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IMPLICIT PREDICTOR CORRECTOR METHODS FOR PDE'S WITH CONVECTION AND DIFFUSION

Diego A. Murio

SUMMARY.

A modified diagonally implicit Runge-Kutta method for solving PDE's with convection, diffusion and chemical kinetic interaction terms is presented. We obtain stability and second order accuracy in both space and time. An application to the numerical solution of a non-linear system of equations is also illustrated.

1. INTRODUCTION.

In this paper we do a Fourier stability analysis on a hybrid scheme based upon Miller's second order diagonally implicit Runge-Kutta method DIRK2 (see [1], [2]), for solving certain PDE's with convection, diffusion and chemical kinetic interaction terms.

In this scheme, we treat implicitly only the "stiff" terms in the equations. We obtain stability and second order accuracy in both space and time by means of a first order "upwind" predictor followed by a second order centered corrector.

A typical equation, written for brevity in its one-dimensional form is

$$\frac{\partial v}{\partial t} = w\,\frac{\partial v}{\partial x} + \varepsilon\,\frac{\partial^2 v}{\partial x^2} + f(v) \tag{1.1}$$

where $v = (v_1, v_2, \ldots, v_n)^T$ describes the n chemical concentrations convected with velocity w and diffusing according to the "diffusion coefficient" ε. The nonlinear term f(v) models the chemical reactions.

The spatially semi-discretized approximation to equation (1.1) is of the form

$$\frac{du}{dt} = Au + B(u) \tag{1.2}$$

where Au involves the first two terms of (1.1) and has long time constants since the grid spacing in the x direction is long, and B(u) is stiff due to the presence of the chemical kinetics.

In §2, the modified 2nd order DIRK method is presented and discussed.

In §3, using von Neumann's method (see e.g. [3]), a region of stability is obtained for a certain simplified model of (1.1) with coefficients frozen constant and with the nonlinear operator B(u) replaced by a linear operator with large negative eigenvalues which commutes with A. We also do not bother about boundary conditions, imagining the equation to hold on the entire real line.

In §4 we discuss the error analysis assuming that the nonlinear operator B has an appropriate Lipschitz constant on the order of unity.

Finally, in section 5 we present some numerical results of interest.

2. NUMERICAL METHOD.

The modified 2nd order DIRK method when applied to (1.2) is given by

Predictor

(2.1a) $$u^{n+1/3} = u^n + \frac{\Delta t}{3} A^* u^n + \frac{\Delta t}{3} B\, u^{n+1/3}$$

Extrapolation

(2.1b) $$u^{(n+1)*} = 3u^{n+1/3} - 2u^n$$

Corrector

(2.1c) $$u^{n+1} = u^n + \frac{3}{4}\Delta t(Au^{n+1/3} + Bu^{n+1/3}) + + \frac{\Delta t}{4} Au^{(n+1)*} + \frac{\Delta t}{4} Bu^{n+1}$$

The finite difference operator A indicates the 2nd order *centered* approximations to the convection and diffusion terms in the PDE. However, for the sake of stability we have had to replace A by A* in the explicit A^*u^n term of the predictor formula (2.1a); this indicates that the wv_x convection term has been approximated by the 1st order noncentered *upwind* difference approximation. The predictor-corrector result however, is 2nd order correct in both space and time.

2.1. PRELIMINARIES.

Given the function $f_k = f(kh)$, k integer, defined at the points $x = kh$ on the whole discrete line, we will use the operators

$$(S_+f)_k = f_{k+1} \qquad \text{(forward shift)}$$

$$(D_+f)_k = \frac{1}{h}(f_{k+1} - f_k) \qquad \text{(forward difference)}$$

$$(D_-f)_k = \frac{1}{h}(f_k - f_{k-1}) \qquad \text{(backward difference)}$$

and

$$(D_0f)_k = \frac{1}{2h}(f_{k+1} - f_{k-1}) \qquad \text{(centered difference)}$$

If we let $u = \{u_k\}$ and assume that $u_k = e^{ikh\theta}$; $i = \sqrt{-1}$, setting $\alpha = h\theta$, we can define *the discrete Fourier transform of* u, $\hat{u}$, by

$$\hat{u}(\alpha) = \sum_k u_k \, e^{ik\alpha} \quad ; \quad 0 \leqslant \alpha \leqslant 2\pi \ .$$

It follows that

$$\widehat{S_+u}(\alpha) = e^{i\alpha}\,\hat{u}(\alpha)$$

If we have a difference operator L, linear with constant coefficients,

$$u^{n+1} = \Big(\sum_{\ell=n_1}^{n_2} a_\ell (S_+)^\ell\Big) u^n = L\,u^n \ ,$$

after applying Fourier transform, we get

$$\widehat{u^{n+1}}(\alpha) = \rho(\alpha)\,\widehat{u^n}(\alpha)$$

If $\rho(\alpha)$, the amplification factor, satisfies the von Neumann condition

$$|\rho(\alpha)| \leqslant 1 + C\Delta t \ , \quad C \text{ constant} \ ,$$

then

$$\|u^{n+1}\|_2 \leqslant e^{Ct + \Delta t}\,\|u^0\|_2$$

and we have stability [4].

3. STABILITY ANALYSIS.

We consider first the equation

(3.1) $$v_t = v_x + \varepsilon\, v_{xx} \quad ; \quad \varepsilon \text{ constant.}$$

Using the modified DIRK method (2.1), we obtain

(3.2) $$u^{n+1} = (I + \Delta t\, A + \frac{\Delta t^2}{2} AA^*)\, u^n \;,$$

where $A^* = D_+ + \varepsilon\, D_+D_-$ and $A = D_0 + \varepsilon\, D_+D_-$. The method is explicit, and we use D_+ as an approximation for v_x in the predictor and D_0 in the corrector. We use the second difference D_+D_- approximation for v_{xx} in both predictor and corrector.

Setting $k = \Delta t$, $h = \Delta x$, $\lambda = k/u$, $\gamma = k/h^2$ and taking Fourier transform in (3.2), we get

(3.3) $$\rho(\alpha) = 1 - \frac{\lambda^2}{2}\sin^2\alpha + (\cos\alpha - 1)\varepsilon\gamma\,(2 + (\cos\alpha - 1)(\lambda + 2\varepsilon\gamma)) + $$
$$+\, i\,\lambda \sin\alpha\,((\cos\alpha - 1)(\frac{\lambda}{2} + 2\varepsilon\gamma) + 1)$$

The area in the $(\varepsilon\gamma,\lambda)$ plane in which $|\rho(\alpha)| \leqslant 1$, the region of stability (RS), has been determined experimentally and it is shown in Figure 1. In the purely convective case $\varepsilon=0$, we have stability if $\lambda \leqslant 2$.

If we plot $\rho(\alpha)$ for a typical value of λ, for example $\lambda = 0.5$ and several values of $\varepsilon\gamma$ we can see how increasing the diffusion affects the amplification factor. We notice that small diffusion helps $\rho(\alpha)$ to remain inside the unit circle (Figures 2 and 3). If we increase the diffusion term, the second loop approaches the boundary again for $\alpha = \pi$ (Figure 4) and finally, if we still increase the diffusion term, the inner loop crosses the boundary and we reach the region of instability (Figure 5).

3.1. THE GENERAL CASE.

We study now the equation

(3.4) $$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} + \varepsilon\,\frac{\partial^2 v}{\partial x^2} + f(v)$$

This is of the form $\frac{\partial u}{\partial t} = Au + B(u)$ after space discretization. The structure of the nonlinear $B(u)$ is well set up for the Newton's method type of solution of our implicit part of the DIRK method. The Jacobian matrix for the nonlinear term is negative definite and with low profile since the chemical kinetic interaction is sparse.

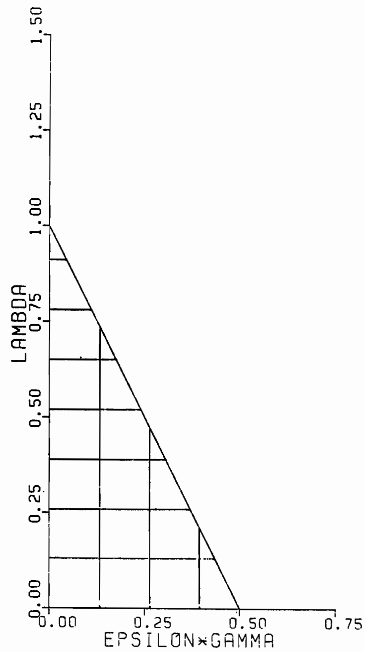


Figure 1

Region of Stability in the $(\varepsilon\gamma, \lambda)$ plane

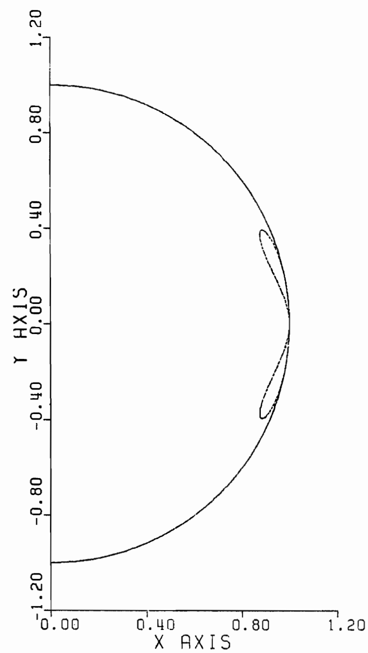


Figure 2

Magnification factor for $\lambda = 0.5$ and $\varepsilon\gamma = 0.0$

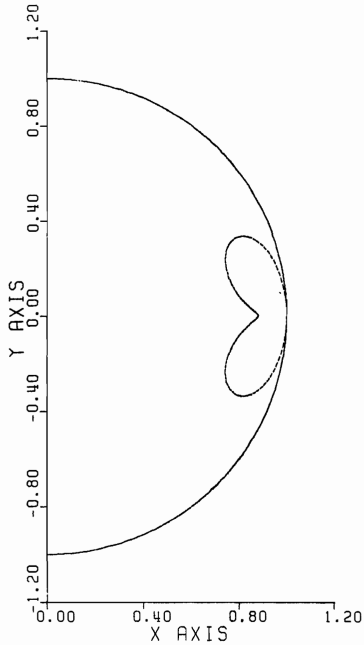


Figure 3

Magnification factor for $\lambda = 0.5$ and $\varepsilon\gamma = 0.1$

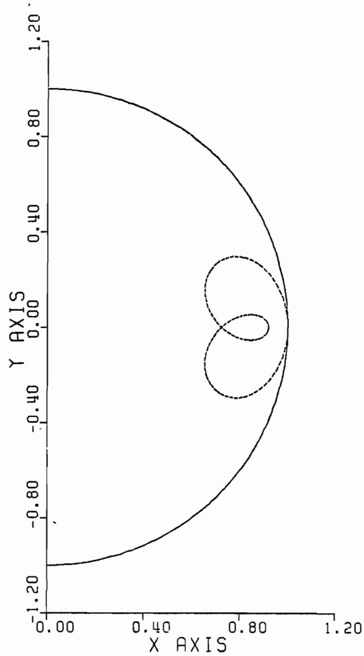


Figure 4

Magnification factor for $\lambda = 0.5$ and $\varepsilon\gamma = 0.2$

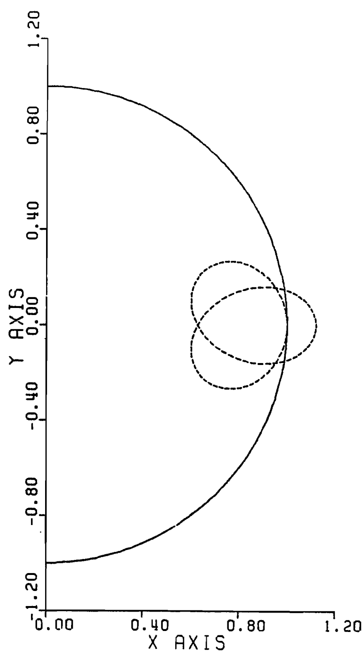


Figure 5

Magnification factor for $\lambda = 0.5$ and $\varepsilon\gamma = 0.3$

In what follows we will instead assume (for the purposes of stability analysis only) that B is linear, commutes with A and A* and has full real negative eigenvalues b, some of which might be expected to be quite large since the chemical kinetic terms may be quite stiff. We want to show that we have stability for all negative b. Using the modified DIRK method (2.1), we obtain

$$u^{n+1} = (I - \frac{k}{4}B)^{-1} [(I - \frac{k}{2}A) + (\frac{3}{2} kA + \frac{3}{4} kB) . \tag{3.5}$$

$$. (I - \frac{k}{3} B)^{-1} (I + \frac{k}{3} A^*)] u^n$$

Using the fact that B has negative eigenvalues b and that it commutes with A and A* and assuming that $(\varepsilon\gamma,\lambda)$ belongs to the region of stability (RS), after taking Fourier transform in (3.5) we get

$$|\rho(\alpha)| \leqslant \frac{1 + \frac{kb}{12} |5 + (\cos\alpha - 1)(3\lambda + 10\varepsilon\gamma) + i\, 5\lambda \sin\alpha|}{1 + \frac{7}{12} kb + \frac{1}{12} k^2 b^2} \tag{3.6}$$

For $(\varepsilon\gamma,\lambda)$ in the region of stability

$$0 \leqslant 3\lambda + 10\varepsilon\gamma \leqslant 5$$

and if

(3.7) $$\lambda \leqslant (24/25)^{1/2} ,$$

it follows that

(3.8) $$|15 + (\cos\alpha - 1)(3\lambda + 10\varepsilon\gamma) + i\, 5\lambda \sin\alpha| \leqslant 7$$

Using (3.6) and (3.8) we obtain stability for all $b \leqslant 0$ if λ satisfies (3.7) and $(\varepsilon\gamma,\lambda)$ belongs to the region of stability.

4. ERROR ANALYSIS.

In this section we perform the error analysis for the equation

(4.1) $$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x} + \varepsilon \frac{\partial^2 v}{\partial x^2} + f(v)$$

where we assume that the solution v is sufficiently smooth in both space and time and also that the nonlinear term f has an appropriate Lipschitz constant on the order of unity.

The modified DIRK method is given by

Predictor

(4.2a) $$u^{n+1/3} = u^n + \frac{k}{3}(D_+ + \varepsilon D_+D_-)u^n + \frac{k}{3} Bu^{n+1/3}$$

Extrapolation

(4.2b) $$u^{(n+1)*} = 3u^{n+1/3} - 2u^n$$

Corrector

(4.2c) $$u^{n+1} = u^n + \frac{3}{4} k(D_0 + \varepsilon D_+D_-)u^{n+1/3} + \frac{3}{4} kBu^{n+1/3} + $$
$$+ \frac{k}{4}(D_0 + \varepsilon D_+D_-)u^{(n+1)*} + \frac{k}{4} Bu^{n+1}$$

The restriction of the smooth true solution v(x,t) to the grid domain will be noted by $v_j^n = v(nk,jh)$. We now try to write v itself as a solution of the difference equations plus a uniformly small local truncation error.

Predictor

(4.3) $$v^{n+1/3} = v^n + \frac{k}{3}(v_x^n + \varepsilon v_{xx}^n) + \frac{k}{3} Bv^{n+1/3} + O(k^2)$$

Since

$$v_x^n = D_+ v^n + O(u)$$

and

$$\varepsilon v_{xx}^n = \varepsilon D_+D_- v^n + O(u^2)$$

After replacing these terms in (4.3) we get

(4.4) $$v^{n+1/2} = v^n + \frac{k}{3}(D_+ + \varepsilon D_+D_-)v^n + \frac{k}{3} Bv^{n+1/3} + + O(k^2 + kh + kh^2)$$

From (4.4) and (4.2a) we have

(4.5) $$v^{n+1/3} = u^{n+1/3} + O(k^2 + kh + kh^2)$$

Extrapolation:

(4.6) $$v^{(n+1)*} = 3v^{n+1/3} - 2v^n = = 3u^{n+1/3} - 2u^n + O(k^2 + kh + kh^2).$$

From (4.2b) and (4.6), we have

(4.7) $$v^{(n+1)*} = u^{(n+1)*} + O(k^2 + kh + kh^2).$$

Corrector:

(4.8) $$v^{n+1} = v^n + \frac{3}{4} k(v_x^{n+1/3} + \varepsilon v_{xx}^{n+1/3} + Bv^{n+1/3}) + + \frac{k}{4}(v_x^{(n+1)*} + \varepsilon v_{xx}^{(n+1)*}) + \frac{k}{4} Bv^{n+1}$$

Using the smoothness of the solution, (4.5) and the fact that B has a Lipschitz constant on the order of unity, the second term in (4.8) can be replaced by

(4.9) $$\frac{3}{4} k(D_0 + \varepsilon D_+D_-)u^{n+1/3} + \frac{3}{4} kBu^{n+1/3} + + O(k^3 + k^2h + k^2h^2 + kh^2)$$

Analogously, using (4.7), the third term in (4.8) can be replaced by

(4.10) $$\frac{k}{4}(D_0 + \varepsilon D_+D_-)u^{(n+1)*} + O(k^3 + k^2h + k^2h^2 + kh^2)$$

From (4.2c), (4.9) and (4.10), assuming k/h bounded, we have

(4.11) $$v^{n+1} = u^{n+1} + k\, O(k^2 + h^2)$$

In this equation, u^{n+1} denotes the approximate value that we would compute in one step of our method. We see that the method has local truncation error which is uniformly bounded by k times $O(k^2 + h^2)$. The method is 2nd order correct both in space and time.

