2\}-\frac{3}{2}(\alpha^2-\beta^2)\ .$$

Substituting (3.60) and (3.63) into (3.62), we may obtain

$$(3.64)\qquad \alpha^2 - \beta^2 = 0\ .$$

This contradicts to (3.3) because we assume that $\alpha \not\equiv \beta$.
Therefore, we have proved that $\alpha = \beta$ identically on N, i.e., N is pseudo-umbilical in E^5. Because $\alpha \equiv \beta$, (3.42), (3.43), (3.44) and (3.45) reduce to

$$(3.65)\qquad \alpha\lambda_2 + 3\delta\omega_3^5(e_1) = 0\ ,$$

$$(3.66)\qquad \beta\lambda_1 + 3\delta\omega_3^5(e_2) = 0\ ,$$

$$(3.67)\qquad (\alpha+\beta)\beta\lambda_1 = -2\delta^2\lambda_1 + 2\delta\eta\lambda_2\ ,$$

$$(3.68)\qquad (\alpha+\beta)\alpha\lambda_2 = -2\delta\eta\lambda_1 - 2\delta^2\lambda_2\ .$$

From (3.67) and (3.68) we obtain

(3.69) $$\lambda_1 = \lambda_2 = 0 .$$

Thus, from (3.30) and (3.31), we have

(3.70) $$\delta\omega_4^5 = 2\gamma\omega_1^2 .$$

From (3.38), (3.39), (3.42) and (3.43), we find

(3.71) $$\omega_3^4 = \omega_3^5 = 0 .$$

Substituting (3.69) and (3.71) into (3.11), (3.14), (3.17) and (3.18), we find

(3.72) $$\alpha = \beta = \text{constant on } N .$$

From (3.12), (3.15), (3.69) and (3.71), we obtain

(3.73) $$d\gamma = \eta\omega_4^5 .$$

From (2.9), (2.10), (3.1) and (3.71), we find

(3.74) $$d\omega_4^5 = -2\gamma\delta\ \omega^1 \wedge \omega^2 .$$

Using (3.13), (3.16), (3.69), (3.70) and (3.71), we have

(3.75) $$d\eta = \left(\frac{\delta^2 - \gamma^2}{\gamma}\right) \omega_4^5 .$$

Taking exterior differentiation of (3.73) and applying (2.9), (2.10), and (3.74), we obtain

(3.76) $$0 = d^2\gamma = -2\gamma\delta\eta\ \omega^1 \wedge \omega^2 .$$

From (3.76) we get

(3.77) $$\eta = 0 .$$

Since (3.74) shows that $\omega_4^5 \neq 0$, (3.75) and (3.77) give $\delta^2 = \gamma^2$. Without loss of generality, we may assume that

(3.78) $$\delta = -\gamma .$$

From (3.70) and (3.78), we find

(3.79) $$\omega_4^5 = -2\omega_1^2 .$$

From (3.73) and (3.77), we see that $\delta = -\gamma$ is a nonzero constant on N. Thus, by the definition of N and continuity, we conclude that N is the whole surface M.

From (2.7), (2.9), (3.1), (3.74), (3.78) and (3.79) we find

$$\alpha^2 = 3\gamma^2 . \tag{3.80}$$

Consequently, we may assume that $\alpha = -\sqrt{3}\,\gamma$. Therefore, by combining (3.71), (3.77), (3.79) and (3.80), we conclude that the connection form (ω_A^B), restricted to N, is given by

$$\begin{pmatrix} 0 & \omega_1^2 & \sqrt{3}\,\gamma\omega^1 & -\gamma\omega^1 & \gamma\omega^2 \\ \omega_2^1 & 0 & \sqrt{3}\,\gamma\omega^2 & \gamma\omega^2 & \gamma\omega^1 \\ -\sqrt{3}\,\gamma\omega^1 & -\sqrt{3}\,\gamma\omega^2 & 0 & 0 & 0 \\ \gamma\omega^1 & -\gamma\omega^2 & 0 & 0 & 2\omega_1^2 \\ -\gamma\omega^2 & -\gamma\omega^1 & 0 & 2\omega_2^1 & 0 \end{pmatrix}$$

This shows that, up to similarity transformations of E^5, M coincides locally with the Veronese surface [4].

Conversely, if, up to similarity transformations of E^5, M is an open portion of the Veronese surface, then M has parallel second fundamental form, i.e., $\bar{\nabla}h \equiv 0$. Thus, by Theorem C of Chen [1], we conclude that M has pointwise planar normal sections. This completes the proof of Theorem 1.

4. SURFACES IN E^4 WITH CONSTANT MEAN CURVATURE.

In this and the next two sections, we will study surfaces in E^4. Assume that M is a surface in E^4 with pointwise planar normal sections. We choose a local field of orthonormal frame $\{e_1,e_2,e_3,e_4\}$ so that, restricted to M, e_3 is in the direction of H, e_1, e_2 are the principal directions of A_3. Then e_3 is perpendicular to $h(e_1,e_2)$. With respect to $\{e_1,e_2,e_3,e_4\}$, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} , \quad A_4 = \begin{pmatrix} \eta & \delta \\ \delta & -\eta \end{pmatrix}$$

Thus we have

$$h(e_1,e_1) = \alpha e_3+\eta e_4 \;, \quad h(e_1,e_2) = \delta e_4 \;, \quad h(e_2,e_2) = \beta e_3-\eta e_4 . \tag{4.1}$$

It is easy to find that the mean curvature, the normal curvature and the Gauss curvature of M in E^4 are given respectively by

$|H| = \frac{1}{2}|\alpha+\beta|$, $K^N = 2(\alpha-\beta)^2\delta^2$ and $K = \alpha\beta - \eta^2 - \delta^2$.

Since M has pointwise planar normal sections, Theorem C implies

$$(4.2)\qquad (\bar{\nabla}_{e_1}h)(e_1,e_1) = \lambda_1 h(e_1,e_2) ,$$

$$(\bar{\nabla}_{e_2}h)(e_2,e_2) = \lambda_2 h(e_2,e_2)$$

for some local functions λ_1, λ_2. Using the same method as before, we have the following

$$(4.3)\qquad e_1(\alpha) = \alpha\lambda_1 + \eta\omega_3^4(e_1) ,$$

$$(4.4)\qquad e_1(\beta) = -\eta\omega_3^4(e_1) - \delta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_2) ,$$

$$(4.5)\qquad e_1(\eta) = \eta\lambda_1 - \alpha\omega_3^4(e_1) + 2\delta\omega_1^2(e_1) ,$$

$$(4.6)\qquad e_1(\delta) = \eta\lambda_2 + (\alpha+\beta)\omega_3^4(e_2) - 2\eta\omega_1^2(e_1) ,$$

$$(4.7)\qquad e_2(\alpha) = -\delta\omega_3^4(e_1) + \eta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_1) ,$$

$$(4.8)\qquad e_2(\beta) = \beta\lambda_2 - \eta\omega_3^4(e_2) ,$$

$$(4.9)\qquad e_2(\eta) = \eta\lambda_2 + \beta\omega_3^4(e_2) + 2\delta\omega_1^2(e_2) ,$$

$$(4.10)\qquad e_2(\delta) = -\eta\lambda_1 + (\alpha+\beta)\omega_3^4(e_1) - 2\eta\omega_1^2(e_2) ,$$

$$(4.11)\qquad 2\alpha\delta\lambda_1 - 3\alpha\eta\lambda_2 - 3\eta\delta\omega_3^4(e_1) - 3\alpha(\alpha+\beta)\omega_3^4(e_2) + \\ + 3(\alpha-\beta)\eta\omega_1^2(e_1) = 0 ,$$

$$(4.12)\qquad (2\alpha-\beta)\eta\lambda_1 - 3(\alpha^2+\alpha\beta+2\delta^2)\omega_3^4(e_1) - 3\eta\delta\omega_3^4(e_2) + \\ + 6(\alpha-\beta)\delta\omega_1^2(e_1) + 3(\alpha-\beta)\eta\omega_1^2(e_2) = 0 ,$$

$$(4.13)\qquad (\alpha-2\beta)\eta\lambda_2 + 3\eta\delta\omega_3^4(e_1) - 3(\alpha\beta+\beta^2+2\delta^2)\omega_3^4(e_2) - \\ - 3(\alpha-\beta)\eta\omega_1^2(e_1) + 6(\alpha-\beta)\delta\omega_1^2(e_2) = 0 ,$$

$$(4.14)\qquad 3\beta\eta\lambda_1 + 2\beta\delta\lambda_2 - 3(\alpha+\beta)\beta\omega_3^4(e_1) + 3\eta\delta\omega_3^4(e_2) - \\ - 3(\alpha-\beta)\eta\omega_1^2(e_2) = 0 .$$

THEOREM 2. *Let* M *be a surface which lies essentially in* E^4. *Then* M *is an open portion of the product surface of two planar circles if and only if* M *has pointwise planar normal sections and constant mean curvature.*

Proof. If M is an open portion of the product surface of two planar circles, then it is easy to check that M has constant mean curvature and pointwise planar normal sections.