5. A NUMERICAL EXAMPLE.

A diffusion-convection reaction process involving three substances of concentrations v_1, v_2 and v_3 respectively, leading to the system

$$\frac{\partial v_1}{\partial t} = -\frac{\partial v_1}{\partial x} + \varepsilon \frac{\partial^2 v_1}{\partial x^2} + v_3 - 10^4 v_1 v_2$$

$$\frac{\partial v_2}{\partial t} = -\frac{\partial v_2}{\partial x} + \varepsilon \frac{\partial^2 v_2}{\partial x^2} + v_3 - 10^4 v_1 v_2$$

$$\frac{\partial v_3}{\partial t} = -\frac{\partial v_3}{\partial x} + \varepsilon \frac{\partial^2 v_3}{\partial x^2} - v_3 + 10^4 v_1 v_2$$

is integrated using the following boundary and initial conditions

$$v_i(0,t) = 0 \ , \ t>0 \ , \ i=1,2,3 \quad , \quad \frac{\partial v_i}{\partial x}(1,t) = 0 \ , \ t>0 \quad , \ i=1,2,3$$

$$v_1(x,0) = \begin{cases} 20x & , \quad 0 < x \leqslant 0.1 \\ -20x+4 & , \ 0.1 < x \leqslant 0.2 \\ 0 & , \ 0.2 < x < 1 \end{cases}$$

$$v_2(x,0) = \begin{cases} 10x & , \quad 0 < x \leqslant 0.1 \\ -10x+2 & , \ 0.1 < x \leqslant 0.2 \\ 0 & , \ 0.2 < x < 1 \end{cases}$$

$$v_3(x,0) = 0 \quad , \quad 0 < x < 1 \ ,$$

and with the physical parameter $\varepsilon = 10^{-2}$. We notice that the first order spatial derivatives are large in relation to the second order ones and, consequently, we expect the solution to behave like the solution of the limiting purely convective case.

We use the modified DIRK method with fixed $\Delta x = 1/20$ and variable step size Δt. Because of the chemical kinetics and in order to save computer time, the initial step size is chosen fairly small (10^{-6}). We impose a rather conservative upper bound (2×10^{-2}) for the maximum size to which the step would be increased, satisfying the stability conditions of section 3. To automatically adjust the step size, the difference between the solution after one step of size Δt and two steps of size $\Delta t/2$ is taken as an estimate of the local truncation error. If this difference is within the prescribed tolerance δ, the step size is doubled. Otherwise the step size is halved. We use Newton's method for the solution of our implicit part of the modified DIRK method, and a solution is accepted when two successive iterations satisfy $\| v^{k+1} - v^k \|_2 \leqslant \delta \times 10^{-1}$. Several runs, with different tolerances, show only the predictable changes in accuracy. However, for $\delta = 10^{-4}$, the average step size is about 4×10^{-3} while for $\delta = 10^{-2}$, the average step size coincides with the maximum step size 2×10^{-2}. The initial data and the solution at different times are shown in Figures 6 to 10. We notice the sharp transition from the initial concentrations (Figure 6) to the state at time $\cong 0.20$ (Figure 8), due to the chemical reactions. In all the plots, the solution for v_1 is represented by a solid curve, for v_2 by a dashed curve and for v_3 by a solid curve with a symbol. Figures 9 and 10 show the outgoing waves travelling with speed less than 1, as expected, due to the presence of diffusion.

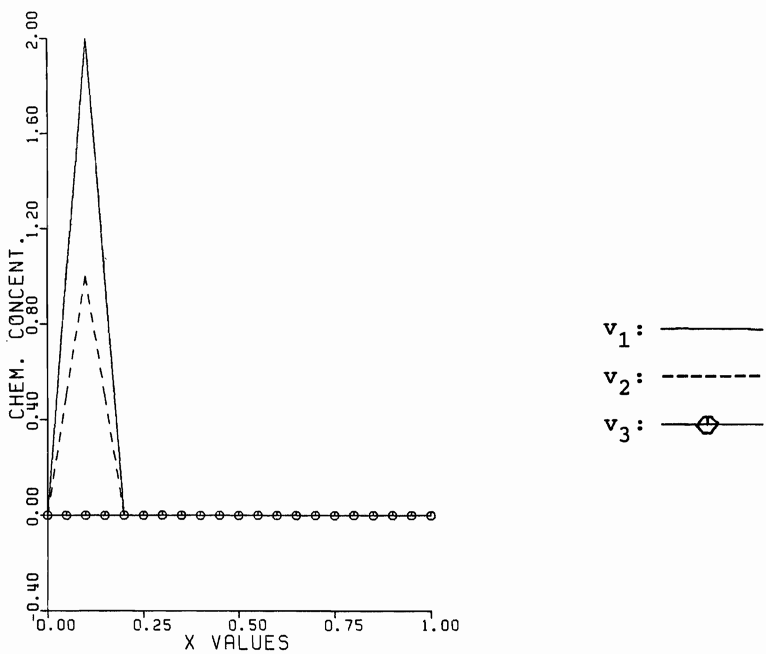


Figure 6

Chemical concentrations at time 0.000

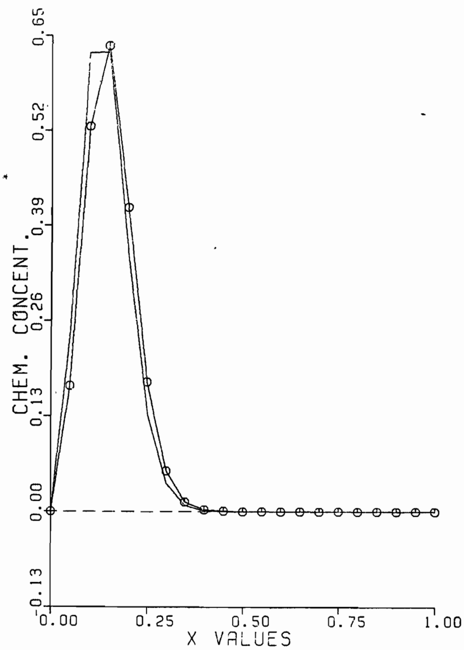
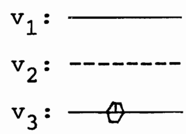


Figure 7

Chemical concentrations at time 0.053



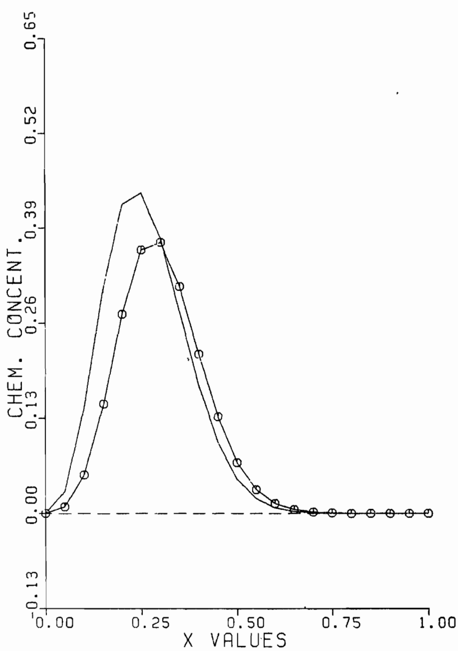


Figure 8

Chemical concentrations at time 0.203

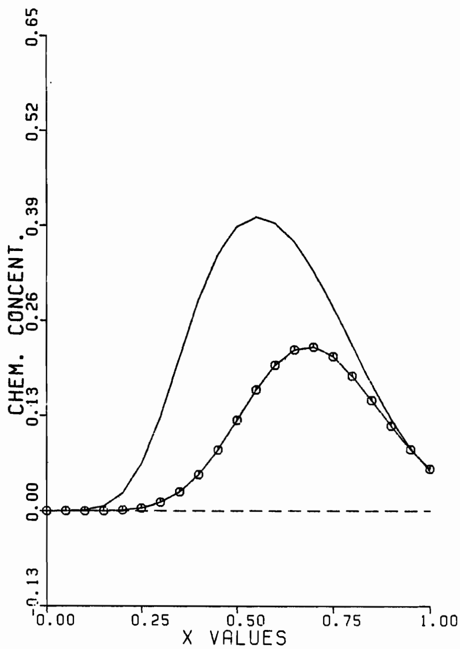
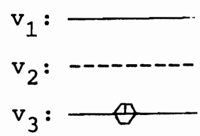


Figure 9

Chemical concentrations at time 0.608



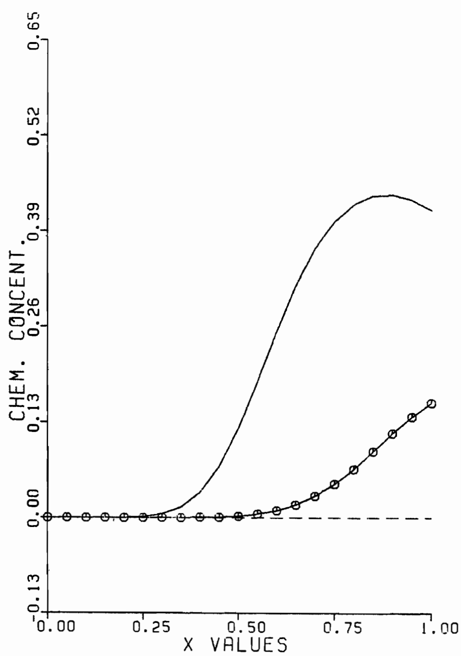


Figure 10

Chemical concentrations at time 1.026

REFERENCES

[1] K. Miller, Math. 228A Notes, Part II on Stiff Equations, Fall' 73, UC Berkeley, unpublished lecture notes.

[2] R. Alexander, *Diagonally implicit Runge-Kutta methods for stiff ODE's*, SIAM J. Num. Anal., Vol.14, N°6(1977) pp.1006-1021.

[3] R.D. Richtmeyer and K.W. Morton, *Difference methods for initial value problems*, second edition, Interscience Publishers, New York, 1967.

[4] G.D. Smith, *Numerical solution of partial differential equations: Finite difference methods*, second edition, Oxford University Press, Oxford, 1978.

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DISCONTINUITY OF MAPPINGS IN BITOPOLOGICAL SPACES

H. Dasgupta and B.K. Lahiri

ABSTRACT. In this paper we introduce the definition of removable discontinuity of mappings in a bitopological space and enquire when such mappings are continuous.

INTRODUCTION.

In this paper, we introduce the definition of removable discontinuity of mappings from one bitopological space (X,P,Q) [2] into another such space and find out conditions when a mapping having at worst a removable discontinuity at a point becomes continuous at that point. A continuous mapping has clearly at worst a removable discontinuity at each point, but we show by an example that the converse is not true. We introduce the definition of removable discontinuity in (X,P,Q) in such a way that when the two topologies P and Q coincide, our definition becomes the same as that of Halfer [1] who establishes various results on discontinuity of mappings in a single topological space For our investigations in (X,P,Q) we require the concepts of local connectedness and connected mappings in a bitopological space, which also we introduce here. Connectedness in a bitopological space has been widely investigated by Pervin [4].

1. KNOWN DEFINITIONS.

DEFINITION 1.1 [2]. A space X where two (arbitrary) topologies P and Q are defined is called a bitopological space and is denoted by (X,P,Q).

DEFINITION 1.2 [4]. A bitopological space (X,P,Q) is called *connected* if and only if X cannot be expressed as the union of two non-empty disjoint sets A and B such that

$$[A \cap cl_P(B)] \cap [cl_Q(A) \cap B] = \emptyset$$

where cl_P and cl_Q denote the closures with respect to P and Q topologies respectively and $\emptyset$ denotes the empty set. If X can be so expressed, then A and B are called separated sets.

NOTE 1.1. If X can be so expressed, we say that A and B are (P,Q)-separated. Throughout the paper we shall follow this convention.

DEFINITION 1.3 [4]. A subset E of (X,P,Q) is called connected if and only if the space (E,P/E,Q/E) is connected.

DEFINITION 1.4 [4]. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be *continuous* if and only if the induced mappings f_1: (X,P) ⟶ (X*,P*) and f_2: (X,Q) ⟶ (X*,Q*) are continuous.

DEFINITION 1.5 [2]. In a bitopological space (X,P,Q), P is said to be *regular* with respect to Q if, for each point $x \in X$ and each P-closed set C such that $x \notin C$, there is a P-open set U and a Q-open set V such that $x \in U$, $C \subset V$ and $U \cap V = \emptyset$. (X,P,Q) is, or P and Q are, pairwise regular if P is regular with respect to Q and vice-versa.

DEFINITION 1.6 [2]. A bitopological space (X,P,Q) is said to be *pairwise Hausdorff* if, for each two distinct points x and y of X, there are a P-open neighbourhood U of x and a Q-open neighbourhood V of y such that $U \cap V = \emptyset$.

2. NEW DEFINITIONS.

DEFINITION 2.1. A bitopological space (X,P,Q) is said to be *locally connected* at a point $x \in X$ if and only if for every pair of P-open set U and Q-open set V each containing x, there exist connected Q-open set C and connected P-open set D such that $x \in C \subset U$ and $x \in D \subset V$. (X,P,Q) is said to be locally connected if and only if it is locally connected at every point of X.

DEFINITION 2.2. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be *connected* if and only if the image of every connected subset of (X,P,Q) is a connected subset of (X*,P*,Q*).

DEFINITION 2.3. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be (P ⟶ Q*) [resp. (Q ⟶ P*)] - closed if and only if the image of every P (resp. Q)-closed subset of (X,P,Q) is a Q* (resp. P*)-closed subset of (X*,P*,Q*).

DEFINITION 2.4. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) has at worst a *removable discontinuity* at a point $p \in X$ if there exists a point $y \in X^*$ such that for each P*-open neighbourhood V_{P^*} and Q*-open neighbourhood V_{Q^*} of y, there are a P-open neighbourhood U_P and Q-open neighbourhood U_Q of p such that

$$f(U_P - \{p\}) \subset V_{P^*} \quad \text{and} \quad f(U_Q - \{p\}) \subset V_{Q^*} .$$

REMARK 2.1. If a function f is continuous at a point of (X,P,Q), then f has at worst a removable discontinuity at that point but the converse is not true as shown by the following example.

EXAMPLE 2.1. Let (X,P,Q) and (X*,P*,Q*) be two bitopological spaces where $X = \{\alpha,\beta,\gamma\}$, $X^* = \{a,b,c\}$, $P = \{X,\emptyset,\{\alpha,\beta\}\}$, $Q = \{X,\emptyset,\{\alpha,\gamma\}\}$ and $P^* = \{X^*,\emptyset,\{a,b\},\{c\}\} = Q^*$.

Let $f: (X,P,Q) \longrightarrow (X^*,P^*,Q^*)$ be given by $f: \alpha \longrightarrow c$, $\beta \longrightarrow b$ and $\gamma \longrightarrow a$.

Here f has a removable discontinuity at α, because there is a point in X*, viz., the point a such that for every P*-open neighbourhood V_{P^*} and for every Q*-open neighbourhood V_{Q^*} of a, there are P-open neighbourhood $\{\alpha,\beta\}$ and Q-open neighbourhood $\{\alpha,\gamma\}$ of α such that

$$f(\{\alpha,\beta\}-\{\alpha\}) \subset V_{P^*} \quad \text{and} \quad f(\{\alpha,\gamma\}-\{\alpha\}) \subset V_{Q^*} .$$

But f is not continuous at α, because {c} is a P*-open neighbourhood of $f(\alpha)$ and there is no P-open neighbourhood U_P of α such that $f(U_P) \subset \{c\}$ and so the induced mapping of f from (X,P) to (X*,P*) is not continuous at α and consequently $f: (X,P,Q) \longrightarrow (X^*,P^*,Q^*)$ is not continuous at α.

3. LEMMAS AND THEOREMS.

LEMMA 3.1. *Let* A *and* B *be respectively two non-empty disjoint* P-*open and* Q-*open subsets of* (X,P,Q). *If* D *is a connected non-empty subset of* (X,P,Q) *such that* $D \subset A \cup B$, *then either* $D \cap A = \emptyset$ *or* $D \cap B = \emptyset$.

Proof. Since $A \cap B = \emptyset$, $A \subset CB$. Also as B is Q-open, CB is Q-closed and so $cl_Q(A) \subset CB$. Hence $B \cap cl_Q(A) = \emptyset$. Similarly, $A \cap cl_P(B) = \emptyset$. As D is a connected subset of (X,P,Q), the space (D,P/D,Q/D) is connected. Let P/D = P* and Q/D = Q*. We write

$$D = (D \cap A) \cup (D \cap B)\ , \quad \text{then}$$

$$(D \cap A) \cap cl_{P*}(D \cap B) = (D \cap A) \cap [D \cap cl_P\ (D \cap B)] \subset$$
$$\subset (D \cap A) \cap cl_P(B) = \emptyset\ .$$

Thus $(D \cap A) \cap cl_{P*}(D \cap B) = \emptyset$. Similarly $(D \cap B) \cap cl_{Q*}(D \cap A) = \emptyset$. Hence if each of the sets $D \cap A$ and $D \cap B$ is non-empty, then the space (D,P^*,Q^*) i.e., the space $(D,P/D,Q/D)$ has a separation and consequently D cannot be a connected subset of (X,P,Q). Hence either $D \cap A = \emptyset$ or $D \cap B = \emptyset$. This proves the lemma.

THEOREM 3.1. *Let* f *be a connected mapping of a locally connected bitopological space* (X,P,Q) *into a pairwise Hausdorff bitopological space* (X^*,P^*,Q^*). *Then if* f *has at worst a removable discontinuity at* p, f *is continuous at* p.

Proof. Here the following cases come up for considerations:

(a) p is an isolated point in (X,P) as well as in (X,Q) ,
(b) p is an isolated point in (X,P) but not in (X,Q) ,
(c) p is an isolated point in (X,Q) but not in (X,P) and
(d) p is neither an isolated point in (X,P) nor in (X,Q).

CASE (a). Let V_{P*} be any P^*-open neighbourhood of $f(p)$ and let U_1 be any P-open neighbourhood of p. As p is an isolated point in (X,P), there is a P-open neighbourhood U_2 of p such that $U_1 \cap U_2 - \{p\} = \emptyset$.
Now $U_1 \cap U_2 = U_p$, say, is a P-open neighbourhood of p. Also, $f(U_p - \{p\}) = \emptyset \subset V_{P*}$ and as $f(p) \in V_{P*}$, $f(U_p) \subset V_{P*}$. Hence the induced mapping of f from (X,P) to (X^*,P^*) is continuous at p.

As p is also an isolated point in (X,Q), we get similarly that the induced mapping of f from (X,Q) to (X^*,Q^*) is also continuous at p. Hence $f: (X,P,Q) \longrightarrow (X^*,P^*,Q^*)$ is continuous at p.

CASE (b). Let y be the point in X^* determined by the definition of removable discontinuity of f. If f is not continuous at p, $y \neq f(p)$ and so as X^* is pairwise Hausdorff, there exist P^*-open set V_{P*} and Q^*-open set V_{Q*} such that $f(p) \in V_{P*}$, $y \in V_{Q*}$ and $V_{P*} \cap V_{Q*} = \emptyset$.
Because f has a removable discontinuity at p, there exists a Q-open neighbourhood U_Q of p such that $f(U_Q - \{p\}) \subset V_{Q*}$.
So $f(U_Q) \subset V_{P*} \cup V_{Q*}$.

Now X is locally connected at p and since p belongs to the Q-open

set U_Q, there is a connected P-open set C_P such that $p \in C_P \subset U_Q$. Similarly as p belongs to the P-open set C_P, there is a connected Q-open set D_Q such that $p \in D_Q \subset C_P$. So, $p \in D_Q \subset U_Q$.
As p is not isolated in (X,Q), $D_Q - \{p\} \neq \emptyset$. Again,
$\emptyset \neq f(D_Q - \{p\}) \subset f(U_Q - \{p\}) \subset V_{Q*}$, which implies that
$f(D_Q) \cap V_{Q*} \neq \emptyset$.
Also as $f(p) \in f(D_Q)$ and $f(p) \in V_{P*}$, $f(D_Q) \cap V_{P*} \neq \emptyset$.
Now as f is connected, $f(D_Q)$ is a connected subset of (X*,P*,Q*).
Thus as $f(D_Q) \subset V_{P*} \cup V_{Q*}$ and as V_{P*} and V_{Q*} are respectively disjoint P*-open set and Q*-open set, by Lemma 3.1, either
$f(D_Q) \cap V_{P*} = \emptyset$ or $f(D_Q) \cap V_{Q*} = \emptyset$, which is a contradiction. Hence f is continuous at p.

The cases (c) and (d) may be dealt with similarly. This proves the theorem.

LEMMA 3.2. *The following three properties are equivalent:*

(1) (X,P,Q) *is a bitopological space such that* P *is regular with respect to* Q.

(2) *For each* $x \in (X,P,Q)$ *and for each* P-*open neighbourhood* U_P *of* x, *there is a* P-*open neighbourhood* V_P *of* x *such that* $x \in V_P \subset$ $\subset cl_Q(V_P) \subset U_P$.

(3) *For each* $x \in (X,P,Q)$ *and each* P-*closed set* A *not containing* x, *there is a* P-*open neighbourhood* V_P *of* x *with* $cl_Q(V_P) \cap A = \emptyset$.

Proof. (1) → (2). Let U_P be given. Then the P-closed set $C U_P$ does not contain x. As in (X,P,Q), P is regular with respect to Q, there is a P-open set V_P and a Q-open set V_Q such that $x \in V_P$ and $C U_P \subset V_Q$ and $V_P \cap V_Q = \emptyset$. Thus $V_P \subset CV_Q$ and so $cl_Q(V_P) \subset CV_Q \subset U_P$.
Hence $x \in V_P \subset cl_Q(V_P) \subset U_P$.

(2) → (3). Using x and its P-open neighbourhood CA, we can find a P-open neighbourhood V_P of x such that $x \in V_P \subset cl_Q(V_P) \subset CA$. Thus $cl_Q(V_P) \cap A = \emptyset$.

(3) → (1). Let A be P-closed and $x \notin A$. We choose a P-open neighbourhood V_P of x such that $cl_Q(V_P) \cap A = \emptyset$. Thus $A \subset C[cl_Q(V_P)]$ and $C[cl_Q(V_P)] \cap V_P = \emptyset$. This proves the lemma.

LEMMA 3.3. *The following properties are equivalent:*

(1) (X,P,Q) *is a bitopological space such that* Q *is regular with respect to* P.

(2) *For each* $x \in (X,P,Q)$ *and for each* Q*-open neighbourhood* U_Q *of* x, *there is a* Q*-open neighbourhood* V_Q *of* x *such that* $x \in V_Q \subset \subset cl_P(V_Q) \subset U_Q$.

(3) *For each* $x \in (X,P,Q)$ *and for each* Q*-closed set* A *not containing* x, *there is a* Q*-open neighbourhood* V_Q *of* x *with* $cl_P(V_Q) \cap A = \emptyset$.

Proof. The proof runs parallel to Lemma 3.2.

THEOREM 3.2. *Let* f *be a function that maps a pairwise regular bitopological space* (X,P,Q) *into a bitopological space* (X^*,P^*,Q^*) *such that* f *is* $(P \to Q^*)$*-closed and* $(Q \to P^*)$*-closed and also for every* $y \in X^*$, $f^{-1}(y)$ *is* P*-closed and* Q*-closed subset of* X. *Then if* f *has at worst a removable discontinuity at* $p \in X$, f *is continuous at* p.

Proof. We should consider the following cases:

(a) p is an isolated point in (X,P) as well as in (X,Q); (b) p is not an isolated point in (X,P) but is an isolated point in (X,Q); (c) p is an isolated point in (X,P) but not an isolated point in (X,Q) and (d) p is neither an isolated point in (X,P) nor an isolated point in (X,Q).

We prove the theorem for the case (b). The other cases are similar.

Let y be the point in X^* determined by the definition of removable discontinuity of f. If f is not continuous at p, $f(p) \neq y$ and so $p \notin f^{-1}(y)$. But $f^{-1}(y)$ is a P-closed set in X and as P is regular with respect to Q, there exists, by Lemma 3.2, a P-open neighbourhood U_P of p such that

$$f^{-1}(y) \cap cl_Q(U_P) = \emptyset.$$

As f is $(Q \to P^*)$-closed, $f(cl_Q(U_P))$ is P^*-closed and as $y \notin f(cl_Q(U_P))$, there is a P^*-open neighbourhood V_{P^*} of y such that

$$V_{P^*} \cap f(cl_Q(U_P)) = \emptyset.$$

From the definition of removable discontinuity, there exists a P-open neighbourhood W_P of p such that

$$f(W_P - \{p\}) \subset V_{P^*}.$$

Since p is not isolated in (X,P), $U_P \cap W_P - \{p\} \neq \emptyset$.

Hence $\emptyset \neq f(W_P - \{p\}) \cap f(cl_Q(U_P)) \subset V_{P^*} \cap f(cl_Q(U_P)) = \emptyset$, a contradiction. Hence f is continuous at p.

REFERENCES

[1] E.HALFER, *Conditions implying continuity of functions*, Proc. Amer.Math.Soc., Vol.11, p.688-691, 1960.

[2] J.C.KELLY, *Bitopological spaces*, Proc. London Math.Soc., Vol. 13, p.71-89, 1963.

[3] W.J.PERVIN and N.LEVINE, *Connected mappings of Hausdorff spaces*, Proc.Amer.Math.Soc., Vol.9, p.488-496, 1958.

[4] W.J.PERVIN, *Connectedness in bitopological spaces*, Indagations Mathematicae, Vol.29, p.369-372, 1967.

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MAPPING THEOREMS IN PARANORMED SPACES

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0. INTRODUCTION.

Let X be a metrizable topological space, Y a linear space, X_o a subset of X and T an application from X_o to Y. By a *mapping theorem* involving these elements we mean the problem of determining sufficient metric-linear conditions in order that the equation

$$\text{(E)} \qquad Tx = 0$$

should have a solution in X_o. The prototype of all mapping results of this kind must be considered the 1971 Browder's theorem [6] proved by a specific *asymptotic direction* technique. As refinements of Browder's original result we quote the 1976 Altman's contribution [1] obtained by a *transfinite induction* argument combined with a *contractor direction* technique, as well as the 1977 Downing-Kirk result [10] proved by a Caristi fixed point procedure (see also Kirk and Caristi [14]). Finally, as further developments in this direction, we must quote those of Altman [2] and, respectively, Cramer and Ray [9] based, essentially, on Ekeland's variational principle [11] and, respectively, Brêzis-Browder ordering principle [3]. A basic assumption of all these contributions is that the range of the application is a Banach space with respect to a suitable norm; it's therefore natural to ask whether this condition cannot be removed. The main aim of the present note is to give a positive answer to this question - more exactly, to state and prove a mapping theorem for applications taking values in a *paranormed space* - the basic instrument of our investigations being a *maximality principle* on (partially) ordered metric spaces appearing (under its quasi-ordered semi-metrical version) as a generalization of the Brêzis-Browder ordering principle we quoted before. As useful particular cases, a couple of mapping theorems for applications whose range is a normed (respectively, a Fréchet) space is given, extending in this way the similar Cramer-Ray result (see the above reference) and, respectively, completing, under this perspective, the result of Turinici [21] obtained by a specific *variable drop* technique. Some further extensions of this last result to *non-metrizable* uniform spaces will be given elsewhere.

1. PRELIMINARIES.

Let Y be a *linear space* (over the real or complex numbers). A function $x \longmapsto \|x\|$ from Y to $[0,\infty)$ will be called a *paranorm* on Y when (a) $\|x\| = 0$ if and only if $x = 0$, (b) $\|x+y\| \leqslant \|x\|+\|y\|$, $x,y \in Y$, (c) $\|-x\| = \|x\|$, $x \in Y$; correspondingly, the couple $(Y,\|\cdot\|)$ will be termed a paranormed space. Evidently, $\|\cdot\|$ induces a *metric* structure on Y by the standard construction $d(x,y) = \|x-y\|$, $x,y \in Y$. An important class of paranorms - largely used in the sequel - is that introduced by the following convention. Letting $r > 0$ and $Z \subset Y$, we shall say the paranorm $\|\cdot\|$ is r-*superadditive* on Z when

$$(1) \qquad r\|\varepsilon x\| + \|(1-\varepsilon)x\| \leqslant \|x\| \quad , \quad 0 \leqslant \varepsilon \leqslant 1 \ , \ x \in Z.$$

Note that (1) necessarily implies $r \leqslant 1$ because, when $r > 1$, we have (for $x\neq 0$ in Z and $\varepsilon \neq 0$ in $[0,1]$) by the triangle inequality (b)

$$r\|\varepsilon x\|+\|(1-\varepsilon)x\| = (r-1)\|\varepsilon x\|+\|\varepsilon x\|+\|(1-\varepsilon)x\| > \|\varepsilon x\|+\|(1-\varepsilon)x\| \geqslant \|x\| \ ,$$

a contradiction with respect to (1). Afirst example of such paranorms is contained in the evident

LEMMA 1. *Every norm is* 1 - *superadditive on every subset of* Y.

As another specific example of r-superadditive paranorms, let $S = \{|\cdot|_i;\ i \in N\}$ be a (denumerable) *sufficient* family of *seminorms* on Y ($|x|_i = 0$, all $i \in N$ imply $x=0$), $L = (\lambda_i;\ i \in N)$ a sequence of strict positive numbers and $A = (\alpha_i;\ i \in N)$ a *summable* family of strict positive numbers ($\alpha_1+\alpha_2+\ldots<\infty$). Define a function $\|\cdot\| = \|\cdot\|(S,L,A)$ from Y to $[0,\infty)$ by

$$(2) \qquad \|x\| = \sum_{i\in N} \alpha_i |x|_i/(\lambda_i + |x|_i) \quad , \quad x \in Y.$$

LEMMA 2. *The function* $x \longmapsto \|x\|$ *defined by* (2) *is a paranorm on* Y. *Moreover,* $(Y,\|\cdot\|)$ *and* (Y,S) *are equivalent as (metrizable) topological spaces (i.e., a sequence* $(y_n;\ n \in N)$ *in* Y *converges (modulo* $\|\cdot\|$) *to* y *in* Y *if and only if it converges (modulo* S) *to* y).

Proof. The first part of the statement is clear if we observe that, for any $i \in N$,

$$|x+y|_i/(\lambda_i+|x+y|_i) \leqslant |x|_i/(\lambda_i+|x|_i)+|y|_i/(\lambda_i+|y|_i) \ , \ x,y \in Y.$$

To prove the second part, let $\varepsilon > 0$ be arbitrary fixed. As $A = (\alpha_i;\ i \in N)$ is a summable family, there exists $m = m(\varepsilon) \in N$ such that

$$\alpha_{m+1} + \alpha_{m+2} + \ldots < \varepsilon/2$$

in which case, denoting

$$\delta = \varepsilon/2 \sum_{i \leq m} \alpha_i/\lambda_i$$

we have at once

$$|x|_i < \delta, \ i \leqslant m \quad \text{implies} \ \|x\| < \varepsilon \ ;$$

conversely, given any $i \in N$, and putting

$$\eta_i = \alpha_i \varepsilon/(\lambda_i + \varepsilon)$$

one clearly obtains

$$\|x\| < \eta_i \quad \text{implies} \quad |x|_i < \varepsilon$$

proving our assertion. Q.E.D.

LEMMA 3. *Let* r *in* (0,1) *be arbitrary fixed. Then, the paranorm* $\|\cdot\|$ *given by* (2) *is* r - *superadditive on*

$$Y(L,r) = \{y \in Y; \ |y|_i \leqslant \lambda_i(r^{-1/2} - 1), \ i \in N\}.$$

Proof. Evidently, in order that (1) be valid it suffices that, for any $i \in N$,

$$(1)' \qquad r\varepsilon|x|_i(\lambda_i+\varepsilon|x|_i) + (1-\varepsilon)|x|_i/(\lambda_i + (1-\varepsilon)|x|_i) \leqslant$$
$$\leqslant |x|_i/(\lambda_i + |x|_i) \ , \ 0 \leqslant \varepsilon \leqslant 1 \ , \ |x|_i \leqslant \lambda_i(r^{-1/2} - 1)$$

or equivalently (denoting $|x|_i = \xi$ and $\lambda_i = \lambda$) that

$$(3) \qquad r\varepsilon/(\lambda+\varepsilon\xi) + (1-\varepsilon)/(\lambda+(1-\varepsilon)\xi) \leqslant 1/(\lambda+\xi) \ ,$$
$$0 \leqslant \varepsilon \leqslant 1 \ , \ 0 \leqslant \xi \leqslant \lambda(r^{-1/2} - 1).$$

Let $f(\varepsilon)$ denote the left member of this inequality. A simple computation yields

$$f'(\varepsilon) = (-\xi(s+1)\varepsilon + s\xi - \lambda(1-s))g(\varepsilon) \quad , \quad 0 \leqslant \varepsilon \leqslant 1,$$

where $g(\varepsilon)$ is strictly positive and $s = r^{1/2}$ so that, a sufficient condition for (3) (equivalent - under the above notation - with $f(\varepsilon) \leqslant f(0)$, $0 \leqslant \varepsilon \leqslant 1$) to be valid is that

$$s\xi - \lambda(1-s) \leqslant 0 \ (\text{or, equivalently, } \xi \leqslant \lambda(s^{-1} - 1))$$

which is just condition appearing in the final part of (3), proving our assertion. Q.E.D.

Let $(X,d,\leqslant)$ be a (*partially*) *ordered* metric space. A sequence $(x_n; n \in N)$ in X will be said to be (a)' *monotone*, when $x_i \leqslant x_j$ for $i \leqslant j$, (b)' *asymptotic*, when $\liminf_{n\to\infty} d(x_n,x_{n+1}) = 0$, (c)' *bounded above*, in case $x_n \leqslant y$, all $n \in N$, for some y in X. Also, the element z in X will be termed *maximal*, provided that $z \leqslant y$ implies z=y. Concerning these notions, the following maximality principle established by the author in [24] will play a central role in the sequel.

LEMMA 4. *Let the ordered metric space* $(X,d,\leqslant)$ *be such that*

(i) *any monotone sequence is asymptotic*

(ii) *any monotone Cauchy sequence is bounded above.*

Then, to any x *in* X *there corresponds a maximal element* z *in* X *with* $x \leqslant z$.

As already pointed out by the author in [20], the above lemma can be formulated in the larger context of quasi-ordered semi-metric spaces, in which case, it may be viewed as a straightforward extension of the "abstract" Brêzis-Browder ordering principle [3] as well as the "uniform" Brøndsted's maximality principle [4]. Moreover, under a pattern discovered by Brøndsted [5] (see also Ekeland [11]) it's possible to formulate this lemma as a fixed point statement, in which situation it appears as an abstract counterpart of the so-called Caristi's fixed point theorem [8,13,16,19]. Finally, an extension of this maximality principle to metrizable uniform spaces may be found in Turinici [21].

2. THE MAIN RESULT.

In what follows, a precise statement of the considerations exposed in the introductory part of the note will be performed. Let (X,d) be a metric space and $(Y,\|\cdot\|)$ a paranormed space. Given a subset X_o of X, let $X_o(x_o,r)$ denote (for $x_o \in X_o$ and $r > 0$) the X_o - *closed sphere* with center x_o and radius r (the subset of all x in X_o with $d(x,x_o) \leqslant r$); also, given the application $T: X_o \longrightarrow Y$ we shall say it is *closed* [10] when $(x_n;\ n \in N)$ in X_o, $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ imply $x \in X_o$ and $Tx = y$. Suppose henceforward (X,d) and $(Y,\|\cdot\|)$ are complete in the usual sense and $T: X_o \longrightarrow Y$ is closed in the above sense. Then, as the main result of the present note, the following "local" mapping theorem can be stated and proved.

THEOREM 1. *Let the element* x_o *in* X_o *with* $Tx_o \neq 0$ *be such that, a couple of functions* $b,f\colon (0,\infty) \longrightarrow (0,\infty)$ *satisfying*

$$(4) \qquad (t-s)b(t) \leqslant f(t)-f(s) \quad , \quad 0 < s < t$$

and a couple of numbers q,r *satisfying* $0 \leqslant q < r \leqslant 1$ *as well as*

(iii) $\|\cdot\|$ *is* r - *superadditive on* $T(X_o')$

where

(5) $$X_o' = X_o(x_o, f(\|Tx_o\|)/(r-q))$$

may be found with the property: for any x *in* X_o' *with* $Tx \neq 0$ *there exist* x' *in* X_o *and* ε *in* (0,1] *with*

(6) $$\|Tx'-(1-\varepsilon)Tx\| \leqslant q\|\varepsilon Tx\| \quad , \quad d(x,x') \leqslant \|\varepsilon Tx\| b(\|Tx\|).$$

Then, necessarily, (E) *has at least a solution in* X_o'.