Now, let M be a surface which lies essentially in E^4. Assume that M has constant mean curvature and pointwise planar normal sections. Then, by using Theorem 4 of [2], we see that $\alpha+\beta \neq 0$. We want to claim that $(\alpha-\beta)\delta = 0$. Assume that $(\alpha-\beta)\delta \neq 0$. If $\eta \neq 0$, then by eliminating $\omega_1^2(e_1)$, $\omega_1^2(e_2)$ from (4.12) and (4.13) with the help of (4.11), (4.14), we have

$$(4.15)\qquad 2[(\alpha+\beta)\eta^2 - 2\alpha\delta^2]\lambda_1 + 2(3\alpha+\beta)\eta\delta\lambda_2 - \\ - 3(\alpha+\beta)^2\eta\omega_3^4(e_1) + 6\alpha(\alpha+\beta)\delta\omega_3^4(e_2) = 0 \quad ,$$

$$(4.16)\qquad -2(\alpha+3\beta)\eta\delta\lambda_1 + 2[(\alpha+\beta)\eta^2 - 2\beta\delta^2]\lambda_2 + \\ + 6(\alpha+\beta)\beta\delta\omega_3^4(e_1) + 3(\alpha+\beta)^2\eta\omega_3^4(e_2) = 0 \ .$$

Combining (4.15) and (4.16), we have

$$(4.17)\qquad [(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2][2\eta\lambda_1 + 2\delta\lambda_2 - 3(\alpha+\beta)\omega_3^4(e_1)] = 0 \ .$$

If $(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2 \neq 0$. We have from (4.11) - (4.17)

$$(4.18)\qquad \omega_1^2 = \frac{2\eta\delta\lambda_1 + (\alpha^2+\alpha\beta+2\delta^2)\lambda_2}{3(\alpha^2-\beta^2)}\,\omega^1 + \\ + \frac{(\alpha\beta+\beta^2+2\delta^2)\lambda_1 - 2\eta\delta\lambda_2}{3(\alpha^2-\beta^2)}\,\omega^2 \ ,$$

$$(4.19)\qquad \omega_3^4 = \frac{2(\eta\lambda_1+\delta\lambda_2)}{3(\alpha+\beta)}\,\omega^1 + \frac{2(\delta\lambda_1-\eta\lambda_2)}{3(\alpha+\beta)}\,\omega^2 \ .$$

If $(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2 = 0$, differentiating this relation, we have, with the help of (4.3) - (4.10),

$$(4.20)\qquad [\alpha(\alpha+\beta)\eta^2 - 2\alpha\beta\delta^2]\lambda_1 + 4\alpha\beta\eta\delta\lambda_2 - \\ - [\alpha(\alpha+\beta)^2 + 2(\alpha-\beta)\delta^2]\eta\omega_3^4(e_1) + \\ + [4\alpha\beta(\alpha+\beta) - (\alpha+\beta)\eta^2 - 2\alpha\delta^2]\delta\omega_3^4(e_2) + 2(\alpha-\beta)^2\eta\delta\omega_1^2(e_1) + \\ + (\alpha-\beta)[(\alpha+\beta)\eta^2 + 2\alpha\delta^2]\omega_1^2(e_2) = 0 \ .$$

$$(4.21)\qquad -4\alpha\beta\eta\delta\lambda_1 + [\beta(\alpha+\beta)\eta^2 - 2\alpha\beta\delta^2]\lambda_2 + \\ + [4\alpha\beta(\alpha+\beta) - (\alpha+\beta)\eta^2 - 2\beta\delta^2]\delta\omega^4(e_\ ) +$$

$$+ [\beta(\alpha+\beta)^2 - 2(\alpha-\beta)\delta^2]\eta\omega_3^4(e_2) +$$
$$+ (\alpha-\beta)[(\alpha+\beta)\eta^2 + 2\beta\delta^2]\omega_1^2(e_1) +$$
$$+ 2(\alpha-\beta)^2\eta\delta\omega_1^2(e_2) = 0 .$$

From (4.11) - (4.14) and (4.20), (4.21), we still have (4.18), (4.19). Because $|H|$ is constant, differentiating the relation $\alpha+\beta$ = constant, we have

$$\alpha\lambda_1 - \delta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_2) = 0 \tag{4.22}$$

$$\beta\lambda_2 - \delta\omega_3^4(e_1) + (\alpha-\beta)\omega_1^2(e_1) = 0 . \tag{4.23}$$

Substituting (4.18), (4.19) into (4.22), (4.23), we get

$$(3\alpha+\beta)\lambda_1 = 0 . \tag{4.24}$$

$$(3\beta+\alpha)\lambda_2 = 0 . \tag{4.25}$$

Thus we have (i) $\lambda_1 = \lambda_2 = 0$, or (ii) $3\alpha+\beta = 0$, $3\beta+\alpha = 0$, or (iii) $3\alpha+\beta = 0$, $\lambda_2 = 0$, or (iv) $3\beta+\alpha = 0$, $\lambda_1 = 0$. If case (i) occurs, (4.18) and (4.19) imply $\omega_1^2 = \omega_3^4 = 0$. In particular, we have $K^N = 0$. Thus, by applying Theorem 5 of Chen [2], we see that M is an open portion of the product surface of two planar circles. In particular, we have $\delta = 0$. This is a contradiction. If case (ii) occurs, we have $\alpha = \beta = 0$. This contradicts to $\alpha+\beta \neq 0$. For case (iii), differentiating $3\alpha+\beta = 0$, we have

$$3e_2(\alpha) + e_2(\beta) = 0 . \tag{4.26}$$

Since $\lambda_2 = 0$, (4.7), (4.8), (4.18), (4.19), and (4.26) imply

$$\eta\delta\lambda_1 = 0 . \tag{4.27}$$

From this we may again obtain a contradiction. The last case is similar to case (iii). Consequently, we have $\eta = 0$.

If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta \neq 0$, then from (4.3) - (4.14) we have $\alpha\beta+\delta^2 = 0$ and

$$\omega_1^2 = \frac{\alpha\lambda_2}{3(\alpha+\beta)}\,\omega^1 - \frac{\beta\lambda_1}{3(\alpha+\beta)}\,\omega^2 , \tag{4.28}$$

$$\omega_3^4 = \frac{2\delta\lambda_2}{3(\alpha+\beta)}\,\omega^1 + \frac{2\delta\lambda_1}{3(\alpha+\beta)}\,\omega^2 . \tag{4.29}$$

Differentiating $\alpha+\beta$ = constant, we have (4.22) and (4.23). By substituting (4.28) and (4.29) into (4.22) and (4.23), we obtain

$$(4.30) \qquad (3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2)\lambda_1 = 0 ,$$

$$(4.31) \qquad (\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2)\lambda_2 = 0 .$$

Thus, (i) $\lambda_1 = \lambda_2 = 0$, or (ii) $3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2 = 0$ and $\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2 = 0$, or (iii) $3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2 = 0$ and $\lambda_2 = 0$, or (iv) $\lambda_1 = 0$ and $\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2 = 0$.

Case (i) contradicts the assumption. Case (ii) implies $\alpha^2 = \beta^2$ which contradicts the assumption too. For case (iii), since $\alpha\beta + \delta^2 = 0$, we obtain

$$(4.32) \qquad 3\alpha^2 + 4\alpha\beta + \beta^2 = 0 .$$

This implies $3\alpha+\beta = 0$. We know that this is impossible. The last case is similar to case (iii).

If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta = 0$, then without loss of generality, we may assume $\beta = 0$. From (4.3) - (4.14), we have

$$(4.33) \qquad e_1(\beta) = -\delta\omega_3^4(e_2) + \alpha\omega_1^2(e_2) = 0 ,$$

$$(4.34) \qquad e_2(\eta) = 2\delta\omega_1^2(e_2) = 0 ,$$

$$(4.35) \qquad 2\delta\lambda_1 = 3\alpha\omega_3^4(e_2) = 0 .$$

These imply $\lambda_1 = 0$ and since $\beta = \eta = 0$, we have $h(e_2,e_2) = 0$. Thus, by (4.2), we may choose $\lambda_2 = 0$. From these we obtain a contradiction. Consequently, we obtain $(\alpha-\beta)\delta = 0$. Thus, $K^N = 0$, from which we obtain Theorem 2 by applying Theorem 5 of Chen [2]. (Q.E.D.)

5. SURFACES IN E^4 WITH CONSTANT NORMAL CURVATURE.

In this section, we give the following classification result.