Proof. Assume by contradiction $Tx \neq 0$ for all x in X_o'; so, given any x in X_o' there exist x' in X_o and ε in (0,1] such that (6) holds. By the first part of this relation

$$\|Tx-Tx'\| \leqslant (1+q)\|\varepsilon Tx\|$$

as well as (if we take (iii) into account)

$$\|Tx'\| \leqslant q\|\varepsilon Tx\|+\|(1-\varepsilon)Tx\| \leqslant (q-r)\|\varepsilon Tx\|+r\|\varepsilon Tx\|+\|(1-\varepsilon)Tx\| \leqslant$$
$$\leqslant (q-r)\|\varepsilon Tx\|+\|Tx\|$$

or, equivalently,

(7) $$\|\varepsilon Tx\| \leqslant (\|Tx\|-\|Tx'\|)/(r-q)$$

so that, by a combination of them

(8) $$\|Tx-Tx'\| \leqslant (1+q)(\|Tx\|-\|Tx'\|)(r-q)$$

(note at this moment that, again by the first part of (6), $Tx \neq Tx'$ (hence, $x \neq x'$) because, otherwise, $Tx=Tx'$ would imply $1 \leqslant q$, a contradiction). On the other hand, the second part of this relation yields, in combination with a consequence of (1) ($tb(t) \leqslant f(t)$, $t > 0$) and a consequence of (7) ($\|\varepsilon Tx\| \leqslant \|Tx\|/(r-q)$)

$$d(x,x') \leqslant \|Tx\| b(\|Tx\|)/(r-q) \leqslant f(\|Tx\|)/(r-q).$$

Now, let us denote

(5)' $$X_o'' = \{y \in X_o';\ d(x_o,y) \leqslant (f(\|Tx_o\|)-f(\|Ty\|))/(r-q)\}$$

and observe that $x \in X_o''$ plus the above inequality implies $x' \in X_o'$ (whence $Tx' \neq 0$) so that, again by the second part of (6), in combination with (4) + (7)

(9) $$d(x,x') \leqslant (f(\|Tx\|)-f(\|Tx'\|))/(r-q)$$

a relation that evidently implies $x' \in X_o''$. Let e denote the "product" metric on X_o

$$e(x,y) = \max\ (d(x,y), \|Tx-Ty\|) \quad , \quad x,y \in X_o$$

and let ≤ indicate the ordering on X_o' defined as

$x \leqslant y$ *if and only if* $d(x,y) \leqslant (f(\|Tx\|)-f(\|Ty\|))/(r-q)$ and $\|Tx-Ty\| \leqslant (1+q)(\|Tx\|-\|Ty\|)/(r-q)$.

We claim conditions (i)+(ii) are fulfilled in $(X_o'',e,\leqslant)$ and this will lead us to the desired contradiction. (Indeed, it will follow then by Lemma 4 that, for the element x_o in X_o'' a maximal element z in X_o'' may be found with $x_o \leqslant z$; on the other hand, by the above developments, a $z' \in X_o''$ may be chosen with $z \leqslant z'$ and $z \neq z'$, contradicting the maximality of z in $(X_o'',\leqslant)$). To this end, let $(x_n; n \in N)$ be a monotone sequence in X_o'', that is

$$(10)\qquad \begin{aligned} &d(x_n,x_m) \leqslant (f(\|Tx_n\|)-f(\|Tx_m\|))/(r-q) \quad \text{and} \\ &\|Tx_n-Tx_m\| \leqslant (1+q)(\|Tx_n\|-\|Tx_m\|)/(r-q) \ , \ n \leqslant m. \end{aligned}$$

As $(f(\|Tx_n\|); n \in N)$ and $(\|Tx_n\|; n \in N)$ are descending (hence Cauchy) sequences on $(0,\infty)$ it immediately follows $(x_n; n \in N)$ and $(Tx_n; n \in N)$ are Cauchy sequences in X and Y respectively. By the completeness - closedness hypothesis, $x_n \longrightarrow x$ and $Tx_n \longrightarrow Tx$ for some x in X_o and this establishes (i) (module e). Moreover, as X_o' is relatively closed in X_o, it also follows $x \in X_o'$ (whence $Tx \neq 0$) in which situation, observing that, as a consequence of (10) (the first part)

$$d(x_n,x_m) \leqslant (f(\|Tx_n\|)-f(\|Tx\|))/(r-q) \ , \ n \leqslant m$$

one immediately derives (letting m tend to infinity)

$$d(x_n,x) \leqslant (f(\|Tx_n\|)-f(\|Tx\|))/(r-q) \ , \ n \in N \ ,$$

and consequently, $x \in X_o''$; in the same time, by an argument similar to the above one, (10) (the second part) gives

$$\|Tx_n-Tx\| \leqslant (1+q)(\|Tx_n\|-\|Tx\|)/(r-q) \quad , \quad n \in N$$

proving $x_n \leqslant x$, $n \in N$, and establishing (ii). Therefore, the proof is complete. Q.E.D.

3. SOME PARTICULAR CASES.

The main result we established in the preceding paragraph appears, at this stage of our exposition, as an "abstract" mapping theorem only, so that, for a number of practical reasons, some "concrete" realizations of it (based on the considerations of §1) were welcomed. As a first step in this direction, let (X,d) be a complete metric space, $(Y,\|\cdot\|)$ a Banach space, X_o a subset of X and $T: X_o \longrightarrow Y$ a closed application then, the following "normed" variant of the main result may be formulated.

THEOREM 2. *Let the element* x_o *in* X_o *with* $Tx_o \neq 0$ *be such that, a number* $q \in [0,1)$ *and a couple of functions* $b,f: (0,\infty) \longrightarrow (0,\infty)$ *satisfying*

(4)' $(1-s/t)b(t) \leqslant f(t)-f(s)$, $t,s > 0$, $qt \leqslant s < t$

may be found with the property: for any x *in* X_o', *where*

(5)' $X_o' = X_o(x_o, f(\|Tx_o\|)/(1-q))$

with $Tx \neq 0$ *there exist* x' *in* X_o *and* ε *in* (0,1] *with*

(6)' $\|Tx'-(1-\varepsilon)Tx\| \leqslant q\varepsilon\|Tx\|$; $d(x,x') \leqslant \varepsilon b(\|Tx\|)$.

Then, (E) *has at least a solution in* X_o'.

Proof. It suffices to observe that, putting $t = \|Tx\|$, $s = (1-(1-q)\varepsilon)t$, one easily obtains $qt \leqslant s < t$ and (by (7)) $\|Tx'\| \leqslant s$, in which case, taking into account (4)', we established (9). The remaining part of the argument follows from the main result with r=1 and this ends the proof. Q.E.D.

Concerning condition (4)' (essentially involved in the theorem) let us observe it is fulfilled in case $t \longmapsto b(t)$ is increasing and $c(t) = \int_0^t (b(s)/s)ds < \infty$, $t > 0$ for, letting $f(t) = c(t/q)$, $t > 0$, we have, for any couple $t,s > 0$, $qt \leqslant s < t$

$$(1-s/t)b(t) \leqslant (b(u)/u)(t/q-s/q) \ , \ s/q \leqslant u \leqslant t/q$$

in which case, the above statement reduces to Theorem 2.1 of Cramer and Ray [9] proved by a Brézis-Browder ordering procedure. Moreover, it was demonstrated by the above quoted authors their contribution represents a considerable refinement of some "abstract" (metrical) mapping theorems established by Browder [6,7], Kirk and Caristi [14], Downing and Kirk [10], Altman [1,2] (see also Turinici [22]) as well as of some "concrete" (differential) mapping theorems established by Pohozhayev [17], Kačurovskii [12], Krasnoselskii [15], Rosenholtz and Ray [18], so that our theorem also extends these results.

Passing to the second particularization, suppose further X is a complete metrizable uniform space under the (denumerable) sufficient family of semi-metrics $D = (d_i;\ i \in N)$, Y is a complete Fréchet space under the sufficient family of seminorms $S = (|\cdot|_i;\ i \in N)$ and $T: X_o \longrightarrow Y$ (X_o a subset of X) is a closed application, then, the following "global" version of the main result can be derived.

THEOREM 3. *Suppose there exist a sequence* $L = (\lambda_i;\ i \in N)$ *in* $(0,\infty)$, *a couple of numbers* q,r *with* $0 \leqslant q < r \leqslant 1$ *and*

(iii)' $|Tx|_i \leqslant \lambda_i(r^{-1/2} - 1)$, $i \in N$, $x \in X_o$

a summable family $A = (\alpha_i;\ i \in N)$ *in* $(0,\infty)$ *as well as a number* $b > 0$ *with the property: for any* x *in* X_o *with* $Tx \neq 0$ *there exist* x' *in* X_o, ε *in* $(0,1]$ *and a couple of injections* φ,ψ *from* N *to itself, with*

(iv) $q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)} < \alpha_i(\lambda_{\varphi(i)}+|\varepsilon Tx|_{\varphi(i)})$ *implies*

$$|Tx'-(1-\varepsilon)Tx|_i \leqslant \lambda_i q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)}/(\alpha_i(\lambda_{\varphi(i)} + |\varepsilon Tx|_{\varphi(i)})-q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)})$$

(v) $b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)} < \alpha_i(\lambda_{\psi(i)}+|\varepsilon Tx|_{\psi(i)})$ *implies*

$$d_i(x,x') \leqslant \lambda_i b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}/(\alpha_i(\lambda_{\psi(i)} + |\varepsilon Tx|_{\psi(i)})-b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}).$$

Then, (E) *has at least a solution in* X_o.

Proof. It suffices to observe that (iv) plus (v) give

$$\alpha_i|Tx'-(1-\varepsilon)Tx|_i/(\lambda_i+|Tx'-(1-\varepsilon)Tx|_i) \leqslant$$
$$\leqslant q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)}/(\lambda_{\varphi(i)}+|\varepsilon Tx|_{\varphi(i)}) \ , \ i \in N$$

and respectively,

$$\alpha_i d_i(x,x')/(\lambda_i+d_i(x,x')) \leqslant$$
$$\leqslant b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}/(\lambda_{\psi(i)}+|\varepsilon Tx|_{\psi(i)}) \ , \ i \in N \ ,$$

so that, if we introduce a metric (paranormed) structure on X (Y) by the convention

$$d(x,y) = \sum_{i\in N} \alpha_i d_i(x,y)/(\lambda_i+d_i(x,y)) \ , \ x,y \in X$$

(and, respectively,

$$\|x\| = \sum_{i\in N} \alpha_i|x|_i/(\lambda_i+|x|_i) \ , \ x \in Y) \ ,$$

conditions of the main result are fulfilled with $X_o' = X_o$, $b(t)=b$, $t > 0$, and $f(t) = b\,t$, $t > 0$, so that by the conclusion of that statement, the proof is complete. Q.E.D.

Concerning the elements involved in this result , a special mention must be made about the functions φ and ψ appearing in (iv) and (v) respectively. Namely, suppose, in particular that $\varphi = \psi =$ the identity, then, the above conditions become

(iv)' $|Tx'-(1-\varepsilon)Tx|_i \leqslant \lambda_i q|\varepsilon Tx|_i/(\lambda_i+(1-q)|\varepsilon Tx|_i)$, $i \in N$

(v)' $b|\varepsilon Tx|_i < \lambda_i+|\varepsilon Tx|_i$ implies

$$d_i(x,x') \leqslant \lambda_i b|\varepsilon Tx|_i/(\lambda_i+(1-b)|\varepsilon Tx|_i)$$

and the corresponding version of Theorem 3 may be compared with a similar one due to the author [21] and proved by a "variable drop" technique. Regarding this last aspect, it's not without importance to ask whether a direct treatment of the problem - based on the initial metrizable (Fréchet) structure of the ambient spaces - may not be given. A partial answer to this question may be found in Turinici [23]; some further extensions to non-metrizable uniform spaces will be given elsewhere.

REFERENCES

[1] ALTMAN,M., *Contractor directions, directional contractors and directional contractions for solving equations*, Pacific J.Math., 62, 1-18, (1976).

[2] ALTMAN,M., *Existence theorems for nonlinear operator equations without continuity condition*, Nonlinear Analysis, TMA, 4, 437-449, (1980).

[3] BRÉZIS,H. and BROWDER,F.E., *A general principle on ordered sets in nonlinear functional analysis*, Adv.in Math.,21, 355-364, (1976).

[4] BRØNDSTED,A., *On a lemma of Bishop and Phelps*, Pacific J.Math. 55, 335-341, (1974).

[5] BRØNDSTED,A., *Fixed points and partial orders*, Proc.Amer.Math. Soc., 60, 365-366, (1976).

[6] BROWDER,F.E., *Normal solvability for nonlinear mappings into Banach spaces*, Bull.Amer.Math.Soc., 77,73-77, (1971).

[7] BROWDER,F.E., *Normal solvability and the Fredholm alternative for mappings into infinite dimensional manifolds*, J.Functional Analysis, 8, 250-274, (1971).

[8] CARISTI,J., *Fixed point theorems for mappings satisfying inwardness conditions*, Trans.Amer.Math.Soc., 215,241-251, (1976).

[9] CRAMER,Jr.W.J. and RAY,W.O., *Solvability of nonlinear operator equations*, Pacific J.Math., 95,37-50, (1981).

[10] DOWNING,D. and KIRK,W.A., *A generalization of Caristi's theorem with applications to nonlinear mapping theory*, Pacific J.Math., 69, 339-346, (1977).

[11] EKELAND,I., *Nonconvex minimization problems*, Bull.Amer.Math.Soc., (N.S.), 1, 443-474, (1979).

[12] KAČUROVSKII,R.I., *Generalization of the Fredholm theorems and of theorems on linear operators with closed range to some classes of nonlinear operators*, (Russian), Dokl.Akad.Nauk SSSR, 197, 520-523, (1971).

[13] KIRK,W.A., *Caristi's fixed point theorem and metric convexity*, Colloq.Math., 36, 81-86, (1976).

[14] KIRK,W.A. and CARISTI,J., *Mapping theorems in metric and Banach spaces*, Bull.Acad.Pol.Sci.,Sér.Sci.Math., 23, 891-894, (1975).

[15] KRASNOSELSKII,M.A., *On several new fixed point principles*, (Russian), Dokl.Akad.Nauk SSSR, 208, 1280-1281, (1973).

[16] PASICKI,L., *A short proof of the Caristi theorem*, Comment. Math. (Prace Matem.), 20, 427-428, (1977/78).

[17] POKHOZHAYEV,S.I., *Normal solvability of non-linear equations*,

(Russian), Dokl.Akad.Nauk SSSR, 184, 40-43, (1969).

[18] ROSENHOLTZ,I. and RAY,W.O., *Mapping theorems for differentiable operators*, Bull.Acad.Pol.Sci.,Sér.Sci.Math., 29, 265-273, (1981).

[19] SIEGEL,J., *A new proof of Caristi's fixed point theorem*, Proc. Amer.Math.Soc., 66, 54-56, (1977).

[20] TURINICI,M., *A generalization of Brézis-Browder's ordering principle*, An.St.Univ."Al.I.Cuza" Iaşi, S.I-a (Mat.), 28, 11-16, (1982).

[21] TURINICI,M., *Constant and variable drop theorems on metrizable locally convex spaces*, Comment.Math.Univ.Carolinae, 23, 383-398, (1982).

[22] TURINICI,M., *Mapping theorems via variable drops in Banach spaces*, Rend.Ist.Lombardo, (A), 114, 164-168, (1980).

[23] TURINICI,M., *Mapping theorems via contractor directions in metrizable locally convex spaces*, Bull.Acad.Pol.Sci., Sér.Sci. Math., 30, 161-166, (1982).

[24] TURINICI,M., *A maximality principle on ordered metric spaces*, Rev.Colomb.Mat., to appear.

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THE RELATIVE GENERALIZED JACOBIAN MATRIX IN THE SUBDIFFERENTIAL CALCULUS

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1. INTRODUCTION.

The purpose of this article is the characterization of the relative generalized jacobian matrix of a locally Lipschitz function which is used in optimization problems with non differentiable data.

An important tool in that way was the concept of generalized jacobian matrix of F at x_o in Clarke's sense [1], we denote by $\tilde{J}F(x_o)$. From its basic definition (recalled at the beginning of Section II), one can see that $\tilde{J}F(x_o)$ takes into account the behaviour of F all around x_o. Actually, it has appeared that, for many purposes, the knowledge of all this information was not quite necessary; what needs to be known is the contribution of F restricted to a subset Q in the construction of $\tilde{J}F(x_o)$.
These considerations gave rise to what we call the generalized jacobian matrix of F relative to Q, denoted (at x_o) by $\tilde{J}_QF(x_o)$. This concept which is defined through a "lim sup" - operation on the collection $\{\tilde{J}F(x): x \in Q\}$ was firstly used in [2] for particular purposes. In Section II, we go into details of the study of $\tilde{J}_QF(.)$: conditions ensuring the connectedness of $\tilde{J}_QF(x_o)$, properties of $\tilde{J}_QF(.)$ as a set-valued mapping and chain rules. Of course, all these properties and calculus rules are mainly derived from those already exhibited for the usual generalized jacobian matrix (which turns out to be $\tilde{J}_QF(.)$ for Q the whole space) [1].

2. THE RELATIVE GENERALIZED JACOBIAN MATRIX.

Let $F: R^n \longrightarrow R^m$ be a locally Lipschitz function on O an open subset of R^n.

The generalized jaçobian matrix of F at $x_o \in O$ denoted by $\tilde{J}F(x_o)$ is the set of matrices defined as the convex hull of all matrices M of the form $M = \lim_{n\to\infty} JF(x_n)$ where x_n converges to x_o in dom F'.

In this definition, dom F' denotes the subset of full measure of O where F is differentiable.

As readily seen, this definition takes into account the behaviour of F all around x_o. In many problems it has appeared that the information is not quite necessary; what it is needed is essentially the behaviour of F relative to a given subset Q.

That let us to introduce a relative generalized jacobian matrix which can be defined on all of cl(Q) (clousure of Q).

DEFINITION 2.1. The generalized jacobian matrix of F relative to Q is defined at x_o by

$$\bigcap_{V \in E(x_o)} cl\{ \bigcup_{x \in Q \cap V} JF(x)\} \qquad (2.2)$$

where $E(x_o)$ denotes the collection of neighborhoods of x_o. The set of matrices defined in (2.2) will be denoted by $\tilde{J}_Q F(x_o)$.

Let $F = (f_1, \ldots, f_m)^t$ be a locally Lipschitz function defined on an open subset $O \subset R^n$, let $x_o \in O$ and Q be a subset of R^n such that $x_o \in cl(Q)$.

We denote by $E_Q(x_o)$ the filter of neighborhoods of x_o for the topology induced on Q. The collection $\{\tilde{J}F(x): x \in O ; E_Q(x_o)\}$ is a filtered family [3, pág.126]. For this family, we may consider the "lim sup" which we will denote by $\tilde{J}_Q F(x_o)$ and to obtain the definition (2.2).

In other words, $\tilde{J}_Q F(x_o)$ consists of all cluster points of sequences of matrices $M_n \in \tilde{J}F(x_n)$; x_n converging to x_o in Q. Clearly, $\tilde{J}_Q F(x_o)$ is empty if $x_o \notin cl(Q)$ and coincides with $\tilde{J}F(x_o)$ whenever $x_o \in Q$. If x_o lies on the boundary of Q (bdQ), $\tilde{J}_Q F(x_o)$ is, as often as not, strictly smaller than $\tilde{J}F(x_o)$.

The properties of the relative generalized jacobian matrix are mainly derived from those of the generalized jacobian matrix; let us some of them:

(P_1) Let $\{Q_\alpha : \alpha \in A\}$ be a finite collection of subsets whose union is Q; then

$$\tilde{J}_Q F(x_o) = \bigcup_{\alpha \in A} \tilde{J}_{Q_\alpha} F(x_o) \qquad (2.3)$$

(P_2) The set-valued mapping $x \rightrightarrows \tilde{J}_Q F(x)$ is locally bounded, that is to say: there exist a neighborhood V of x_o and a constant K such that $\max\{|||M||| , M \in \tilde{J}_Q F(x), x \in V\} \leq K$ where $|||.|||$ is the norm used for topologize the vector space of mxn matrices. (2.4)

(P_3) Clearly, $\tilde{J}_Q F(x_o)$ is a nonempty compact subset of the vector space of mxn matrices. (2.5)

(P_4) $\tilde{J}_Q F(.)$ is an upper semicontinuous set-valued mapping in the sense that if x_n converges to x_o in cl(Q) and M_n converges to M_o with

$$M_n \in \tilde{J}_Q F(x_n) \quad \text{for all } n \ , \quad \text{then } M_o \in \tilde{J}_Q F(x_o) \tag{2.6}$$

Indeed, $\tilde{J}_Q F(.)$ is the smallest upper semicontinuous extension to cl(Q) of the set-valued mapping $\tilde{J}F(.)$ restricted to Q.

(P_5) Note that, unlike $\tilde{J}F(x_o)$, $\tilde{J}_Q F(x_o)$ is not generally convex when $x_o \in bd(Q)$.

However, its convex hull can be expressed in a way generalizing the basic definition (2.1) of the generalized jacobian matrix. F is actually differentiable except on a set of null measure; let E denote the set contained in dom F' such that its complementary set in O is of null measure. Denoting by Q' the set $Q \cap E$, the collection $\{\tilde{J}F(x) : x \in Q' \ ; E_Q(x_o)\}$ is also a filtered family. Therefore we may carry out for this collection the "lim sup" process as done in definition (2.1).

If Q is open, we have that cl(Q') = cl(Q) and the basic definition of the $\tilde{J}F(x)$ yields:

$$co\{\tilde{J}_Q F(x_o)\} = co\{J_{Q'} F(x_o)\} \tag{2.7}$$

for all x_o.

When Q is not open, a more general relation would be that:

$$co\{\tilde{J}_Q F(x_o)\} = \bigcap_{\substack{Q \subset U \\ U \text{ open}}} co\{J_{U'} F(x_o)\} \tag{2.8}$$

(P_6) In default of convexity, $\tilde{J}_Q F(x_o)$ enjoys a weaker topological property: namely connectedness, under precisely a local connectedness assumption on Q at x_o.

PROPOSITION 2.2. *Suppose there exists a basis* $\{V_n\}_{n \in N}$ *for the neighborhood system of* x_o *such that* $Q \cap V_n$ *is connected for all* n. *(When* $x_o \in Q$ *, this assumption merely means that* Q *is locally connected at* x_o*). Then* $\tilde{J}_Q F(x_o)$ *is connected.*

Proof. Without loss of generality, we can suppose that $V_{n+1} \subset V_n$ for all n. Now, $\tilde{J}F(x)$ is nonempty and connected (since convex) for all $x \in Q \cap V_n$.

Moreover, $\tilde{J}F(.)$ is an upper semicontinuous set-valued mapping. Therefore, the image set $A_n = \bigcup_{x \in Q \cap V_n} JF(x)$ is connected [4, Theorem 5]. Thus, $\tilde{J}_Q F(x_o)$ which is the intersection of a decreasing sequence $\{cl\, A_n\}_{n \in N}$ of compact sets is connected. (q.e.d.)

Concerning chain rules on relative generalized jacobian matrices, we state the following results.

THEOREM 2.3. *Let* $f = \varphi \circ F$ *where* $\varphi: R^m \to R$ *is continuously differentiable at* $F(x_o)$. *Then*

$$\partial_Q(\varphi \circ F)(x_o) = \{M^t . \nabla\varphi(F(x_o)) : M \in \tilde{J}_Q F(x_o)\} \tag{2.9}$$

Proof. This property is directly derived from the corresponding one on generalized jacobian matrices [5,6].

Interesting properties are the following:

(P_7) Let φ be continuously differentiable at $F(x_o)$ with $\nabla\varphi(F(x_o)) \neq 0$, let $\tilde{J}_Q F(x_o)$ be surjective. Then $0 \notin \tilde{J}_Q(\varphi \circ F)(x_o)$.

By considering φ defined by $\varphi(y_1, \ldots, y_m) = y_i$ we have the following "projection" property:

$$\partial_Q f_i(x_o) = \{x_i^* : (x_1^*, \ldots, x_i^*, \ldots, x_m^*) \in \tilde{J}_Q^t F(x_o)\} \tag{2.10}$$

When φ is simply Lipschitz around x_o, things are less pleasant. By applying earlier results [7] we obtain the following result:

THEOREM 2.4. *Let* φ *be Lipschitz in a neighborhood of* $F(x_o)$. *Then*

$$\partial_Q(\varphi \circ F)(x_o) \subset co\{M^t u : u \in \partial_{F(Q)} \varphi(F(x_o)) \ ; \ M \in \tilde{J}_Q F(x_o)\} \tag{2.11}$$

Proof. We begin by recalling the following mean-value theorem:

Let $F: R^n \to F^m$ be locally Lipschitz on an open subset O of R^n, let [a,b] be a segment in O with $a \neq b$. Then there exist real numbers α_k, vectors c_k, matrices M_k; (k = 1,...,m) such that $\alpha_k \geq 0$; $c_k \in (a,b)$; $M_k \in \tilde{J}F(c_k)$ for all k, $\sum_{1 \leq k \leq m} \alpha_k = 1$ and

$$F(b) - F(a) = \sum_{1 \leq k \leq m} \alpha_k M_k(b-a) \tag{2.12}$$

When the segment [a,b] is not in O, one can assert a formula analogous to (2.12) when the open subset O is connected. In such that a case, two points a and b in O can be joined by a locally Lipschitz path σ, $\sigma: [0,1] \to O$ locally Lipschitz on (a,b); $\sigma(0) = a$; $\sigma(1) = b$. Then by application of a chain rule [8, Proposition 4.9] we have that $F(b)-F(a) \in co\{\tilde{J}F(\sigma(t))\partial\sigma(t) : t \in (0,1)\}$.

Let us consider a sequence $\{x_n\}$ converging to x_o and a sequence $\{\lambda_n\} \in R_+^*$ converging to 0.

We set $E_n = \{\varphi(F(x_n + \lambda_n d)) - \varphi(F(x_n))\}\lambda_n^{-1}$.

According to the above theorem, there exist $F_n \in (F(x_n), F(x_n+\lambda_n d))$ and $u_n \in \partial\varphi(F_n)$ such that

$$E_n = \langle F(x_n + \lambda_n d) - F(x_n) , u_n \rangle \lambda_n^{-1} \tag{2.13}$$

Now, by applying the same mean-value theorem to F, we get that

$$(F(x_n + \lambda_n d) - F(x_n))\lambda_n^{-1} = \sum_{1\le k\le m} a_{k,n} M_{k,n} d \tag{2.14}$$

where $a_{k,n} \geqslant 0$; $\sum_{1\le k\le m} a_{k,n} = 1$ and $M_{k,n} \in \tilde{J}F(c_{k,n})$ for some $c_{k,n} \in (x_n, x_n + \lambda_n d)$.

Since the set-valued mappings $\partial\varphi$ and JF are upper-semicontinuous, we may suppose that $u_n \to u \in \partial\varphi(F(x_o))$; $a_{k,n} \to a_k$ for all k ($\sum_{1\le k\le m} a_k = 1$); $M_{k,n} \to M_k \in \tilde{J}F(x_o)$ for all k.

Consequently, we derive from (2.13) and (2.14) that

$$\overline{\operatorname{Lim}}\sup_{n\to\infty} E_n \leqslant \widehat{\max}\{ \langle M^t u, d\rangle : u \in \partial\varphi(F(x_o)) ; M \in \tilde{J}F(x_o)\}.$$

Hence, $\partial(\varphi \circ F)(x_o) \subset co\{M^t u : u \in \partial\varphi(F(x_o)) ; M \in \tilde{J}F(x_o))\}$ and the result (2.11) is thereby proved. (q.e.d.)

Now, let F be continuously differentiable at x_o. Then by [9, §13] the following inclusion is valid

$$\partial_Q(\varphi \circ F)(x_o) \subset J^t F(x_o)\partial_{F(Q)}\varphi(F(x_o)) .$$

Therefore, we note:

(P_8) Let F be continuously differentiable at x_o. If $0 \notin \partial_{F(Q)}\varphi(F(x_o))$ and if $JF(x_o)$ is surjective, then $0 \notin \partial_Q(\varphi \circ F)(x_o)$. Under the same assumption on $JF(x_o)$, if $0 \notin co\{\partial_{F(Q)}\varphi(F(x_o))\}$ then $0 \notin co\{\partial_Q(\varphi \circ F)(x_o)\}$.

REFERENCES

[1] Telma Caputti - *The plenary hull of the generalized jacobian matrix and the inverse function theorem in subdifferential calculus.*

[2] J-B. Hiriart-Urruty - *Refinements of necessary optimality conditions in nondifferentiable programming I* - Applied Mathematical Optimization 5, 1979, Pág.63-82.

[3] C. Berge - *Espaces topologiques, fonctions multivoques* - Dunod, Paris (1966).

[4] J-B. Hiriart-Urruty - *On the extensions of two theorems in general topology* - Technical Notae (1978).

[5] J-B. Hiriart-Urruty - *Gradients généralisées de fonctions composées - Applications* - Note aux Comptes Rendus Acad. Sc. Paris, tomo 285, Série A, (1977). Pág.781-784.

[6] J-B. Hiriart-Urruty - *News concepts in nondifferentiable programming* - Journées d'Analyse Non Convexe (Université de Pau, Mai 1977). Bulletin Société Math. France, Mémoire 60 (1979). Pág.57-85.

[7] J-B. Hiriart-Urruty - *Théorèmes de valeur moyenne en analyse nondifférentiable: cas des fonctions localment lipschitz a valeurs vectorielles* - Note aux Comptes Rendus Acad. Sc. Paris, tomo 287, Série A (1978), Pág.751-753.

[8] B.H. Pourciau - *Analysis and optimization of lipschitz continuous mappings* - Journal of Optimization Theory and Applications, Vol.22, N°3 (1977), Págs.311-351.

[9] F.H. Clarke - *Generalized gradients of lipschitz functionals* - Advances in Mathematics, Vol.40, N°1, (1981), Págs.52-62.

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A NOTE ON THE EXTENSION OF LIPSCHITZ FUNCTIONS

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1. INTRODUCTION.

In many areas including optimization problems as well as some important questions of analysis, we have to deal with functions F satisfying a Lipschitz property only on a subset S of the whole space E. It is important to know whether F can be extended to E preserving such a property, that is whether there exists a function F_S, defined and possessing a Lipschitz property on all of E, which coincides with F on S. For such a problem, an explicit formula for the extension was given forty - five years ago by E.J.McShane [1] but one can propose here an alternative extension obtained by performing the infimal convolution of two functions associated with the data of the problem. Although conceptually identical to McShane's procedure, the extension by infimal convolution is more suitable for minimization problems. The difference will also appear to be relevant when comparing generalized gradients of the respective functions.

The first section is introductory; the second section deals with the definition and basic properties of the space of Lipschitz functions on a subset. In Section III we introduce the extension process. Section IV is devoted to comparison results between the generalized gradient of the extended function and that of the initial function. In view of applications, we consider in Section V problems dealing with optimization of the extended function. In particular, it will be proved that the search for a global or local minima of F on S is equivalent to the same problem on E with the extension as objective function.

2. LIPSCHITZ FUNCTIONS.

Let E be a real Banach space and let $\|.\|$ denote the norm of E.

Given a nonempty subset S of E, F: E $\to \bar{R}$ (the extended reals) is said to be Lipschitz on S with Lipschitz constant $r \geq 0$ if F is finite on S and if

$$|F(x) - F(y)| \leq r \|x-y\| \quad \text{for all} \quad x,y \text{ in } S \qquad (2.1)$$

The class of all such functions is denoted by $L_r^{ip}(S)$. The class of all $L_r^{ip}(s)$ for $r \geqslant 0$ is the class of Lipschitz functions on S and is denoted by $L^{ip}(S)$. It is evident that $F \in L^{ip}(S)$ only in the case where

$$|||F||| = \sup\{ \frac{|F(x)-F(y)|}{\|x-y\|} \ ; \ y,x \text{ in } S \ ; \ x \neq y\} < \infty \tag{2.2}$$

$|||F|||$ is the least number r such that (1.1) holds for F.

Suppose that $x \in S$ and define

$$|||F|||_{\overline{x}} = |F(\overline{x})| + |||F||| \quad \text{for all} \quad F \in L^{ip}(S).$$

Then $(L^{ip}(S), |||.|||_{\overline{x}})$ is a Banach space [2].

Since only the values of F on S are relevant for our purpose, we will make a constant use of F defined on E by

$$\overline{F}(x) = F(x) \quad \text{if} \quad x \in S \quad ; \quad +\infty \quad \text{if not} \tag{2.3}$$

In particular, the Lipschitz property of F on S may be expressed in terms of the infimal convolution $F_{S,r}$ which appears as the result of a sort of regularization as follows:

Let F be non identically $(+\infty)$ or $(-\infty)$ on S; then $F \in L_r^{ip}(S)$ if and only if

$$\overline{F} \,\square\, r\,\|.\| = F \quad \text{on} \quad S \tag{2.4}$$

which is, furthermore, equivalent to

$$\overline{F} \,\square\, r\,\|.\| \geqslant F \quad \text{on} \quad S \tag{2.5}$$

where the symbol $\square$ denotes infimal convolution defined by: Let g and h be two functions from E into $\overline{R}$, the infimal convolution of g and h is a function, denoted by $g \,\square\, h$, which assigns to $x \in E$ the value

$$\inf_{u \in E} \{g(u) + h(x-u)\} .$$

The general properties of this binary operation, particularly those related to convex analysis are developed in [3].

3. EXTENSION OF THE RANGE OF A LIPSCHITZ FUNCTION.

Let S be a nonempty subset of E and let $F \in L_r^{ip}(S)$. In 1934, McShane showed that such a function F could be extended to the whole space E by preserving a Lipschitz condition. Actually, his procedure yielded an explicit formula for the extension $F^{S,r}$ which was

$$F^{S,r}(x) = \sup_{u\in S} \{F(u) - r \|x-u\|\} \qquad (3.1)$$

$F^{S,r}$ turns out to be Lipschitz on E with r as Lipschitz constant and coincides with F on S.

We define another extension which is conceptually related to McShane's one [1]. The definition of the extended function $F_{S,r}$ comes naturally from paragraph 2 as

$$F_{S,r} = \overline{F} \,\square\, r \|.\| \qquad (3.2)$$

In more explicit way

$$F_{S,r}(x) = \inf_{u\in S} \{F(u) + r \|x-u\|\} \text{ for all } x \text{ in } E .$$

Clearly if $F \in L_r^{ip}(S)$ then $F_{S,r} \in L_r^{ip}(S)$ and coincides with F on S.

4. THE GENERALIZED GRADIENT OF THE EXTENDED FUNCTION.

Given a function F Lipschitz in a neighborhood of $x_o \in E$, the generalized gradient of F at x_o in Clarke's sense [5] is a subset of E* (topological dual space of E) denoted by $\partial F(x_o)$ and defined as follows:

$$\partial F(x_o) = \{x^* \in E^*: \langle x^*,d\rangle \leqslant F^\circ(x_o;d) \text{ for all } d \in E\} \qquad (4.1)$$

where

$$F^\circ(x_o;d) = \operatorname*{Lim\,sup}_{\substack{x\to x_o\\ \lambda\to 0}} \frac{F(x+\lambda d) - F(x)}{\lambda} \qquad (4.2)$$

The definition of the generalized gradient for an arbitrary function requires some preliminary definitions. Let E be a real Banach space, let A be a subset of E and let $u_o \in cl(A)$ (closure of A).

DEFINITION 4.3. δ is tangent direction to A at u_o if and only if for every sequence $\{u_n\} \subset A$ converging to u_o and for every sequence $\{\lambda_n\} \subset R_+^*$ converging to 0, there exists a sequence $\{\delta_n\}$ converging to δ such that $u_n + \lambda_n \delta_n \in A$ for all n.

The cone of all tangent directions to A at u_o is the *tangent cone to* A at u_o and will be denoted by $T_A(u_o)$. Its polar cone, i.e. the set of n in E* such that $\langle n,\delta\rangle \leqslant 0$ for all $\delta \in T_A(u_o)$ is called the *normal cone* to A at u_o and will be denoted by $N_A(u_o)$.