THEOREM 3. *Let* M *be a surface which lies essentially in* E^4. *Then* M *is an open portion of the product surface of two planar circles if and only if* M *has pointwise planar normal sections and constant normal curvature.*

Proof. Let M be a surface which lies essentially in E^4. Assume M has constant normal curvature and pointwise planar normal sections. As mentioned in the proof of Theorem 2 we may assume that $\alpha+\beta \neq 0$. We want to claim that $(\alpha-\beta)\delta = 0$. Assume that $(\alpha-\beta)\delta \neq 0$. Because, $(\alpha-\beta)\delta$ = constant, we have

$$(5.1)\qquad \delta[e_i(\alpha) - e_i(\beta)] + (\alpha-\beta)e_i(\delta) = 0\ ,\quad i = 1,2\ .$$

Assume that $\eta \neq 0$. Using (4.3) - (4.10) and (4.18), (4.19), we obtain from (5.1),

$$(5.2)\qquad \delta(5\alpha-3\beta)\lambda_1 - \eta(\alpha+\beta)\lambda_2 = 0\ ,$$

$$(5.3)\qquad -\eta(\alpha+\beta)\lambda_1 + \delta(3\alpha-5\beta)\lambda_2 = 0\ .$$

From these, we know that either $\lambda_1 = \lambda_2 = 0$ or $\lambda_1^2 + \lambda_2^2 = 0$ and

$$(5.4)\qquad \delta^2(15\alpha^2 + 15\beta^2 - 34\alpha\beta) = \eta^2(\alpha+\beta)^2\ .$$

The first case implies that $\omega_3^4 = 0$ which gives $(\alpha-\beta)\delta = 0$. In the second case, we differentiate (5.4) to obtain

$$(5.5)\qquad \delta\lambda_1 = \eta\lambda_2\quad ,\quad \eta\lambda_1 = -\delta\lambda_2\ ,$$

where we have used (4.3) - (4.10) and (4.18), (4.19). From (5.5) we find $\eta^2+\delta^2 = 0$ which contradicts to the assumption. Consequently, we have $\eta = 0$.

If $\alpha\beta \neq 0$ and $(\alpha-\beta)\delta \neq 0$, then, from (4.3)-(4.14), we have (4.28) and (4.29) and $\alpha\beta+\delta^2 = 0$. Differentiating K^N, we find

$$(5.6)\qquad (3\alpha-\beta)\beta e_i(\alpha) + (\alpha-3\beta)\alpha e_i(\beta) = 0\quad ,\quad i = 1,2\ .$$

Using (4.3), (4.4), (4.7), (4.8), (4.28) and (4.29), we have from (5.6),

$$(5.7)\qquad (5\alpha-3\beta)\lambda_1 = (3\alpha-5\beta)\lambda_2 = 0\ .$$

Since $\alpha\beta+\delta^2 = 0$, $5\alpha-3\beta$ and $3\alpha-5\beta$ are nonzero. Thus, $\lambda_1 = \lambda_2 = 0$. This will give a contradiction. If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta = 0$, then, by the same argument as given in section 4, we also have a contradiction. Thus, we have $(\alpha-\beta)\delta = 0$, i.e., $K^N = 0$. Therefore, by Theorem 5 of Chen [2], M is an open portion of the product surface of two planar circles. The converse of this is clear. (Q.E.D.)

6. SURFACES IN E^4 WITH CONSTANT GAUSS CURVATURE.

THEOREM 4. *Let* M *be a surface which lies essentially in* E^4. *If* M *has pointwise planar normal sections and constant Gauss curvature, then* M *has vanishing Gauss curvature.*

Proof. Let M be a surface which lies essentially in E^4. Assume that M has constant Gauss curvature K and pointwise planar normal sections. We may assume that $\alpha+\beta \neq 0$ by Theorem 4 of [2]. If $(\alpha-\beta)\eta\delta \neq 0$, then, by differentiating K, we have

$$\beta e_i(\alpha) + \alpha e_i(\beta) - 2\eta e_i(\eta) - 2\delta e_i(\delta) = 0 , \quad i = 1,2 . \tag{6.1}$$

Using (4.3) - (4.10), (4.18), (4.19) and (6.1) we find

$$(\alpha\beta - \eta^2 - \delta^2)\lambda_1 = (\alpha\beta - \eta^2 - \delta^2)\lambda_2 = 0 . \tag{6.2}$$

From this, we may conclude that $K = \alpha\beta - \eta^2 - \delta^2 = 0$.

If $(\alpha-\beta)\delta \neq 0$, $\alpha\beta \neq 0$, but $\eta = 0$, then we have (4.28), (4.29) and $\alpha\beta+\delta^2 = 0$. Differentiating $K = \alpha\beta-\delta^2 =$ constant, we have

$$\beta e_i(\alpha) + \alpha e_i(\beta) - 2\delta e_i(\delta) = 0 , \quad i = 1,2 . \tag{6.3}$$

From (4.3), (4.4), (4.6), (4.7), (4.8), (4.10), (4.28) and (4.29), we have

$$(\alpha\beta-\delta^2)\lambda_1 = (\alpha\beta-\delta^2)\lambda_2 = 0 . \tag{6.4}$$

Thus, we have $\alpha\beta-\delta^2 = 0$ which contradicts $\alpha\beta+\delta^2 = 0$. If $(\alpha-\beta)\delta \neq 0$ but $\eta = \alpha\beta = 0$, then by a similar argument as given in section 4, we have a contradiction too.

When $(\alpha-\beta)\delta = 0$, $K^N = 0$. In this case, Theorem 5 of [2] implies that M is an open portion of a flat torus. Thus, $K = 0$. (Q.E.D.)

REFERENCES

[1] B.Y. Chen, *Submanifolds with planar normal sections*, Soochow J. Math., 7 (1981), 19-24.

[2] B.Y. Chen, *Differential geometry of submanifolds with planar normal sections*, Ann. Mat. Pura Appl., 130 (series IV), (1982), 59-66.

[3] B.Y.Chen, *Classification of surfaces with planar normal sections*, J. Geometry, 20 (1983), 122-127.

[4] S.S. Chern, M. DoCarmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields, Springer-Verlag, (1970), 60-75.

[5] Y. Hong, C.S. Houh, and G.Q. Wang, *Some surfaces with pointwise planar normal sections*, to appear.

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SECTIONAL VORONOI TESSELLATIONS

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Dedicated to L.A. Santaló, by way of whose delightful 'Introduction to Integral Geometry' [16] I was first exposed in 1959 to the beauties of geometry and randomness combined.

ABSTRACT. Formulae for the expected mean s-content of s-facet per polytope in the Voronoi random polytopal tessellation V of R^d, with respect to a homogeneous Poisson point process basis, are derived. s-flat sections of V yield a new class of random s-dimensional polytopal tessellations, whose properties are explored for s = 1,2,3.

1. INTRODUCTION & SUMMARY.

The area of random tessellations is an important one in stochastic geometry, and some of the earliest work is due to L.A. Santaló [12, 14,15]. An s-flat section of an ergodic homogeneous and isotropic random polytopal tessellation of R^d is a similar such tessellation in the s-flat as containing space ($1 \leqslant s \leqslant d-1$). The main interest in this paper is in exploring properties of such sectional tessellations.

In section 2, Santaló's basic formula for the expected mean projections of the isotropic uniform random section of a domain, in terms of the mean projections of the domain itself, finds useful application; in particular to sectional tessellations. The most rewarding specific random tessellations as regards sectioning are the Voronoi tessellations V considered in Section 3. An explicit formula for the mean s-content $E\{L_s\}$ of s-facet per polytope of V is derived; the case s=0 gives the mean number of vertices. Sectional Voronoi tessellations are examined in Section 4, with exact mean sectional values being obtained for s = 1,2 and asymptotic ones as $d \to \infty$ for s=3. In fact, an s-section of homogeneous V is stochastically equivalent to an s-section of a corresponding inhomogeneous (s+1)-dimensional structure. In Section 5, this aspect is explored in some detail in the line section case s=1, with an integral expression being given for the interval length distribution. Finally, in Section 6, generalized Voronoi tessellations V_n, involving the nearest n particles to a point, rather analogous formula for $E\{L_s\}$ to that obtained for V in Section 3, and an integral expression for the volume moments in s-sections, are derived.

Some of the results have been stated elsewhere [8,9], but without proofs.

PRELIMINARIES. $Q_d(x,r)$ represents the closed ball with centre x, radius r, in euclidean d-space R^d, with boundary sphere $\partial Q_d(x,r)$. $|\ldots|_m$ is used for appropriate measure, of dimension m, e.g. $|Q_d(x,r)|_d = \upsilon_d r^d$ where $\upsilon_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$, and $|\partial Q_d(x,r)|_{d-1} = \sigma_d r^{d-1}$ where $\sigma_d = 2\pi^{d/2}/\Gamma(\frac{d}{2})$.