Let F: E $\longrightarrow \overline{R}$ be finite at x_o and Lipschitz around x_o. Starting from the geometric concept of tangent cone, the generalized directional derivative of F at x_o is defined by

$$d \to F^\circ(x_o;d) = \inf\{\mu \in R : (d,\mu) \in T_{\text{epi}F}(x_o,F(x_o))\} \tag{4.4}$$

The relationship with the normal cone is given as follows:

$$\partial F(x_o) = \{x^* \in E^* : (x^*,-1) \in N_{\text{epi}F}(x_o,F(x_o))\} \tag{4.5}$$

For the indicator function of a subset S

$$\delta(x/S) = 0 \quad \text{if} \quad x \in S \quad ; \quad \delta(x/S) = +\infty \quad \text{if} \quad x \notin S$$

one has $\delta(x_o/S) = N(S;x_o)$. For more details on what has been recalled above, see [3].

Concerning the generalized gradients of $\overline{F}$ and $F_{S,r}$ such as defined in the previous paragraph 3, we have a general comparison result:

THEOREM 4.6. *Let* x_o *in* S. *Then,*

a) *for all* $r \geq |||F|||$, $\partial\overline{F}(x_o) \subset \partial F_{S,r}(x_o) + N(S;x_o)$.

b) *for all* $r \geq |||F|||$, $\partial F_{S,r}(x_o) \subset \partial\overline{F}(x_o) \cap rB^*$ *where* B^* *denotes the closed unit ball in* E^**:*

Proof. a) Since $F_{S,r}$ coincides with F on S, we have that $\overline{F} = F_{S,r} + \delta(./S)$. Then the announced result follows from the calculus rule giving an estimate of the generalized gradient of the sum of two functions [6, Theorem 2].

b) $F_{S,r}$ is Lipschitz with constant r, therefore

$$F^\circ_{S,r}(x_o;d) \leq r\|d\| \text{ for all } d \text{ and } \partial F_{S,r}(x_o) \subset rB^*$$

If x_o is in int(S), $F_{S,r} = \overline{F} = F$ in a neighborhood of x_o; thus

$$\partial F_{S,r}(x_o) = \partial\overline{F}(x_o) = \partial F(x_o)$$

Let now, x_o in $S \cap \text{bd}(S)$; we have $F_{S,r}(x_o) = \overline{F}(x_o) = F(x_o)$; the inclusion

$$\partial F_{S,r}(x_o) \subset \partial\overline{F}(x_o)$$

is then equivalent to the following one

$$T_{\text{epi}\overline{F}}(x_o,F(x_o)) \subset T_{\text{epi}F_{S,r}}(x_o,F(x_o)) \tag{4.7}$$

Let (d,μ) in $T_{\text{epi}\overline{F}}(x_o,F(x_o))$. We consider a sequence $\{x_n\}$ converging to x_o and a sequence $\{\lambda_n\} \subset R^*_+$ converging to 0. With $\{x_n\}$ and $\{\lambda_n\}$ we associate a sequence $\{\overline{x}_n\} \subset S$ such that $\overline{x} \in M(\lambda_n^2,x_n)$ for all n where $M(.,.)$ is given as in Theorem 5.1.b).

Since $r > |||F|||'$, $\{\overline{x}_n\}$ converges to x_o, therefore the sequence $\{(\overline{x}_n, F(\overline{x}_n))\}$ converges to $(x_o, F(x_o))$ in epi $\overline{F}$. Since $(d,\mu) \in$ $\in T_{epi\overline{F}}(x_o, F(x_o))$, there exists a sequence $\{(d_n,\mu_n)\}$ converging to (d,μ) such that $\overline{x}_n + \lambda_n d_n \in S$ and

$$F(\overline{x}_n + \lambda_n d_n) \leqslant F(\overline{x}_n) + \lambda_n \mu_n \text{ for all } n.$$

Due to the Lipschitz property of $F_{S,r}$, we get that

$$F_{S,r}(x_n + \lambda_n d_n) \leqslant F(\overline{x}_n) + r \, \|x_n - \overline{x}_n\| + \lambda_n \mu_n$$

Since $\overline{x}_n \in M(\lambda_n^2, x_n)$, we have that

$F_{S,r}(x_n + \lambda_n d_n) \leqslant F_{S,r}(x_n) + \lambda_n(\mu_n + \lambda_n)$. Hence, since $(d_n, \lambda_n + \mu_n) \to (d,\mu), (d,\mu) \in T_{epiF_{S,r}}(x_o, F(x_o))$ and the inclusion (4.7) is proved. (q.e.d.)

5. OPTIMIZATION OF LIPSCHITZ FUNCTIONS.

Given S a nonempty subset of E and $F \in L_r^{ip}$ we consider the problem of minimizing (at least locally) F on S: (P) minimize F on S.

A device for converting the constrained optimization problem (P) into an unconstrained one is to consider

(P*) minimize $\overline{F}$ on E.

Of course, x_o is a local minimum of F on S if and only if x_o is a local minimum of F on E.

Similar properties hold for the extended function $F_{S,r}$ with the advantage that $F_{S,r}$ is finite and Lipschitz over all E.

THEOREM 5.1. *Let* S *be closed in* E.

a) x_o *is a global minimum of* F *on* S *if and only if* x_o *is a global minimum of* $F_{S,r}$ *on* E $(r > 0)$.

b) x_o *is a local minimum of* F *on* S *if and only if* x_o *is a local minimum of* $F_{S,r}$ *on* E *whenever* $r > |||F|||$.

Proof. a) Let x_o in S such that $F(u) \geqslant F(x_o)$ for all u in S. Clearly, $F_{S,r}(x) = \inf_{u \in S} \{F(u) + r\,\|x-u\|\} = F(x_o)$ for all x in E.

Conversely, let x_o be a global minimum of $F_{S,r}$ on E. The only thing to prove is that x_o necessarily belongs to S. For that, we suppose that $d_S(x_o) = a > 0$ (distance from x_o to S).

Let $\bar{x}$ in S be such that

$$F_{S,r}(x_o) > F(\bar{x}) + r \|\bar{x}-x_o\| - \frac{ra}{2} \tag{5.2}$$

Since $F_{S,r}$ agrees with F on S and $\bar{x} \in S$, we have that

$$F(\bar{x}) \geqslant F_{S,r}(x_o) \quad \text{and} \quad \|\bar{x}-x_o\| \geqslant a$$

That is inconsistent with inequality (5.2); hence $a=0$ and since S is closed $x_o \in S$.

b) Let x_o in S be a local minimum of F on S. So, there exists $\rho > 0$ such that $F(u) \geqslant F(x_o)$ whenever $u \in S$ and $\|u-x_o\| \leqslant \rho$.

There exists $\rho_o > 0$ and $\varepsilon_o > 0$ such that

$\|x-x_o\| \leqslant \rho_o$; $\varepsilon \leqslant \varepsilon_o \Rightarrow \|\bar{x}-x_o\| \leqslant \rho$ for all $\bar{x}$ in $M(\varepsilon,x)$, where

$$M(\varepsilon,x) = \{u \in S / F(u) + r \|x-u\| \leqslant F_{S,r}(x) + \varepsilon\}$$

with $\varepsilon \geqslant 0$ and x in E and $M(x) = M(0,x)$.

Clearly, $M(\varepsilon,x)$ is nonempty for all x in E and all $\varepsilon > 0$. Furthermore, if $x \in S$, $M(x)$ contains x and is reduced to $\{x\}$ whenever $r > |\!|\!|F|\!|\!|$.

Generally speaking, computing $F_{S,r}(x)$ gives rise to an abstract optimization problem. It is important to know the behaviour of the set of solutions or of approximate solutions.

Consequently, $F_{S,r}(x) \geqslant F(x_o)$ whenever $\|x-x_o\| \leqslant \rho_o$. Conversely, let us prove that x_o local minimum of $F_{S,r}$ on E is in S. Then exists $\rho > 0$ such that $F_{S,r}(x) \geqslant F_{S,r}(x_o)$ if $\|x-x_o\| \leqslant \rho$. Let us suppose that $d_S(x_o) = a > 0$; we set $\varepsilon < r/2 \min\{\rho,a\}$ and we choose $\bar{x}$ in S satisfying $F_{S,r}(x_o) > F(\bar{x}) + r\|\bar{x}-x_o\| - \varepsilon$.

Let $\theta = 1/2 \min \{\rho,a\}$ and $x^* = x_o + \theta \dfrac{\bar{x} - x_o}{\|\bar{x} - x_o\|}$. We have that $F_{S,r}(x^*) > F(\bar{x}) + r \|\bar{x}-x_o\| - \varepsilon$. Since $x^* \notin S$ and $\|\bar{x}-x_o\| = \|\bar{x}-x^*\| + \theta$ we deduce from (5.2) that $r\theta < \varepsilon$; hence the contradiction from the choice of ε. (q.e.d.)

REMARK. Let $f: O \to R$ be a function defined on an open subset O of R^n and Lipschitz in a neighborhood of x_o. Let $B(x_o,\varepsilon)$ be a closed ball around x_o of radius included in O.

We denote by r the Lipschitz constant of f on $B(x_o,\varepsilon)$ and we set

$f^* = f + \delta_{B(x_o,\varepsilon)}$ where $\delta_{B(x_o,\varepsilon)}$ is the indicator function of $B(x_o,\varepsilon)$ defined by

$$\delta_{B(x_o,\varepsilon)}(x) = \begin{cases} 0 \text{ if } x \text{ is in } B(x_o,\varepsilon) \\ +\infty \quad \text{otherwise.} \end{cases}$$

Now, we perform the infimal convolution of f^* and the function $r\|.\|$, that is

$$\tilde{f}(x) = \inf \{f^*(x_1) + r\,\|x_2\| : x_1 + x_2 = x\} .$$

It is easy to see that by performing this operation we produce a function $\tilde{f}$ such that

i) $\tilde{f}$ is Lipschitz on the whole space with Lipschitz constant r.

ii) $\tilde{f}(x) = f(x)$ when x is in $B(x_o,\varepsilon)$.

iii) $\lim\limits_{\|x\|\to\infty} f(x) = +\infty$.

Now, if x_o is a local minimum of f on S, x_o becomes a global minimum of f^* on S. In order to isolate x_o, we may substitute

$$\tilde{\tilde{f}}: x \longrightarrow \tilde{f}(x) + \|x-x_o\|^2 \quad \text{for} \quad \tilde{f}(x) .$$

In a neighborhood of x_o, $\tilde{\tilde{f}}$ is differentiable at x whenever f is differentiable at x. Thus, is no trouble in the calculation of the generalized jacobian matrix and if $F = (f,f_1,\ldots,f_m)^t$ and $\tilde{\tilde{F}} = (\tilde{\tilde{f}},f_1,\ldots,f_m)^t$ then

$$\tilde{J}_S(F;x_o) = \tilde{J}_S(\tilde{\tilde{F}},x_o) \qquad [4]$$

So, one can safely regard x_o as a unique and strong global minimum of f on S and suppose that $\lim\limits_{\|x\|\to\infty} f(x) = +\infty$.

REFERENCES

[1] E.J. McShane. *Extension of range of functions*. Bulletin of the American Mathematical Society, 40 (1934) p.837-843.

[2] R.H. Martin. *Nonlinear Operators and Differential Equations in Banach Spaces*. John Wiley & Sons (1976).

[3] R.T. Rockafellar. *Convex Analysis*. Princeton University Press (1970).

[4] T. Caputti. *The relative generalized jacobian matrix in the subdifferential calculus*.

[5] T. Caputti. *The plenary hull of the generalized jacobian matrix and the inverse function theorem in the subdifferential calculus*. Rev. UMA. Vol.30 - N°3/4 - 1982-1983. pp.190-196.

[6] R.T. Rockafellar. *Directionally Lipschitzian functions and subdifferential calculus*. Proceedings of the London Mathematical Society - Vol.39(3), 1979, pp.331-355.

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COMPARACION DE SOLUCIONES DE LAS ECUACIONES DE PRANDTL EN EL CASO ESTACIONARIO BIDIMENSIONAL

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SUMMARY.

Solutions $\tilde{u} > 0$, $\tilde{v}$ of Prandtl's boundary layer equations (1,1),(1,2) for steady viscous flows along a flat plate are considered (cf. [1], [2]). It is customary that the qualitative study of the solutions to these equations be performed by means of the classical maximum principles for parabolic equations, applied either directly to (1,1), (1,2) or through the von Mises transformation (cf. [3] and references therein). This application requires the smoothness of solution - and derivatives - up to the boundaries, including some matching at the leading edge x=0 of the plate for the initial profile (2,2), and the explicit statement of condition (2,3).

Although the latter may be a sensible condition physically speaking, conditions on the behaviour of the solution near the leading edge are far from satisfactory. In this paper the von Mises transformed equation (3) is employed, (2,3) is replaced by a boundedness condition on the behaviour of u, u_η at $\eta \to \infty$ (cf. (6,1),(7,5)) which is a consequence of the boundedness of u in the newtonian case, and (2,2) is defined in the L^1 sense: $\|u(x,.) - u_0(.)\|_{L^1_{loc}} \longrightarrow 0$ as $x \to 0+$ (cf. (4,3) for its use for comparison purposes).

Two comparison theorems (Sections III and IV) are presented which are valid in a class of solutions (4,1),(4,2),(4,3) that includes the classical solutions of [4] (this class was introduced in [5], cf. also [6]). The restrictive condition $U'(x) \geqslant 0$ of Section III is removed in IV by the introduction of (7,3),(7,4): Theorem 2 of IV furnishes uniqueness of solutions with initial profile monotone increasing (cf. Section V, Corollaries).

The method of proof applies also to certain fluids with non-newtonian constituting terms.

I. Las ecuaciones de L. Prandtl, de la capa límite sobre una placa plana que ocupa el semieje $x \geqslant 0$ son ([1],[2]) (1,1) $\widetilde{u}\widetilde{u}_x + \widetilde{v}\widetilde{u}_y = (\nu\widetilde{u}_y)_y + U(x)U'(x)$,

(1,2) $\widetilde{u}_x + \widetilde{v}_y = 0$

La ecuación de Bernoulli $2p(x) + U^2(x) =$ constante relaciona la presión $p(x)$ con la velocidad $U(x) > 0$ del fluido lejos de la placa, considerado no viscoso. Las condiciones sobre las componentes $\widetilde{u},\widetilde{v}$ de la velocidad en la capa límite donde el fluido tiene viscosidad cinemática ν son

(2,1) $\widetilde{u}(x,0) = 0$, $\widetilde{v}(x,0) = \widetilde{v}_0(x)$

(2,2) $\widetilde{u}(0,y) = \widetilde{u}_0(y)$ (perfil inicial)

(2,3) $\lim_{y\to+\infty} \widetilde{u}(x,y) = U(x)$ uniforme en $[0,X]$, cualquiera sea X.

En el caso tratado habitualmente, en que $\widetilde{u},\widetilde{v}$ son soluciones clásicas, $\widetilde{v}(x,0) = 0$, $\widetilde{u}_0'(0) > 0$, $\widetilde{u}$, $\widetilde{u}_x$, $\widetilde{u}_y$, continuas en $[0,X] \times [0,\infty)$, basta suponer $U' > 0$ para que $u > 0$ y $u_y(x,0) > 0$, es decir, no haya reflujo ni separación.

En este trabajo consideraré exclusivamente el caso laminar sin reflujo, eso es, $\widetilde{u}(x,y) > 0$ en $(0,X) \times (0,+\infty) =: \widetilde{G}$.

Es sabido ([3]) que, si $\widetilde{u}$ es acotada, por ejemplo, entonces de $\lim_{y\to\infty} \widetilde{u}(0,y) = U(0)$ resulta $\lim_{y\to\infty} \widetilde{u}(x,y) = U(x)$ uniforme en $[0,X]$.
Esto sugiere la posibilidad de obviar la condición en $y = +\infty$, como se verá más adelante ((6,1) y (7,5)).

La condición $\widetilde{v}(x,0) = \widetilde{v}_0(x)$ permite considerar casos de succión ($\widetilde{v}_0 \leqslant 0$) o soplido ($\widetilde{v}_0 \geqslant 0$) de la capa límite sobre la placa; el caso en que ésta es impenetrable ($\widetilde{v}_0 = 0$) es el más corriente. En [4] se presentan diversos resultados de existencia de solución clásica para (1,1),...,(2,3), probándose en uno de ellos la existencia de soluciones $\widetilde{u} > 0$, aún en el caso en que $U'(x) \leqslant 0$, con tal que $\widetilde{v}_0(x)$ sea negativa y suficientemente grande en módulo en los puntos donde $U'(x) < 0$. Vale decir, si la succión es suficientemente intensa en los puntos donde $p'(x) > 0$ ("gradiente de presiones desfavorable"), la capa límite no se separa, y no hay reflujo, claro.

La suposición $\widetilde{u} > 0$ permite introducir la transformación de variables de von Mises:

$$\eta = \int_0^y \widetilde{u}(x,s)\,ds - \int_0^x \widetilde{v}(s,0)\,ds \quad , \quad \widetilde{u}(x,y) = u(x,\eta)$$

Sea G el nuevo dominio en las variables (x,η), donde $x \in (0,X)$ y $\eta > - \int_0^x \tilde{v}(s,0)\, ds$.

La ecuación de la continuidad (1,2) queda automáticamente satisfecha, y (1,1) se escribe ahora

(3) $$u_x = (\nu(uu_\eta))_\eta + U(x)U'(x)/u$$

Es una ecuación de tipo parabólico, que degenera pues $u=0$ en $\eta = - \int_0^x \tilde{v}(s,0)\, ds$ - curva que corresponde a $y=0$ -, y que exhibe una fuente no lineal, con singularidad sobre dicha curva.

A pesar de estas complicaciones emplearé la ecuación (3) en lugar de la más conocida $(u^2 - U^2)_x = \nu.u.(u^2 - U^2)_{\eta\eta}$ y arrastraré la expresión $\nu(uu_\eta)$, que sugiere la posibilidad de aplicar estos resultados a fluidos con términos de tensiones más complejos que el newtoniano $\nu.\tilde{u}_{yy}$.

II. El objeto de este trabajo es comparar dos perfiles de velocidades $u_1(x,\eta)$, $u(x,\eta)$ que provienen vía la transformación de von Mises de dos soluciones $\tilde{u}_1,\tilde{v}_1$ y $\tilde{u},\tilde{v}$ de (1,1) (1,2) de datos $U_1(x)$, $U(x)$, $\tilde{u}_1(x,0) = \tilde{u}(x,0) = 0$, $\tilde{v}_1(x,0)$, $\tilde{v}(x,0)$ y perfiles iniciales $\tilde{u}_1(0,y)$, $\tilde{u}(0,y)$ respectivamente.

Se sobreentiende que $u_1(x,\eta)$ y $u(x,\eta)$ son soluciones de las correspondientes ecuaciones (3).

El propósito es obtener condiciones bajo las cuales $u_1(x,\eta) \leqslant u(x,\eta)$, y esto resultará de demostrar que $D = \{(x,\eta): u_1(x,\eta) > u(x,\eta)\}$ es vacío.

Es claro que, en las variables de von Mises, u_1 y u están definidas en dominios $G_1 = \{(x,\eta): x \in (0,X),\ \eta > - \int_0^x \tilde{v}_1(s,0)\, ds\}$ y $G = \{(x,\eta): x \in (0,X),\ \eta > - \int_0^x \tilde{v}(s,0)\, ds\}$ donde la hipótesis $\tilde{v}_1(x,0) \leqslant \tilde{v}(x,0)$ implica que $G_1 \subset G$. Como $u_1 > 0$, $u > 0$ es claro que $D \subset G_1$.

Por regla general se consideran soluciones clásicas de (3). Cabe precisar, sin embargo, que de las soluciones u_1,u supondré que son continuas y que $\chi_{G_1}.u_{1x}$, $\chi_G.u_x$, $\chi_{G_1}\nu(u_1u_{1\eta})$ y $\chi_G\nu(uu_\eta)$ pertenecen a

(4,1) $L^1_{loc}((0,X);L^1_{loc}(-\infty,+\infty))$

al igual que $\chi_{G_1}(\nu(u_1u_{1\eta}))_\eta$ y $\chi_G(\nu(uu_\eta))_\eta$ y las ecuaciones se verifican en casi todo (x,η).

Con estas hipótesis existen los límites

(4,2) $\lim_{\eta\downarrow-\int_0^x \tilde{v}_1(s,0)ds} u_1(x,\eta)$ y $\lim_{\eta\downarrow-\int_0^x \tilde{v}(s,0)ds} u(x,\eta)$

que serán nulos para casi todo x de acuerdo a (2,1).

La condición $u_1(0,\eta) \leq u(0,\eta)$ sobre los perfiles iniciales se entenderá así:

(4,3) $\|(u_1(x,.)-u(x,.))^+\|_{L^1(-\infty,m)} \longrightarrow 0$, $x \longrightarrow 0+$, para todo $m>0$.

La asunción del perfil inicial para $x \longrightarrow 0+$ en norma L^1 tiende a obviar condiciones de compatibilidad entre $u(0,\eta)$ y $u(x,0)$ en $(0,0)$, eso es, en el borde de ataque de la placa.

El siguiente resultado preliminar ilustrará el método empleado

III. TEOREMA 1. *Si* $u_1(x,\eta) > 0$, $u(x,\eta) > 0$, $x \in [0,X]$, *son acotadas* $(\leq M)$ *cuando* $\eta \to \infty$; *si* $\tilde{v}_1(x,0) \leq \tilde{v}(x,0)$, $x \in [0,X]$, $u_1(0,\eta) \leq$ $\leq u(0,\eta)$ *según* (4,3) *y*

(5,1) $U_i(x)U_1'(x) \leq U(x)U'(x)$, $U(x)U'(x) \geq 0$,

entonces $u_1(x,\eta) \leq u(x,\eta)$ *en* G_1.

Demostración. Sea $D = \{(x,\eta): u_1(x,\eta) > u(x,\eta)\}$. Es claro que $D \subset G_1$; sea $\chi = \chi_D$ su función característica. Tomando

$$n > m > \max\{-\int_0^x v_1(s,0)\,ds\ ,\ x \in [0,X]\}\ , \quad \text{sea}$$

$$T(\eta) = T_{n,m}(\eta) = 1 \text{ en } (-\infty,m)\ ,$$
$$= 0 \text{ en } (n,+\infty)\ ,$$
$$= \text{lineal en } (m,n).$$

Multiplicando por $T(\eta)$ a

$$(u_1-u)_x = \{\nu(u_1u_{1\eta}) - \nu(uu_\eta)\}_\eta + \frac{U_1U_1'}{u_1} - \frac{U\,U'}{u}$$

e integrando sobre D entre x = a y x , se obtiene (luego de aplicar el teorema de Fubini a $\chi.(u_1-u)_x$ (cf. [5], [6])):

$$\int_{-\infty}^{\infty} T(\eta)(u_1-u)^+(x,\eta)\,d\eta - \int_{-\infty}^{\infty} T(\eta).(u_1-u)^+(a,\eta)\,d\eta \leq$$

$$\leq \int_a^x ds \int_{-\infty}^{\infty} \chi(s,\eta)\{T(\eta).(\nu(u_1u_{1\eta}) - \nu(uu_\eta))\}_\eta d\eta +$$

$$+ \int_a^x ds\,(n-m)^{-1} \int_m^n \chi(s,\eta)(\nu(u_1u_{1\eta}) - \nu(uu_\eta))\,d\eta +$$

$$+ \int_a^x ds \int_{-\infty}^{\infty} \chi(s,\eta).T(\eta)(U_1U_1' - UU')u_1^{-1}\,d\eta +$$

$$+ \int_a^x ds \int_{-\infty}^{\infty} (-UU')^+(u_1u)^{-1}.T(\eta).(u_1-u)^+(s,\eta)\,d\eta .$$

En el 2° miembro de (6), el cuarto sumando es nulo por hipótesis (y (6) fue escrita así para ser empleada en el Teorema 2); el tercer sumando es ≤ 0 y la eventual singularidad de u_1^{-1} no afecta los razonamientos que siguen. También es ≤ 0 el primero: para cada $s \in (a,x)$ fijo, el dominio de integración en η es una unión de intervalos donde $u_1 > u$, y en los extremos de los cuales $u_1 = u$: es fácil ver que entonces los términos integrados en η en esos intervalos son ≤ 0 en los extremos derechos y ≥ 0 en los izquierdos, resultando ≤ 0 la diferencia.

Resta estudiar el segundo término, donde $n > m$ pueden elegirse en forma arbitraria: se ve, integrando $\frac{\nu}{2}((u_1^2)_\eta - (u^2)_\eta)$ y usando la acotación de u_1,u que

$$\frac{\nu/2}{n-m} \left|\int_a^x [\chi(u_1^2 - u^2)(s,n) - \chi(u_1^2 - u^2)(s,m)]\,ds\right| \leq \frac{\nu/2}{n-m}.X.4M^2$$

y sigue que tomando m arbitrario y n suficientemente grande

$$\|(u_1 - u)^+(x,.)\|_{L^1(-\infty,m)} \leq \int_{-\infty}^{\infty} T_{n,m}(\eta)(u_1-u)^+(x,\eta)\,d\eta \leq$$

$$\leq \int_{-\infty}^{\infty} T_{n,m}(\eta)(u_1-u)^+(a,\eta)d\eta + \varepsilon/2 \leq \|(u_1-u)^+(a,.)\|_{L^1(0,n)} + \varepsilon/2$$

Por (4,3) haciendo $a \to 0+$ resulta entonces

$$\|(u_1-u)^+(x,.)\|_{L^1(-\infty,m)} = 0 \text{ , para todo } x \in (0,X) \text{ y } m > 0 \text{ ,}$$

de donde D es vacío y $u_1 \leq u$ en G_1.

Es de destacar que no se usó la condición habitual $\lim_{n\to\infty} u(x,\eta) = U(x)$, y que aún la hipótesis de acotación de u_1 y u es claramente excesiva, siendo suficiente suponer para cada función una cota independiente de a y x para

(6,1) $$\int_a^x (n-m)^{-1} \int_m^n \chi(s,\eta)\ \nu(uu_\eta)\ d\eta\ ds$$

que tienda a cero para $n,m \to \infty$.

Asimismo, el término νuu_η del tensor de tensiones podría ser una función creciente de uu_η, tal como lo sugiere la notación usada $\nu(uu_\eta)$: por ejemplo $\nu(uu_\eta) = \text{const.}|uu_\eta|^{k-1}.uu_\eta$ que corresponde al caso de fluidos no newtonianos con una ley constitutiva potencial; en variables físicas cons.$|\tilde{u}_y|^{k-1}.\tilde{u}_y$. Más aún, podría admitirse que $\nu = A(x,\eta,u,u_\eta)$ bajo ciertas restricciones sobre A; el primer término de (6) es $\leqslant 0$ en estos casos (cf. [5], [6]) y bastará imponer una condición que controle el correspondiente promedio (6,1) (ver comunicación [7]).

La hipótesis (5,1), $U(x)U'(x) \geqslant 0$, importa suponer para el flujo u,v un "gradiente favorable de presiones" $p'(x) \leqslant 0$, de acuerdo a la fórmula de Bernoulli. Es sabido que - al menos para soluciones clásicas de (1,1) (1,2), caso newtoniano - esto implica $u(x,\eta) > 0$.

En el teorema siguiente voy a prescindir de esta condición $UU' = -p'(x) \geqslant 0$, pero deberé introducir otras hipótesis que permitan tratar el término singular de la ecuación (3).

IV. TEOREMA 2. *Sean* $u_1(x,\eta) > 0$, $u(x,\eta) > 0$ *soluciones de respectivas ecuaciones* (3) *en* G_1, G, *con*

(7,1) $U_1(x)U_1'(x) \leqslant U(x)U'(x)$, $x \in [0,X]$,

(7,2) $\tilde{v}_1(x,0) \leqslant \tilde{v}(x,0)$, $x \in [0,X]$, *y por lo tanto* $G_1 \subset G$;

(7,3) *Sea* $u(x,\eta) \geqslant \alpha > 0$ *si* $\eta \geqslant \eta_\alpha$, *para todo* $x \in [0,X]$ (*sería una consecuencia obvia de* (2,3) *si* $U(x) > 0$ *en* $[0,X]$) ;

(7,4) $u_1(0,\eta) \leqslant u(0,\eta+\varepsilon) =: u_\varepsilon(0,\eta)$ *para todo* $\varepsilon > 0$ *suficientemente pequeño* (*en el sentido* (4,3));

(7,5) *Finalmente, sea* $J(x) = J_{m,n}(x)$ *una cota independiente de* a *para el término*

$$\int_a^x ds(n-m)^{-1} \left|\int_m^n \chi(s,\eta)(\nu(u_1u_{1\eta}) - \nu(uu_\eta))\ d\eta\right|$$

tal que $J_{m,n}(x)$ *tiende a cero acotada por una función integrable cuando* $m,n \to \infty$ *(para el caso newtoniano, basta la acotación de* u_1, u *como ya se ha visto).*
Entonces $u_1(x,\eta) \leq u(x,\eta+\varepsilon)$, *para todo* $\varepsilon > 0$ *suficientemente pequeño, es decir,* $u_1(x,\eta) \leq u(x,\eta)$.

Demostración. La función $u_\varepsilon(x,\eta) = u(x,\eta+\varepsilon)$ está definida en $G_\varepsilon = \{(x,\eta): \eta > -\int_0^x \tilde{v}(s,0)\,ds - \varepsilon\}$ y verifica allí la ecuación (3). Sea $D = \{(x,\eta): u_1(x,\eta) > u_\varepsilon(x,\eta)\}$: como $D \subset G_1$, D se encuentra a una distancia positiva del borde inferior de G_ε:

$$-\int_0^x \tilde{v}_1(s,0)\,ds \geq -\int_0^x \tilde{v}(s,0)\,ds > -\int_0^x \tilde{v}(s,0)\,ds - \varepsilon\ ;$$

resulta u_ε estrictamente positivo en $\bar{D}$ y por la condición (7,3), $\delta = \delta_\varepsilon = \inf\{u_\varepsilon(x,\eta);(x,\eta) \in D\} > 0$.
Llevando estas consideraciones a (6) y poniendo
$M = \max\{(-UU')^+: x \in [0,X]\}$ se obtiene, con $a > 0$ y
$\|.\| = \|.\|_{L^1(-\infty < \eta < \infty)}$,

$$\|(T(u_1-u_\varepsilon)^+)(x,.)\| \leq \|(T(u_1-u_\varepsilon)^+(a,.)\| + J_{m,n}(x) + M\delta^{-2}\int_a^x \|(T(u_1-u_\varepsilon)^+)(s,.)\|\,ds\ .$$

Esta es una desigualdad de tipo Gronwall de la cual surge

$$\|(T(u_1-u_\varepsilon)^+)(x,.)\| \leq e^{M\delta^{-2}(x-a)}\|T(u_1-u_\varepsilon)^+(a,.)\| + J_{m,n}(x) + M\delta^{-2}\int_a^x e^{M\delta^{-2}(x-s)}J_{m,n}(s)\,ds\ .$$

Tomando entonces m arbitrario y $n > m$ suficientemente grande, los dos últimos sumandos de (6) son $< o(m,n)$, y

$$\begin{aligned}\|(u_1-u_\varepsilon)^+(x,.)\|_{L^1(-\infty,m)} &\leq \|(T(u_1-u_\varepsilon)^+)(x,.)\| \leq \\ &\leq e^{M^{\delta-2}x}\|(T(u_1-u_\varepsilon)^+)(a,.)\| + o(m,n) \leq \\ &\leq e^{M\delta^{-2}x}\|(u_1-u_\varepsilon)^+(a,.)\|_{L^1(-\infty,n)} + o(m,n)\end{aligned}$$

y haciendo $a \to 0+$ resulta $\|(u_1-u_\varepsilon)^+(x,.)\|_{L^1(-\infty,m)} = 0$, m arbitrario, luego $u_1(x,\eta) \leq u(x,\eta+\varepsilon)$ en G_1, para todo $\varepsilon > 0$ suficientemente pequeño. El teorema queda así demostrado.

V. COMENTARIOS.

La condición (7,4) se cumple si $u_1(0,\eta) \leq u(0,\eta)$ y una al menos de las funciones es monótona (no decreciente). Si ambas son no decrecientes, entonces también sus correspondientes $\tilde{u}_1(0,y)$, $\tilde{u}(0,y)$ lo son y recíprocamente; la condición (7,4) sobre los perfiles iniciales (en las variables de von Mises) parece menos restrictiva y más manejable que la condición clásica de que uno de los perfiles $\tilde{u}_1(x,y)$, $u(x,y)$ sea cóncavo, condición ésta que aparece naturalmente para aplicar el principio del máximo a (3) escrita en la forma $(u^2)_x = \nu u(u^2)_{\eta\eta} + (U^2)'$, en el caso newtoniano.

COROLARIO 1 AL TEOREMA 2. Si $u(0,\eta)$ es monótona, la solución del problema del perfil inicial es única.

Un candidato natural para ocupar el papel de $u(x,\eta)$ es $U_1(x)$, la velocidad del flujo potencial no viscoso supuesto que $U_1(x) \geq \alpha > 0$ en $[0,X]$. Entonces $D = \{u_1 > u := U_1\} \subset G_1 \subset G$, (7,2) es superflua y (7,3), (7,4) y (7,5) inmediatas. Se tiene entonces el

COROLARIO 2 AL TEOREMA 2. Sea $U(x) \geq \alpha > 0$, $x \in [0,X]$, $\tilde{u}(x,y) > 0$ y $\tilde{u}(0,y) = u(0,\eta) \leq U(0)$.

Entonces $\tilde{u}(x,y) = u(x,\eta) \leq U(x)$, $x \in [0,X]$.

Es decir, la velocidad del flujo no viscoso nunca es excedida corriente abajo, supuesta la ausencia de reflujo. Además, siempre en ausencia de reflujo, el perfil $u(x,\eta)$ responde "instantáneamente" (en el mismo x) a una caída de la velocidad $U(x)$.

Es interesante destacar que la condición (7,5) ha reemplazado a (2,3). Si se define $w := uu_\eta$, es fácil ver que w satisface - hipótesis de suavidad mediante - a la ecuación

$$w_x = (u(\nu(w))_\eta)_\eta .$$

Esta ecuación es del tipo ya discutido, pero carece de término que contenga a $U(x)U'(x) = -p(x)$.

Se puede ver entonces que, si se imponen condiciones sobre el crecimiento de u y u_η en $\eta \to +\infty$, el número de cambios de signos de w (i.e., los de u_η) no puede exceder al de $w(0,\eta) = u(0,\eta)\, u_\eta(0,\eta)$ (ver [5]). En particular, si $u(0,\eta)$ es monótona creciente, también tendrá que serlo $u(x,\eta)$.

Con un procedimiento similar al del Corolario 2, tomando primero

$u_1 := \delta U(x)$ y $0 < \delta < 1$, y luego $u := \delta U(x)$ y $\delta > 1$, en intervalos donde $U'(x) \geqslant 0$, se puede probar que, en ellos, $\lim_{\eta\to\infty} u(x,\eta) = U(x)$. Un argumento análogo es aplicable a intervalos donde $U'(x) \leqslant 0$.

BIBLIOGRAFIA

[1] L. Prandtl, *The Mechanics of Viscous Fluids, en Aerodynamic Theory*, vol.III, editada por W.F.Durand, 1935.

[2] H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, 1960.

[3] W. Walter, *On the asymptotic behaviour of solutions of the Prandtl boundary layer equations*, MRC Technical Summary Report N°1056, Madison, Wisconsin, March 1970.

[4] O.A. Oleinik, *Mathematical Problems of Boundary Layer Theory*, Uspehi Mat. Nauk 23(1968), p.3-65. También Lecture Notes, University of Minnesota, 1969.

[5] J.E. Bouillet, C. Atkinson, *A Generalized Diffusion Equation: Radial Symmetries and Comparison Theorems*, Journal of Mathematical Analysis and Applications 95(1), August 1983, p.37-68.

[6] J.E. Bouillet, *El criterio de unicidad de Osgood y la desigualdad de Gronwall para ecuaciones parabólicas con estructura de divergencia*, aparecerá en Mathematicae Notae.

[7] J.E. Bouillet, J.C. Pedraza, *Un criterio de unicidad para una ecuación parabólica con estructura de divergencia en dominios no acotados*, Comunicación a la XXXIII Reunión Anual de la U.M.A., Tucumán, 1983, Rev. Un. Mat. Argentina 31, 1-2 (1983), p.58-59.