2. FLAT SECTIONS OF RANDOM TESSELLATIONS.

The following result is essentially due to Santaló [17; Section 5], but the form we present here is that given in [4; Relation (2.31T)]. Suppose X is a compact subset of R^d, and that $M_i\{X\}$ denotes its mean i-projection, i.e. the mean i-dimensional Lebesgue measure of its orthogonal projection onto an isotropic i-subspace in R^d ($i = 0,\ldots,d$; with $M_o \equiv 1$, $M_d \equiv |X|_d$). For smooth convex bodies, the mean projections equal, apart from constant factors, the quermassintegrals of integral geometry [4; Relation (2.27T)]. Let F_s be an isotropic uniform random (IUR) s-flat hitting X, i.e. governed by restricted and normalized invariant s-flat measure in R^d. Then $X \cap F_s$ is a random s-dimensional compact subset, which has its own set of (random) mean projections $M_j^{(s)}$ *with respect to* F_s *as containing space*, and we have the striking result

$$(2.1) \qquad E\, M_r^{(s)}\{X \cap F_s\} = M_{d-s+r}\{X\}/M_{d-s}\{X\} \qquad (0 \leqslant r \leqslant s \leqslant d) .$$

This extends to a corresponding result relating to a finite aggregate of compact subsets $\{_iX\}$ ($i = 1,\ldots,n$) each $\subset X$, as follows. If the scalar or vector Z is some domain characteristic, then the aggregate mean value of Z is defined as

$$E\{Z\} = n^{-1} \sum_{i=1}^{n} {}_iZ .$$

The (random) sectional mean $\hat{E}\{M_r^{(s)}\}$ for m independent IUR s-flat sections of X is also defined in the obvious way as the sum of the $M_r^{(s)}$ values for each flat/subset intersection, divided by the total number of such intersections; then, as $m \to \infty$, almost surely

$$E\{M_r^{(s)}\} \longrightarrow E\{M_{d-s+r}\}/E\{M_{d-s}\}\ . \tag{2.2}$$

[7; Sections 5,6].

Although this result holds for rather general ${}_iX$, in this paper we shall only be concerned with the specific case where they form a (polytopal) tessellation, i.e. each point of X (apart from boundaries $\partial_i X$) lies in one and only one ${}_iX$ and, apart from edge effects on ∂X, the ${}_iX$ are d-dimensional convex polytopes.

Since X is arbitrary, (2.2) may be extended as an almost sure identity

$$E\{M_r^{(s)}\} = E\{M_{d-s+r}\}/E\{M_{d-s}\} \tag{2.3}$$

for an ergodic homogeneous and isotropic random polytopal tessellation in R^d [9; Section 3.4.6], where $E\{M_i\}$ are ergodic mean polytope values and $E\{M_r^{(s)}\}$ is the corresponding mean value for an arbitrary s-flat section of the tessellation.

CONSISTENCY OF (2.3). These formulae are consistent in the following sense. Write T_d for the random tessellation in R^d, T_s for the sectional random tessellation $T_d \cap F_s$ and T_t for $T_s \cap F_t$, where $t < s$ and $F_t \subset F_s$. Then the values $E\{M_i^{(t)}\}$ for T_t may be obtained either by double application of (2.3), or alternatively by a single application of (2.3) with s=t. Equating these, there results a set of consistency relations between the $E\{M_i^{(t)}\}$.

As an example, consider the random polytopal tessellation $P_d(\rho)$ determined by isotropic Poisson hyperplanes of intensity ρ in R^d, $\mathbf{P}_\rho(d-1,d)$ ([9; Section 3.4.6]; see also [3; Chapter 6]). $\mathbf{P}_\rho(d-1,d)$ is characterized by the property that the number of hyperplanes hitting any compact $X \subset R^d$ has a Poisson $(\rho M_1\{X\})$ distribution [9; Theorem 1]. For $P_d(\rho_d)$

$$E\{M_r\} = 2^r \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d-r}{2}+1)} \left\{\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})\rho_d}\right\}^r \tag{2.4}$$

[9; Relation (62) with t=d, s=r]. Now

$$P_d(\rho_d) \cap F_s = P_s(\rho_s)$$

for which, by (2.3), (2.4) holds with d replaced by s, and

$$\rho_s = \{\Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{d}{2})/\Gamma(\tfrac{s}{2})\Gamma(\tfrac{d+1}{2})\}\rho_d\ .$$

MEAN CROSS-SECTION OF HIT AGGREGATE. Besides the sectional tessellation $\mathcal{T}_s = \mathcal{T}_d \cap F_s$, another quantity of interest is the union U of polytopes of $\mathcal{T}_d$ hit by F_s. We now derive a formula for the mean (d-s)-content, $E\{V_{d-s}\}$, of the intersection of U with orthogonal (d-s)-flats F_{d-s}. Suppose the generic 'f' denotes ergodic densities of polytopes of $\mathcal{T}_d$, in which each polytope has equal weight. Now the 'chance' F_s hits any specific polytope T of $\mathcal{T} \propto M_{d-s}\{T\}$, so that the aggregate of cells hit by F_s has ergodic densities $\propto M_{d-s} f(M_{d-s},.)$. Hence the mean d-volume V_d of each is

(2.5) $$E'\{V_d\} = E\{M_{d-s}\ V_d\}/E\{M_{d-s}\} .$$

For $\mathcal{T}_s$, by (2.3) ,

(2.6) $$E\{V_s\} = E\{V_d\}/E\{M_{d-s}\} .$$

It follows from (2.5), (2.6) that

$$E\{V_{d-s}\} = E\{\text{d-content of U per unit s-content of } F_s\} =$$
$$= E'\{V_d\}/E\{V_s\} = E\{M_{d-s}V_d\}/E\{V_d\} ,$$

which is the expectation of M_{d-s} for a V_d-weighted random member of $\mathcal{T}_d$.

THE POLYTOPAL CHARACTERISTICS $Y_j^{(k)}$. Actually, (2.3) applies to random aggregates of quite general random subsets of R^d. When specializing to tessellations, the facet structure of the polytope boundaries permits (2.3) to be replaced by a larger system of such basic relations. Writing $T_{t,i}$ $(i = 1,\dots,N_t)$ for the N_t t-facets of a convex polytope T, we define

$$Y_j^{(k)}\{T\} = \sum_{i=1}^{N_k} M_j^{(k)}\{T_{k,i}\} .$$

Defining L_r to be the sum of the r-contents of the N_r r-facets of T, we have the special cases

$$Y_r^{(r)} = L_r \quad , \quad Y_r^{(d)} = M_r \quad , \quad Y_o^{(r)} = N_r \qquad (0 \leqslant r \leqslant d) .$$

(2.1) is replaced by the larger system

(2.7) $$E\ Y_r^{(s+u-d)}\{T \cap F_s\} = \kappa_d(s,u) Y_{d-s+r}^{(u)}\{T\}/M_{d-s}^{(d)}\{T\} , \quad (0 \leqslant r \leqslant s+u-d \leqslant s \leqslant d),$$

where

$$\kappa_d(s,u) = \Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{u+1}{2})/\Gamma(\tfrac{s+u-d+1}{2})\Gamma(\tfrac{d+1}{2})$$

[7; Section 10]. As (2.1) becomes (2.3) for a random tessellation, so

(2.7) must be replaced by the same relation having two expectations on the right side. An important formula for convex polytopes is

$$(2.8) \qquad M_s\{T\} = \{\Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{d-s+1}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}\sum_{i=1}^{N_s} L_{s,i}\psi_{s,i} \quad ,$$

where $\psi_{s,i}$ is the normalized (so that the total angle at an s-facet, in the orthogonal (d-s)-subspace, is 1) exterior angle at the s-facet $T_{s,i}$ [4; Relation (2.18T)].

3. VORONOI TESSELLATIONS.

In geometrical statistical applications, it is desirable to have a variety of specific random tessellations, for modelling purposes. A natural source of such models are three dimensional flat sections of higher dimensional tessellations. As we have just seen, sectioning P tessellations leads to nothing new. However, this is not the case for the other basic class of specific tessellations, the Voronoi (sometimes Thiessen, or Dirichlet) tessellations. We now determine basic properties of Voronoi tessellations, before considering their flat sections in Section 4.

The basic building block for a Voronoi tessellation is an underlying stochastic point process. For simplicity, we shall simply take the latter as the homogeneous Poisson point process $\mathbf{P}_\rho(0,d)$ of intensity ρ in R^d, for which the number of point *particles* falling in any measurable set X has a Poisson $(\rho|X|_d)$ distribution, and realizations in disjoint sets are mutually independent. Each point $x \in R^d$ has an (almost surely well-defined) nearest particle of $\mathbf{P}_\rho(0,d)$. The set of all x with the same nearest particle is (almost surely) the intersection of a finite number of (open) halfspaces in mutual general position, and so is a (simple convex) polytope T_x [1;p.58]; $x \in T_x$ and may be regarded as its *nucleus* (particle). Being simple, every s-facet of T_x lies in the boundaries of $\binom{d-s}{d-t}$ t-facets of T_x $(0 \leqslant s \leqslant t \leqslant d)$.
The aggregate of such polytopal *cells* constitutes a random tessellation $V = V(d)$ of R^d, which is ergodic, homogeneous and isotropic. $V(1)$ is a sequence of random intervals in R^1. It is easily analysed, with the interval distribution being $\Gamma(2,2\rho)$, i.e. the distribution of the sum of two independent exponential (2ρ) random variables. For discussions of $V(2)$ and $V(3)$, the reader may consult [2] and [8], respectively.

BASIC (ALMOST SURE) PROPERTIES OF V. As with all polytopal tessellations, each (d-1)-facet bounds two cells, but in this case it is a por the two associated nuclei. More generally, each s-facet lies in the boundaries of d-s+1 cells (s = 0,...,d-1): tessellations having this property we call *normal*, because real-life tessellations for d = 1,2,3 commonly possess this property. Moreover, for V, each s-facet is a portion of the s-flat all of whose points are equidistant from the associated d-s+1 nuclei. In particular, each vertex (0-facet) is a vertex of d+1 cells and is the circumcentre of the circumsphere through the associated d+1 nuclei.

Now for some notation. For particles $x_o,\dots,x_{d-s}$ in general position in R^d,

$$F_s = \{y: |yx_o|_1 = |yx_1|_1 = \dots = |yx_{d-s}|_1\}$$

is the equidistant s-flat (s = 0,...d-1). $y \in F_s$ lies in the common s-facet of the cells with nuclei $x_o,\dots,x_{d-s}$ iff the unique d-sphere centre y through $x_o,\dots,x_{d-s}$ contains no other particles of $\mathbf{P}_\rho(0,d)$.

THE VALUE OF $E\{L_s\}$ FOR $V(d)$. If obvious interest are the ergodic distributions and moments of characteristics of the members of $V(d)$. Writing $V_d \equiv L_d^{(d)}$, one obvious one is

$$E\{V_d\} = \rho^{-1} , \tag{3.1}$$

true whatever the underlying (ergodic) stochastic point process. We shall now derive the values of the other $E\{L_s\}$, and apply them in investigating sectional Voronoi tessellations (Section 4).

The method relies heavily on a re-parametrization of $x_o,\dots,x_{d-s}$ - supposed to have general position in R^d - which lie in a unique (d-s)-flat F_{d-s}. Write ∇_{d-s} for (d-s)! times the (d-s)-content of the (d-s)-simplex with vertices $x_o,\dots,x_{d-s}$, and suppose $Q_{d-s}(z,R)$ is the unique (d-s)-sphere through $x_o,\dots,x_{d-s}$. Then we have, in polar coordinates within F_{d-s},

$$\vec{zx_i} = R\, u_i^{(d-s)}$$

for unit vectors $u_i^{(d-s)}$; write $dO_i^{(d-s)}$ for the volume element of a unit sphere in F_{d-s} corresponding to $u_i^{(d-s)}$ (i = 0,...,d-s). Finally, write $F_{s(0)}$ for the s-subspace orthogonal to F_{d-s}, and parallel to $F_s = \{z\} + F_{s(0)}$.

Now the (d-s+1)-set $x_o,\dots,x_{d-s}$ is alternatively parametrized by

$$z,\ R,\ F_{s(0)},\ u_o^{(d-s)},\dots,u_{d-s}^{(d-s)}$$

and we have the corresponding integral geometric density relationship

$$(3.2)\qquad dx_o \ldots dx_{d-s} = \nabla_{d-s}^{s+1} R^{d(d-s)-1}\, dz\, dR\, dF_{s(0)}\, dO_o^{(d-s)} \ldots dO_{d-s}^{(d-s)},$$

due to Blaschke & Petkantschin [9; Relation [74)]. Next, we express points y in F_s in terms of polar coordinates $(S, v^{(s)})$ within F_s with respect to z as origin, so that

$$|y\, x_i|_1 = (R^2 + S^2)^{1/2} \equiv T \qquad (i = 0,\ldots d-s)$$

say, and

$$\begin{aligned}(3.3)\qquad & \Pr\{y \in \text{associated s-facet of } V | \text{particles at } x_o,\ldots,x_{d-s}\} \\ &= \Pr\{\text{int } Q_d(y,T) \text{ contains no particles}\} \\ &= \exp(-\rho\, \upsilon_d T^d)\ .\end{aligned}$$

The probability element for particles of $\mathbf{P}_\rho(0,d)$ in $dx_o,\ldots,dx_{d-s}$ is $\rho^{d-s+1}\, \Pi_{i=1}^{d-s}\, dx_i$, so that the probability element for a (d-s+1)-set of particles within limitations dz, dR, $dF_{s(0)}, dO_o^{(d-s)},\ldots,dO_{d-s}^{(d-s)}$ is

$$\rho^{d-s+1}\, \nabla_{d-s}^{s+1}\, R^{d(d-s)-1}\, dz\, dR\, dF_{s(0)}\, dO_o^{(d-s)} \ldots dO_{d-s}^{(d-s)}\ .$$

Now consider the contribution ℓ_s from given particles at $x_o,\ldots x_{d-s}$ to the total s-facet content. We may write

$$\ell_s = \iint I(S,v^{(s)}) S^{s-1}\, dS\, dO^{(s)}$$

where $I(S,v^{(s)})$ indicates that $(S,v^{(s)})$ lies in an s-facet of V. Hence, by the complete independence of Poisson point processes,

$$\begin{aligned}(3.4)\qquad & E\{\ell_s | \text{particles at } x_o,\ldots,x_{d-s}\} \\ &= \iint E\{I(S,v^{(s)})\} S^{s-1}\, dS\, dO^{(s)} \\ &= \sigma_s \int \exp(-\rho\upsilon_j T^d)\, S^{s-1}\, dS\ .\end{aligned}$$

It follows from (3.3) and (3.4) that

$$(3.5)\qquad E\{\ell_s \text{ from particle (d-s+1)-sets with circumcentre in dz}\} =$$

$$= \frac{\rho^{d-s+1} dz}{(d-s+1)!} \underbrace{\int dF_{s(0)}}_{J_1} \underbrace{\int_0^\infty\!\!\int_0^\infty R^{d(d-s)-1}\, S^{s-1}\, \exp(-\rho\upsilon_d T^d) dR\, dS}_{J_2}$$

$$\underbrace{\int \cdots \int \nabla_{d-s}^{s+1}\, dO_o^{(d-s)} \ldots dO_{d-s}^{(d-s)}}$$

$$\equiv \chi_s \, dz \ ,$$

say. The (d-s+1)! factor arises because with total integration every particle (d-s+1)-set is counted this many times. By [6; Relation (12)]

$$J_1 = \sigma_{d-s+1} \cdots \sigma_d / \sigma_1 \cdots \sigma_s \ , \tag{3.6}$$

while

$$J_2 = \frac{\Gamma(d-s+\frac{s}{d})}{d(\rho\upsilon_d)^{d-s+(s/d)}} \cdot \frac{\Gamma(\frac{d(d-s)}{2})\Gamma(\frac{s}{2})}{2\,\Gamma(\frac{d(d-s)+s}{2})} \ . \tag{3.7}$$

As for J_2, this is σ_{d-s}^{d-s+1} times the mean value of V_{d-s}^{s+1} for d-s+1 particles chosen independently and uniformly on the unit sphere in R^{d-s}. Its value,

$$J_3 = \frac{\Gamma(\frac{d^2-sd+s+1}{2})}{\Gamma(\frac{d^2-sd}{2})} \left\{\frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{d+1}{2})}\right\}^{d-s} \frac{\Gamma(\frac{s+2}{2})\ldots\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\ldots\Gamma(\frac{d-s-1}{2})}\,\sigma_{d-s}^{d-s+1} \ , \tag{3.8}$$

is derived in [6; Theorem 2], essentially by manipulation of the basic Blaschke-Petkantschin formula (3.2). Now χ_s in (3.5) is the average s-content of s-facet per unit d-content of R^d. Hence, since each s-facet is an s-facet of d-s+1 distinct cells of V, we have

$$E\{L_s\} = (d-s+1)\,\chi_s\,E(V_d) \tag{3.9}$$

which, by (3.1) and (3.5) - (3.8) ,

$$= \frac{2^{d-s+1}\pi^{(d-s)/2}\Gamma(\frac{d^2-sd+s+1}{2})\Gamma(\frac{d}{2}+1)^{d-s+(s/d)}\Gamma(d-s+\frac{s}{d})}{(d-s)!\ d\Gamma(\frac{d^2-sd+s}{2})\ \Gamma(\frac{d+1}{2})^{d-s}\ \Gamma(\frac{s+1}{2})\ \rho^{s/d}} \qquad (0 \leqslant s \leqslant d).$$

Special cases are, for s=d, (3.1) and, for s=0, the mean number of vertices

$$E\{N_o\} = \frac{2^{d+1}\pi^{\frac{d-1}{2}}}{d^2}\,\frac{\Gamma(\frac{d^2+1}{2})}{\Gamma(\frac{d^2}{2})}\left\{\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})}\right\}^d \qquad (d = 1,2,\ldots) \ .$$

No other ergodic distributions or moments of V are known. Obvious targets are formulae for $E\{M_s\}$ and $E\{N_s^{\cdot}\}$.

Each vertex of V is the circumcentre of a set of d+1 particles of $\mathbf{P}_\rho(0,d)$, the convex hull of which is a simplex. It turns out that the aggregate of such simplices is a random tessellation - the Delaunay tessellation [11]. Its ergodic distribution and the values of $E\{V_d^k\}$

are derived in [9; Relations (76),(77)].

4. SECTIONAL VORONOI TESSELLATIONS.

Our main concern in this paper is with the sectional Voronoi tessellations

$$V(s,d) = V(d) \cap F_s$$

for an arbitrary s-flat F_s. Note that, in this notation, $V = V(d) = V(d,d)$. Providing the intersections are nonvoid, F_s intersects cells of V in simple s-polytopes and t-facets in (s+t-d)-facets. As expected, each such (s+t-d)-facet lies in the boundaries of $s-(s+t-d)+1 = d-t+1$ cells of $V(s,d)$. Thus, topologically, $V(s,d)$ has the same properties relative to F_s as V has relative to R^d, and is a normal tessellation.

We now investigate the application of (2.3) to $V(s,d)$.

<u>s = 1</u>: Hence F_1 intersects the polytope boundaries of V in an ergodic stationary (= homogeneous) stochastic point process. $V(1,d)$ comprises the intervals so formed, and the obvious goal here is to determine the (ergodic) interval length (L) distribution. (2.3) reduces to one relation, viz.

$$E\{L\} = E\{V_d\}/E\{M_{d-1}\} \tag{4.1}$$

which, by [4; Relation (2.21)],

$$= \{2\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})/\Gamma(\tfrac{d}{2})\}\ E(V_d)/E(L_{d-1})$$

which, by (3.9),

$$= \frac{\Gamma(d-\frac{1}{2})\cdot\Gamma(\frac{d+1}{2})^2}{(d-1)!\,\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)^{1-(1/d)}\Gamma(2-\frac{1}{d})\rho^{1/d}}\ .$$

Note that, as $d \to \infty$,

$$E\{L\} \to (2e)^{-\frac{1}{2}} = 0.4289\ .$$

This limiting process is examined in closer detail, and an integral expression for the distribution of L is given, in the next section.

<u>s = 2</u>: Here $V(2,d)$ is a planar tessellation, and (2.3) yields the two relations

$$E\{A\} = E\{V_d\}/E\{M_{d-2}\}\ , \tag{4.2}$$

$$\pi^{-1}E\{B\} = E\{M_{d-1}\}/E\{M_{d-2}\} \tag{4.3}$$

(A = area, B = perimeter). Simple geometric considerations in a 2-flat orthogonal to any (d-2)-facet show that the sum of the three exterior angles there is 1/2. It follows from (2.8) that, for V ,

$$E\{M_{d-2}\} = E\{L_{d-2}\}/6(d-1)$$

and so (4.2) , (4.3) become

$$E\{A\} = \frac{3\ d\ \Gamma(\frac{3d}{2}-1)\ \Gamma(\frac{d+1}{2})^3}{\pi\Gamma(\frac{3d-1}{2})\ \Gamma(\frac{d}{2}+1)^{3-(2/d)}\ \Gamma(3-\frac{2}{d})\ \rho^{2/d}}, \tag{4.4}$$

$$E\{B\} = \frac{6(d-1)!\ \Gamma(\frac{d+1}{2})\Gamma(\frac{3d}{2}-1)\ \Gamma(2-\frac{1}{d})}{\Gamma(d-\frac{1}{2})\ \Gamma(\frac{d}{2}+1)^{1-(1/d)}\ \Gamma(\frac{3d-1}{2})\ \Gamma(3-\frac{2}{d})\ \rho^{1/d}}. \tag{4.5}$$

Of course, because each vertex in $V(2,d)$ is vertex of three polygons, the mean number of vertices

$$E\{N\} = 6\ .$$

As $d \longrightarrow \infty$,

$$E\{A\} \longrightarrow 3^{\frac{1}{2}}/\pi e = 0.2028$$

and

$$E\{B\} \longrightarrow (6/e)^{\frac{1}{2}} = 1.486\ .$$

The reader may check the consistency of (4.1) and (4.4), (4.5), by considering a line section of $V(2,d)$.

The dimensionless aggregate polygon rotundness measure

$$\theta = 4\pi\ E\{A\}/E\{B\}^2\ ,$$

as a function of d, is of interest. As d increases from 2 to ∞, it increases from 0.785 to 1.155, suggesting that the polygons become more rotund on average as d increases (cf. [5; p.119]).

<u>s = 3</u>: Here application of (2.3) (and (2.7)) yields

$$E\{V\} = E\{V_d\}/E\{M_{d-3}^{(d)}\} \tag{4.6}$$

$$E\{S\} = \{2\Gamma(\tfrac{d}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}E\{L_{d-1}^{(d)}\}/E\{M_{d-3}^{(d)}\} \tag{4.7}$$

$$E\{M_1^{(3)}\} = E\{M_{d-2}^{(d)}\}/E\{M_{d-3}^{(d)}\} \tag{4.8}$$

$$E\{L_1^{(3)}\} = \{2/(d-1)\}\ E\{L_{d-2}^{(d)}\}/L\{M_{d-3}^{(d)}\} \tag{4.9}$$

$$E\{N_0^{(3)}\} = \{\Gamma(\tfrac{d-2}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}E\{L_{d-3}^{(d)}\}/E\{M_{d-3}^{(d)}\}\ , \tag{4.10}$$

where S = surface area. Note that, by Euler's formula and the fact that each vertex lies in three faces, $N_1^{(3)} = (3/2)N_o^{(3)}$ and $N_2^{(3)} =$ $= (N_o^{(3)}) + 2$. Unfortunately, there appears to be no geometrical identity governing the exterior angles at (d-3)-facets, allowing $E\{M_{d-3}^{(d)}\}$ to be determined by means of (2.8). However, we know from (3.3) with $s = d-3$ that the conditional orientation density of the four particles generating a (d-3)-facet of V is proportional to ∇_3^{d-2} on the orthogonal 3-sphere. As $d \to \infty$, this distribution tends to degeneracy, in which the four particles form an (isotropically oriented) equilateral tetrahedron. Consider now the consequences for the interior and exterior angles at (d-3)-facets of V. The interior angles actually correspond to the spherical Voronoi division of the 3-sphere generated by the particle orientations, and so each tends to 1/4. The exterior angles are those of the dual regions on the 3-sphere [13; p.708]; by this duality

$$A + B^* = A^* + B = \frac{1}{2} \cdot$$

From this it follows that each exterior angle tends to

$$\psi = \frac{1}{8} - \frac{3}{4\pi} \sin^{-1}(\tfrac{1}{3}) ,$$

so that, by (2.8), as $d \to \infty$,

$$E\{M_{d-3}^{(d)}\}/E\{L_{d-3}^{(d)}\} \to \{\Gamma(\tfrac{d}{2}-1)/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}\ \psi \quad .$$

Application of this and other formulae to (4.6) - (4.10) yields

$$\begin{aligned}
&E\{V\} \to 1/16\pi e^{3/2}\ \psi && = 0.1012\\
&E\{S\} \to 1/2^{3/2}\pi e\ \psi && = 0.9437\\
&E\{M_1^{(3)}\} = E\{L_1^{(3)}\}/12 \to 1/16(3e)^{1/2}\ \psi && = 0.4989\\
&E\{N_2^{(3)}\} \to 2 + (1/2\psi) && = 13.39\\
&E\{N_1^{(3)}\} \to 3/2\ \psi && = 34.19\\
&E\{N_o^{(3)}\} \to 1/\psi && = 22.79
\end{aligned}$$

as $d \to \infty$. Real-life observational and experimental models have indicated the common ocurrence of random normal tessellations with values of $E\{N_2^{(3)}\}$ between 13 and 15. Thus, assuming that $E\{N_2^{(3)}\}$ for $V(3,d)$ decreases monotonically from 15.54 to 13.39 as d increases from 3 to ∞, the random tessellations $\{V(3,d)\}$ $(d = 3,4,...)$ may be advanced as natural stochastic models for these phenomena. For further details, see [8; Section 6].

<u>$s \geq 4$</u>: The above cases s = 1,2,3 are those of obvious practical signi-

carried out for general s, with its asymptotics involving an equilateral (s+1)-simplex inscribed in an s-sphere.

5. REDUCED DIMENSION STOCHASTIC EQUIVALENCE.

The effect of any particle x in $\mathbf{P}_\rho(0,d)$ on $V(s,d)$ is only by way of its nearest point y of F_s and the distance $|xy|_1$. Hence $V(s,d)$ is stochastically equivalent to $V(s,s+1)$ with respect to a new $\mathbf{P}(0,s+1)$ density which is that of $\mathbf{P}_\rho(0,d)$ collapsed by rotation onto a half-F_{s+1} with bounding s-flat F_s. This density is $\rho\ \sigma_{d-s} r^{d-s-1}$, where r denotes orthogonal distance from F_s; note that it is inhomogeneous. We illustrate this stochastic equivalence in the case s=1 by considering, in the first instance, the case in which the particles form an inhomogeneous Poisson process in R^2 of intensity $\rho(y)$ $(y \geqslant 0)$, with $F_1 =$ = the x-axis.

We begin by exploring the joint distribution of two particles in R^2, given that they give rise, as the intersection of their perpendicular bisector with Ox, to an endpoint H of the interval process $V(1,2)$ on Ox. The necessary and sufficient condition for this to occur is shown in Fig.1.

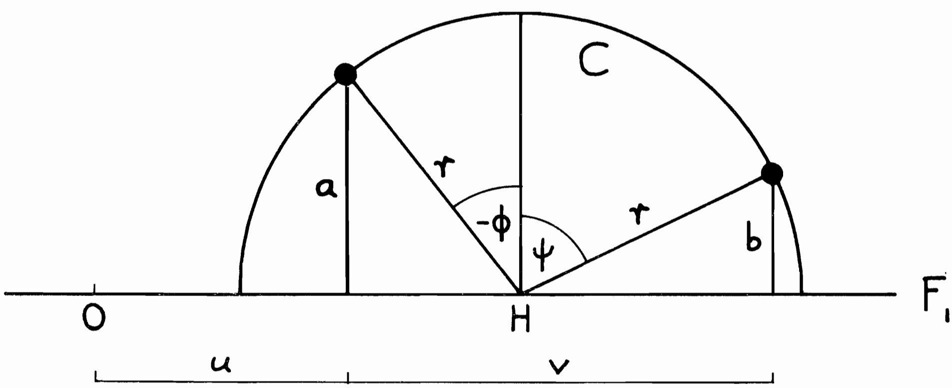

Fig.1. Geometry of an interval end point H of $V(1,d)$.

That is, there are two particles on the semicircle C with centre H, radius r, and none within C. We suppose those particles have angular coordinates $\phi,\psi(-\pi/2 \leqslant \phi \leqslant \psi \leqslant \pi/2)$ with respect to the orthogonal to Ox at H, and are interested in the joint distribution of $(r;\phi,\psi)$ given that the two particles give rise to H. The method is elementary, and uses the alternative coordinates shown in Fig.1, i.e. particles at (u,a) and (u+v,b) (a,v,b > 0). We have

(5.1) Pr{particles in (du,da) and ($d\overline{u+v}$,db), and none in int C} =

$$= \rho(a)\ \rho(b)\ du\ da\ dv\ db\ \exp\{-2\int_0^r \rho(y)(r^2-y^2)^{1/2}dy\}\ .$$

Integrating this respect to u over an interval of unit length, we obtain $E\{L\}^{-1} f(a,v,b)\ da\ dv\ db$, where the joint density f(a,v,b) relates to a random such configuration on F_1. (Here, and below, f(*) means 'density of *'). Thus, transforming to the polar coordinates (r,ϕ,ψ) (Fig.1), we have the ergodic density

$$(5.2) \qquad f(r,\phi,\psi) \propto \rho(r\cos\phi)\rho(r\cos\psi)\ r^2(\sin\psi - \sin\phi)$$
$$\exp\{-2\int_0^r \rho(y)(r^2-y^2)^{1/2}\ dy\}\ .$$

We now specialise to $V(1,d)$, for which $\rho(y) = \rho\sigma_{d-1}y^{d-2}$. Substitution of this in (5.2) shows that r and (ϕ,ψ) are independent, with normalized marginal probability densities

$$f(r) = \frac{d}{\Gamma(2-\frac{1}{d})}\left\{\frac{\rho\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\right\}^{2-\frac{1}{d}} r^{2d-2} \exp\left\{-\frac{\rho\pi^{d/2}r^d}{\Gamma(\frac{d}{2}+1)}\right\} \qquad (r \geqslant 0)\ ,$$

$$f(\phi,\psi) = \frac{(2d-2)!}{2^{2d-1}\ (d-2)!^2}\ (\cos\phi\cos\psi)^{d-2}\ (\sin\psi - \sin\phi) \quad (-\tfrac{\pi}{2}\leq\phi\leq\psi\leq\tfrac{\pi}{2}).$$

A swift integration gives

$$E\{r\} = \frac{1}{\Gamma(2-\frac{1}{d})}\left\{\frac{\Gamma(\frac{d}{2}+1)}{\rho\pi^{d/2}}\right\}^{\frac{1}{d}} \sim (d/2\pi e)^{1/2} \quad \text{as} \quad d \longrightarrow \infty\ ,$$

while the mean projected particle separation onto Ox is

$$E\{r(\sin\psi - \sin\phi)\} = E\{r\}.E\{\sin\psi - \sin\phi\} = E\{L\}$$

given in (4.1).

To investigate the limiting behaviour of the distributions of r, (ϕ,ψ), we consider the new variables

$$R = \{(2e)^{\frac{d}{2}}\ \pi^{\frac{d-1}{2}}\ /\ d^{\frac{d+1}{2}}\}\ r^d$$

$$\alpha = d^{1/2}\ \phi \quad , \quad \beta = d^{1/2}\ \psi\ .$$

Then it is easily shown that, as $d \longrightarrow \infty$,

(i) the distribution of $R \to \Gamma(2,\rho)$, so that $r/E\{r\} \to 1$ in probability;

$$f(\alpha,\beta) \to (2\pi^{\frac{1}{2}})^{-1}(\beta-\alpha)\exp\{-\tfrac{1}{2}(\alpha^2+\beta^2)\} \qquad (-\infty < \alpha \leqslant \beta < \infty) \quad ,$$

with limiting marginal density

$$f(\alpha) = (2\pi^{\frac{1}{2}})^{-1}[e^{-\alpha^2} - (2\pi)^{\frac{1}{2}}\alpha e^{-\alpha^2/2}\{1-\Phi(\alpha)\}] \qquad (-\infty < \alpha < \infty).$$

For the marginal of β, note that β and $-\alpha$ have the same distribution. Note also that, as $d \to \infty$, the projection of the segment joining the two particles onto $F_1 \sim r(\psi-\phi) \sim (2\pi e)^{-\frac{1}{2}}\ (\beta-\alpha)$.

This approach may be extended to two adjacent semicircles, resulting in an integral expression for the distribution of interval length L in $V(1,d)$.

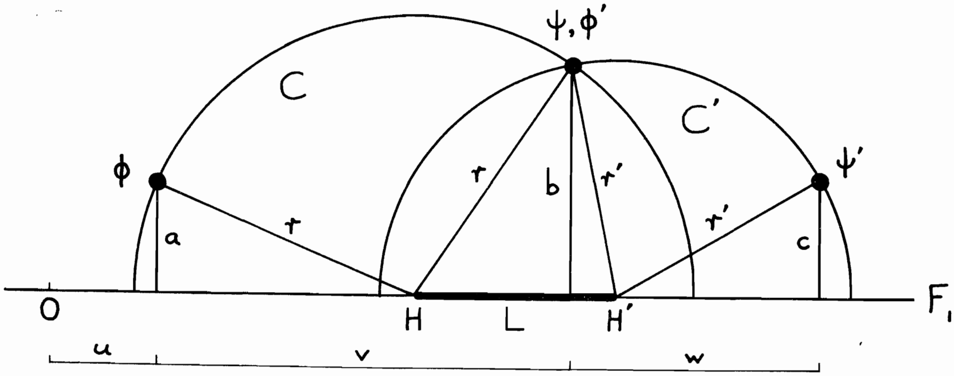

Fig.2. Geometry of an interval HH' of $V(1,d)$.

Fig.2 shows the geometry. We have particles P_1,P_2,P_3 at the points (u,a), $(u+v,b)$ and $(u+v+w,c)$, respectively $(a,b,c,v,w > 0)$. P_1 and P_2 determine the semicircle C with centre $H \in F_1$, as above, and likewise P_2 and P_3 determine the semicircle C' with centre H' also $\in F_1$. The Voronoi geometry requires that there are no other particles within $U = C \cup C'$, so that HH', of length L, is a typical interval of $V(1,d)$. Then, with Poisson intensity $\rho(y)$, the analogue of (5.1) is

$$\Pr\{\text{particles in } (du,da),\ (d\overline{u+v},db) \text{ and } (d\overline{u+v+w},dc)\ ,$$
$$\text{and none in int } U\}$$
$$= \rho(a)\rho(b)\rho(c)\ du\ da\ dv\ db\ dw\ dc \quad \exp\{-\int_0^{\max(r,r')} \rho(y)\ell(y)\ dy\ ,$$

where $\ell(y)$ is the length of the intersection of a line parallel to, and distant y from, F_1 with U, and r,r' are the radii of C,C'. Again integration with respect to u over a unit interval gives

$E\{L\}^{-1}$ $f(a,b,c,v,w)\,da\,db\,dc\,dv\,dw$. Next we switch from (a,b,c,v,w) to $(L,\phi,\psi,\phi',\psi')$, where

P_1 has polar coordinates (r,ϕ) with respect to H

P_2 has polar coordinates $\begin{cases}(r,\psi) \text{ with respect to H} \\ (r',\phi') \text{ with respect to H'}\end{cases}$

P_3 has polar coordinates (r',ψ') with respect to H'

(fig.2, cf. also Fig.1). The transformation relations are

$$a = L\cos\phi\cos\phi'/\sin(\psi-\phi')$$
$$b = L\cos\psi\cos\phi'/\sin(\psi-\phi')$$
$$c = L\cos\psi\cos\psi'/\sin(\psi-\phi')$$
$$v = L\cos\phi'(\sin\psi-\sin\phi)/\sin(\psi-\phi')$$
$$w = L\cos\psi\,(\sin\psi'-\sin\phi')/\sin(\psi-\phi')$$

with

$$\frac{\partial(a,b,c,v,w)}{\partial(L,\phi,\psi,\phi',\psi')} = \frac{L^4(\cos\psi\cos\phi')^2}{\sin\phi\sin\psi'\sin^7(\psi-\phi')}\,.$$

$$.\ \{\cos\psi\cos(\psi-\phi')+\sin\phi\sin(\psi-\phi')-\cos\phi'\}\{\cos\phi'\cos(\psi-\phi')-$$
$$-\sin\psi'\sin(\psi-\phi')-\cos\psi\}.$$

Thus in principle we have the joint density

$$f(L,\phi,\psi,\phi',\psi') = E\{L\}\left|\frac{\partial(a,b,c,v,w)}{\partial(L,\phi,\psi,\phi',\psi')}\right|\ \rho(a)\rho(b)\rho(c)$$
$$\exp\{-\int_0^{\max(r,r')}\rho(y)\ell(y)\,dy\}$$

where

$$r = L\cos\phi'/\sin(\psi-\phi')\quad,\quad r' = L\cos\psi/\sin(\psi-\phi')\,.$$

Finally, the marginal density of L results on integrating $f(L,\phi,\psi,\phi',\psi')$ over the (ϕ,ψ,ϕ',ψ')-set

$$[-\pi/2 \leqslant \phi \leqslant \psi \leqslant \pi/2] \cap [-\pi/2 < \phi' \leqslant \psi' < \pi/2] \cap [\phi' \leqslant \psi]\,.$$

In the $V(1,d)$ case, when $\rho(y) = \rho\sigma_{d-1}y^{d-2}$, E{L} is given by (4.1) and, as may be anticipated from Fig.2, the integrations with respect to ϕ (from $-\pi/2$ to ψ) and ψ' (from ϕ' to $\pi/2$) are elementary, being finite series in closed form.

6. GENERALIZED VORONOI TESSELLATIONS.

to a *single* particle of the underlying point process. However, each point of R^d (almost surely) has a well-defined set of nearest n particles (n = 2,3,...). Similarly the set of points with the same nearest n particles constitutes a simple convex polytope, and the aggregate of such polytopal cells is a generalized Voronoi tessellation V_n of R^d (see [5,10] for discussions of the case d=2); thus $V = V_1$. Like V and $V(s,d)$, V_n is a normal random tessellation.

One piece of the previous theory extends effortlessly to V_n. We have (cf. Section 3 for notation): for particles $x_0,\dots,x_{d-s}$ a point $y \in F$, lies in an associated s-facet of V_n iff int $Q_d(y,T)$ contains n-1 particles of $\mathbf{P}_\rho(0,d)$, an event of probability

$$(\rho\upsilon_d T^d)^{n-1} \exp(-\rho\upsilon_d T^d)/(n-1)!\ .$$

With this modification, the theory of Section 3 carries over, to yield (suffix n $\sim V_n$)

$$\frac{E_n\{L_s\}}{E_n\{V_d\}} = \frac{\Gamma(d+n-s-1+\frac{s}{d})}{(n-1)!\,\Gamma(d-s+\frac{s}{d})}\ \frac{E\{L_s\}}{E\{V_d\}} =$$

$$= \frac{2^{d-s+1}\,\pi^{\frac{d-s}{2}}\,\Gamma(\frac{d^2-sd+s+1}{2})\,\Gamma(\frac{d}{2}+1)^{d-s+\frac{s}{d}}\,\Gamma(d+n-s-1+\frac{s}{d})}{(n-1)!\ (d-s)!\ d\Gamma(\frac{d^2-sd+s}{2})\,\Gamma(\frac{d+1}{2})^{d-s}\,\Gamma(\frac{s+1}{2})}\,\rho^{1-\frac{s}{d}}\ .$$

Unfortunately, $E_n\{V_d\}$ is only known in two cases, viz.

$$E_n\{V_1\} = 1/\rho\ ,$$

$$E_n\{V_2\} = 1/(2n-1)\rho \qquad [5;\ \text{Theorem } 10.1]\ .$$

Apart from this it is known that, because of the one-to-one correspondence between (d-1)-facets of V and cells of V_2,

$$E_2\{V_d\} = 2\ E\{V_d\}/E\{N_{d-1}\}\ ,$$

where of course the value of $E\{N_{d-1}\}$ is also unknown.

We may also section V_n, and naturally write

$$V_n(s,d) = V_n(d,d) \cap F_s\ .$$

It is possible to write down an integral expression for the ergodic moments of V_s for $V_n(s,d)$, as we now show. Select an arbitrary point $0 = x_o$ as origin in F_s. It lies in a random polytope T_o of $V_n(s,d)$ whose distribution is that of a uniform random member of $V_n(s,d)$ weighted by V_s. Thus the kth order moment of $V_s(T_o)$ is

$$E_{n,V_s}\{V_s^k\} = E_n\{V_s^{k+1}\}/E_n\{V_s\} .$$

Now we may write

$$V_s(T_o) = \int_{F_s} I(x)\, dx$$

where I(x) indicates that $x \in T_o$. Thus, in the usual way,

$$E_{n,V_s}\{V_s^k\} = E \int_{F_s} \cdots \int_{F_s} I(x_1)\ldots I(x_k)\, dx_1 \ldots dx_k$$

$$= \int_{F_s} \cdots \int_{F_s} E\{I(x_1)\ldots I(x_k)\}\, dx_1 \ldots dx_k$$

$$= \int_{F_s} \cdots \int_{F_s} \Pr\{x_1,\ldots,x_k \text{ all } \in T_o\}\, dx_1 \ldots dx_k .$$

Now

$$\Pr\{x_1,\ldots,x_k \text{ all } \in T_o\} = \Pr\{x_o,\ldots,x_k \in \text{some cell of } V_n\}$$

and the latter event occurs iff there are particles of $\mathbf{P}_\rho(0,d)$ at $y_1,\ldots,y_n$ and

$$U = \operatorname{int} \bigcup_{i=0}^{k} \bigcup_{j=1}^{n} Q_d(x_i, |y_j - x_i|_1)$$

contains no particles of $\mathbf{P}_\rho(0,d)$. Thus

$$E_{n,V_s}\{V_s^k\} = \int_{F_s} \cdots \int_{F_s} \int_{R^d} \cdots \int_{R^d} e^{-\rho|U|_d}\, dy_1 \ldots dy_n\, dx_o \ldots dx_k .$$

Further progress seems unlikely, given the complex nature of the ball union U.

REFERENCES

[1] B. GRÜNBAUM, *Convex Polytopes*, Wiley, New York, 1967.

[2] A.L. HINDE & R.E. MILES, *Monte Carlo estimates of the distributions of the random polygons of the Voronoi tessellation with respect to a Poisson process*, J.Static.Comput.Simul. 10, 1980, 205-223.

[3] G. MATHERON, *Random Sets and Integral Geometry*, Wiley, New York, 1975.

[4] R.E. MILES, *Poisson flats in Euclidean spaces, Part I: A finite number of random uniform flats*, Adv.Appl.Prob. 1, 1969, 211-237.

[5] R.E. MILES, *On the homogeneous planar Poisson point process*, Math.Biosciences 6, 1970, 85-127.

[6] R.E. MILES, *Isotropic random simplices*, Adv.Appl.Prob. 3, 1971, 353-382.

[7] R.E. MILES, *Multidimensional perspectives on stereology*, J.Microsc. 95, 1972, 181-196.

[8] R.E. MILES, *The random division of space*, Suppl.Adv.Appl.Prob. 4, 1972, 243-266.

[9] R.E. MILES, *A synopsis of 'Poisson flats in Euclidean spaces'*, Pp.202-227 in 'Stochastic Geometry' (ed. E.F. Harding & D.G. Kendall), Wiley, London, 1974.

[10] R.E. MILES & R.J. MAILLARDET, *The basic structures of Voronoi and generalized Voronoi polygons*, 'Essays in Statistical Science' (ed. J. Gani & E.J. Hannan), Applied Probability Trust, J. Appl.Prob. 19A, 1982, 97-111.

[11] C.A. ROGERS, *Packing and Covering*, Cambridge U.P. (Math.Tract N°54), 1964.

[12] L.A. SANTALÓ, *Valor medio del número de partes en que una figura convexa es dividida por n rectas arbitrarias*, Rev.Unión Mat. Argentina 7, 1941, 33-37.

[13] L.A. SANTALÓ, *Integral formulas in Crofton's style on the sphere and some inequalities referring to spherical curves*, Duke Math,J. 9, 1942, 707-722.

[14] L.A. SANTALÓ, *Valor medio del número de regiones en que un cuerpo del espacio es dividido por n planos arbitrarios*, Rev.Unión Mat.Argentina 10, 1945, 101-108.

[15] L.A. SANTALÓ, *Sobre la distribución de planos en el espacio*, Rev.Unión Mat.Argentina 13, 1948, 120-124.

[16] L.A. SANTALÓ, *Introduction to Integral Geometry*, Hermann, Paris (Act.Sci.Indust. N°1198) 1953.

[17] L.A. SANTALÓ, *Sur la mesure des espaces linéaires qui coupent un corps convexe et problèmes qui s'y rattachent, Colloque sur les questions de réalité en géométrie*, Liège 177-190, Georges Thone, Liège; Masson et Cie, Paris, 1955.

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