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THE HOLOMORPHIC FUNCTIONAL CALCULUS

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INTRODUCTION.

The classic holomorphic functional calculus was invented thirty years ago by Arens and Calderón [2]. Since then, it has proven to be an invaluable tool in the study of Banach algebras. It has also attracted a great deal of attention in itself, and many versions and alternative proofs have appeared.

Craw [3] produced the first version of a global functional calculus. By this we mean a morphism applying every function holomorphic near the spectrum of an algebra onto an element of the algebra.

In this paper we give another presentation of the global functional calculus. Our proof differs from Craw's, and also from Taylor's [5], in that we make no use of the classic functional calculus.

We start off by considering finitely determined open sets in §1, and holomorphic functions defined on the topological dual of an algebra in §2. In §3 a notion similar to polynomial convexity is introduced. We then describe, in §4, the set of germs of holomorphic functions over the spectrum as a direct limit of sets of holomorphic functions over open polynomially convex subsets of $\mathbf{C}^n$, and give our version of the functional calculus in §5. Throughout, A denotes a complex commutative unitary Banach algebra.

§1. We shall consider the topological dual A' of A with the weak *-topology. Thus, if γ_0 is an element of A' there is a basis for neighborhoods of γ_0 made up of sets like

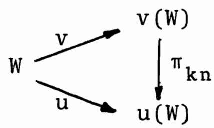
$$U_{\gamma_0} = \{\gamma \in A': |\gamma(a_i)-\gamma_0(a_i)| < 1,\ i = 1,\dots,n\} .$$

The elements $a_1,\dots,a_n$ may be chosen to be linearly independent. Once this is done, define

$$u = \hat{a}_1 \times \dots \times \hat{a}_n : A' \longrightarrow \mathbf{C}^n \qquad (u(\gamma) = (\gamma(a_1),\dots,\gamma(a_n)))$$

u is a linear continuous function, and because of the linear independence of the a_i, it is onto, and therefore open. Hence $u(U_{\gamma_0})$ is the open polydisc of $\mathbf{C}^n$ centered in $u(\gamma_0)$, and with radius one. Note that for γ to belong to U_{γ_0}, only its behaviour over $a_1,\ldots,a_n$ is relevant. We say that U_{γ_0} is finitely determined by $a_1,\ldots,a_n$, or by u.

Now if W is any open set in A', we say it is finitely determined by $v = \hat{b}_1 \times \ldots \times \hat{b}_k$ (the b_i are independent) if $W = v^{-1}(v(W))$. Of course the non-trivial inclusion is $v^{-1}(v(W)) \subset W$, which says that if γ behaves over $b_1,\ldots,b_k$ as an element of W, then γ belongs to W. For any open set U in $\mathbf{C}^k$, $v^{-1}(U)$ is finitely determined by v. We think of W as an infinite cilinder over the open set v(W) of $\mathbf{C}^k$. Different uples may determine the same open set; for example A' is determined by any uple. We need to partially order the uples (or the u's) determining a given W in A'. This will be done as follows: $u \leqslant v$ when the diagram



commutes. Here π_{kn} is the projection to the first n coordinates. We shall need the following facts about finitely determined open sets.

PROPOSITION. *Let* W' *be finitely determined, and* W *finitely determined by* u. *Then there is a* $v \geqslant u$ *which determines both* W *and* W'.

Proof. If $u = \hat{a}_1 \times \ldots \times \hat{a}_n$ and W' is finitely determined by $b_1,\ldots,b_k$, let F be the subspace of A generated by $a_1,\ldots,a_n,b_1,\ldots,b_k$. Let $a_1,\ldots,a_n,a_{n+1},\ldots,a_m$ be a basis of F, and put $v = \hat{a}_1 \times \ldots \times \hat{a}_m$. The following facts are elementary.

PROPOSITION. *Finite unions of finitely determined open sets are finitely determined open sets.*

PROPOSITION. *All compact sets of* A' *have a basis for neighborhoods which are finitely determined open sets.*

§2. We now consider holomorphic functions over open subsets of A'. We say f: U $\longrightarrow$ **C** is holomorphic when

i) the complex directional derivatives

$$\lim_{\lambda\to 0} \frac{f(x+\lambda y) - f(x)}{\lambda}$$ exist for all x in U and y in A'.

ii) f is locally bounded.

The set O(U) of such functions over U form an algebra. Note that the Gelfand transforms of elements of A belong to O(A'). We call ß the subalgebra of O(A') which these elements generate.

The local boundedness condition which we ask of these functions makes them depend locally on just finite variables: every point in A' has a finitely determined neighborhood, that is, an infinite cilinder with a finite dimensional base. As we move in this cilinder, only a finite number of variables are bounded. The functions, holomorphic and bounded, must be constant as of the rest of the variables. To state this more clearly, we have the following

PROPOSITION. *The following are equivalent:*

i) f: U $\longrightarrow$ **C** *is holomorphic.*

ii) *For every* $\gamma \in U$ *there are: a neighborhood* U_γ *of* γ, *linearly independent elements* $a_1,\dots,a_n$ *of* A, *and* $F \in O(u(U_\gamma))$ *such that* $f = Fu$ *over* U_γ; *where* $u = \hat{a}_1 \times \dots \times \hat{a}_n$.

Proof. (Allan, [1])

i) $\Rightarrow$ ii) Let $\gamma \in U$. There is a neighborhood

$$U_{\gamma_0} = \{\gamma \in A' : |\gamma(a_i)-\gamma_0(a_i)| < 1, \quad i = 1,\dots,n\} \subset U$$

with $a_1,\dots,a_n$ linearly independent, over which f is bounded. Set $u = \hat{a}_1 \times \dots \times \hat{a}_n : A' \longrightarrow \mathbf{C}^n$. u is continuous, linear and open. We want to define a function $F \in O(u(U_{\gamma_0}))$ such that $Fu = f$. Let

$$F(z) = f(\gamma) \quad , \quad \text{if} \quad z = u(\gamma)$$

F is well defined: suppose $\gamma, \gamma' \in U_{\gamma_0}$ with $u(\gamma) = u(\gamma')$. For all $\lambda \in \mathbf{C}$, $u(\gamma'-\gamma_0) = u(\lambda\gamma+(1-\lambda)\gamma'-\gamma_0)$, so

$$|\lambda\gamma(a_i)+(1-\lambda)\gamma'(a_i)-\gamma_0(a_i)| = |\gamma'(a_i)-\gamma_0(a_i)| < 1, \quad i = 1,\dots,n.$$

Therefore $\lambda\gamma+(1-\lambda)\gamma' \in U_{\gamma_0}$ for every $\lambda \in \mathbf{C}$, then

$$\alpha: \mathbf{C} \longrightarrow \mathbf{C} \ , \quad \alpha(\lambda) = f(\lambda\gamma+(1-\lambda)\gamma')$$

is an entire bounded function; so it is constant.

$$f(\gamma') = \alpha(0) = \alpha(1) = f(\gamma).$$

The continuity and the existence of partial derivatives of F is easily verified, so by Hartog's theorem, F is holomorphic.

ii) $\Rightarrow$ i) is simple.

In the situation of the proposition, we shall say that f is finitely determined by $a_1,\dots,a_n$, or by u, over U_{γ_0}.

If W is a finitely determined open subset of A', and $f \in O(W)$, we say f is finitely determined by u over W if W is finitely determined by u and there is an $F \in O(u(W))$ such that $f = Fu$ over W. Finitely determined functions of O(W) form a subalgebra which we denote F(W). It is easily verified that the following holds.

PROPOSITION. *If* W *is finitely determined and* $f \in O(W)$ *is bounded, then any* u *that determines* W*, determines* f.

There are, however, unbounded elements in F(W). To clarify the structure of F(W), consider for u and v determining W with $u \leq v$,

$$f_{uv}: O(u(W)) \longrightarrow O(v(W)), \quad f_{uv}(g) = g\pi_{kn} .$$

These f_{uv} are a direct system and it is not hard to verify that $F(W) = \varinjlim_{u} O(u(W))$: the mappings $O(u(W)) \longrightarrow F(W)$ given by $f \longrightarrow fu$ induce a map $\varinjlim_{u} O(u(W)) \longrightarrow F(W)$ which is an isomorphism.

We shall consider O(u(W)) endowed with the topology of uniform convergence over compact subsets of u(W), and F(W) with the direct limit topology. This topology is finer than the topology of uniform convergence on compact subsets of W.

§3. We need in A', a notion analogous to the notion of polynomial convexity in $\mathbf{C}^n$. Define, for each subset B of A',

$$\widetilde{B} = \{\gamma \in A' : |P(\gamma)| \leq \sup_{b\in B} |P(b)| , \quad \text{for all } P \in \beta\}$$

We say B is strongly β-convex if $\widetilde{B} = B$, and β-convex if $\widetilde{K} \subset B$ for all compact subsets K of B. Note that the spectrum of A, X(A), is strongly β-convex: if $\gamma \in \widetilde{X(A)}$, let $a,b \in A$ and $P_{ab} = \hat{a}\hat{b}-\widehat{ab} \in \beta$.

$$|P_{ab}(\gamma)| \leq \sup_{X(A)} |P_{ab}| = 0$$

so $\gamma(a)\gamma(b)-\gamma(ab) = 0$ for all $a,b \in A$, and $\gamma \in X(A)$.

For finitely determined open subsets of A', β-convexity and polynomial convexity are related as follows.

PROPOSITION. *Let* W *be open in* A', *finitely determined by* $u = \hat{a}_1 \times \dots \times \hat{a}_n$. *Then the following are equivalent:*

i) W *is* β-*convex*.

ii) u(W) *is polynomially convex*.

Proof. i) ⇒ ii) Let H be compact, contained in u(W). We must show that its polynomially convex hull $\hat{H}$ is a subset of u(W).

Define $\sigma: \mathbf{C}^n \longrightarrow A'$ by $\sigma(z) = \sum_{i=1}^{n} z_i \phi_i$, where $\phi_i(a_j) = \delta_{ij}$ and $\phi_i = 0$ over the rest of a basis B for A that extends $a_1, \dots, a_n$. Then $u\sigma$ is the identity over $\mathbf{C}^n$, and σ is continuous. Let K = σ(H). K is compact, u(K) = H and $K \subset u^{-1}(u(W)) = W$.

We must verify, then, that $\widehat{u(K)} \subset u(W)$. Since $\tilde{K} \subset W$, $u(\tilde{K}) \subset u(W)$ and it will be enough to show that $\widehat{u(K)} \subset u(\tilde{K})$. Let $z_0 \in \widehat{u(K)}$. Then

$$|P(z_0)| \leqslant \sup_{u(K)} |P(z)| \text{ for all } P \in \mathbf{C}[X_1, \dots, X_n]$$

Now let $\gamma_0 = \sigma(z_0)$. $u(\gamma_0) = z_0$, and we must see that $\gamma_0 \in \tilde{K}$, that is,

$$|Q(\gamma_0)| \leqslant \sup_{\gamma \in K} |Q(\gamma)| \text{ for all } Q \in \beta$$

It is not true that, given Q ∈ β, there is a $P \in \mathbf{C}[X_1, \dots, X_n]$ with Q = Pu. However, there is a polynomial $P \in \mathbf{C}[X_1, \dots, X_n]$ which makes the equality valid over $\sigma(\mathbf{C}^n)$, which is what we really need.

To show the existence of such P, say $\hat{b}_1, \dots, \hat{b}_k$ are the "coordinates" appearing in Q. Then there are $a_1, \dots, a_m$ in B, which generate all b_j and amongst which we may find $a_1, \dots, a_n$. There is a polynomial $\overline{P} \in \mathbf{C}[X_1, \dots, X_m]$ for which $Q = \overline{P}(\hat{a}_1, \dots, \hat{a}_m)$. Then

$$Q(\sigma(z)) = \overline{P}(\hat{a}_1, \dots, \hat{a}_m)(\sigma(z)) = \overline{P}(\sigma(z)(a_1), \dots, \sigma(z)(a_m)) =$$

$$= \overline{P}(\sigma(z)(a_1), \dots, \sigma(z)(a_n), 0, \dots, 0)$$

Let $P(X_1, \dots, X_n) = \overline{P}(X_1, \dots, X_n, 0, \dots, 0)$. Then Q = Pu over $\sigma(\mathbf{C}^n)$, and

$$|Q(\gamma_0)| = |P(u(\gamma_0))| = |P(z_0)| \leqslant \sup_{\gamma \in K} |P(u(\gamma))| = \sup_{\gamma \in K} |Q(\gamma)|.$$

Therefore $\gamma_0 \in \tilde{K}$.

ii) ⇒ i) If K is a compact subset of W, let H = u(K). H is compact, contained in u(W), so $\hat{H} \subset u(W)$. We want to show that $\tilde{K} \subset W$. Since $u^{-1}(\hat{H}) \subset u^{-1}(u(W)) = W$, it will be enough to see $\tilde{K} \subset u^{-1}(\hat{H})$, that is, $u(\tilde{K}) \subset \hat{H}$. This is easily verified once we note that Pu ∈ β for all $P \in \mathbf{C}[X_1,\ldots,X_n]$.

In the preceding proof we have shown the validity of the equality $\widehat{u(K)} = u(\widetilde{K})$, for compact sets $K = \sigma(H)$, with H compact in $\mathbf{C}^n$. This equality for arbitrary compact subsets of A' is false. For example, we know the spectrum X(A) is a compact β-convex subset of A', but $u(X(A)) = sp(a_1,\ldots,a_n)$ is not, in general, polynomially convex.

This fact is an important setback in the construction of a holomorphic functional calculus, for $sp(a_1,\ldots,a_n)$ will not have a basis for neighborhoods whose elements are polynomially convex open subsets of $\mathbf{C}^n$. In the classical functional calculus, this difficulty is overcome by the Arens-Calderón trick. In this version, what we need is the following.

PROPOSITION. *Let* K *be a compact* β*-convex subset of* A'. *Then* K *has a basis for neighborhoods made up of* β*-convex, finitely determined open sets.*

Proof. K has a basis for neighborhoods made up of finitely determined open sets. Let W be such a neighborhood, determined by $u = \hat{a}_1 \times \ldots \times \hat{a}_n$. Also, let $c > 0$ be such that $K \subset D = \{\gamma \in A' : \|\gamma\| \leqslant c\}$.

Given $P \in \beta$, let $K_P = \{\gamma \in A' : |P(\gamma)| \leqslant \sup_K |P|\}$. K is β-convex, so $K = \bigcap_P K_P$. Since $D \cap K_P$ is compact for each P, there are $P_1,\ldots,P_k$ with

$$K \subset D \cap K_{P_1} \cap \ldots \cap K_{P_k} \subset W .$$

Let $v = \hat{a}_1 \times \ldots \times \hat{a}_n \times \ldots \times \hat{a}_m \geqslant u$, such that v determines P_i for $i = 1,\ldots,k$; that is, there are polynomials $Q_1,\ldots,Q_k \in \mathbf{C}[X_1,\ldots,X_m]$ with $P_i = Q_i v$. Let

$$v(K)_{Q_i} = \{z \in \mathbf{C}^m : |Q_i(z)| \leqslant \sup_{v(K)} |Q_i|\}$$

For every i, this set is polynomially convex and $v(K_{P_i}) \subset v(K)_{Q_i}$. Put

$$K_0 = v(D) \cap v(K)_{Q_1} \cap \ldots \cap v(K)_{Q_k} .$$

K_0 is a polynomially convex compact set, for D is compact and v(D) is polynomially convex: to see this let V be the subspace of A generated by $a_1,\ldots,a_m$. Its dual V' may canonically be thought of as a quotient of A'. Factoring v through this quotient we obtain an isomorphism $\bar{v}: V' \longrightarrow \mathbf{C}^m$. We may then identify $D' = \{x \in V' : \|x\| \leqslant c\}$ with $\bar{v}(D') = v(D)$. Now if $z \in \mathbf{C}^m - v(D)$, $z = \bar{v}(x)$ with $\|x\| > c$. Let

$L: V' \longrightarrow \mathbf{C}$ be linear, with norm one and such that $|L(x)| = \|x\|$, and $Q = L\bar{v}^{-1}: \mathbf{C}^m \longrightarrow \mathbf{C}$. $Q \in \mathbf{C}[X_1,\dots,X_m]$ and

$$|Q(z)| = |L\bar{v}^{-1}\bar{v}(x)| = |L(x)| = \|x\| > c = \sup_{D'}|L| = \sup_{v(D)}|Q|$$

Therefore for all $z \in \mathbf{C}^m - v(D)$ there is a Q with $|Q(z)| > \sup_{v(D)}|Q|$; $v(D)$ is polynomially convex.

We also have $v(K) \subset K_0 \subset v(W)$. The first inclusion because $K \subset D \cap K_{P_1} \cap \dots \cap K_{P_k}$ implies

$$v(K) \subset v(D \cap K_{P_1} \cap \dots \cap K_{P_k}) \subset v(D) \cap v(K)_{Q_1} \cap \dots \cap v(K)_{Q_k} = K_0$$

and to verify the second, let $z \in K_0$, $\gamma \in D$ with $z = v(\gamma)$. We have

$$|P_i(\gamma)| = |Q_i v(\gamma)| = |Q_i(z)| \leqslant \sup_{v(K)} |Q_i| = \sup_K |P_i|$$

for $i = 1,\dots,k$; that is, $\gamma \in D \cap K_{P_1} \cap \dots \cap K_{P_k} \subset W$, and $z \in v(W)$.

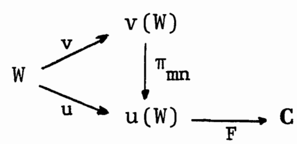
Now let U be a polynomially convex open subset of $\mathbf{C}^m$ such that $v(K) \subset K_0 \subset U \subset v(W)$. Then $K \subset v^{-1}(U) \subset v^{-1}(v(W)) = W$, and $v^{-1}(U)$ is a finitely determined open subset of A', and it is β-convex thanks to the preceding proposition.

§4. We return now to holomorphic functions over A'. If we have two open sets $U \subset V$, we also have the restriction mapping $O(V) \longrightarrow O(U)$. Fix a compact subset K of A'. Its open neighborhoods are partially ordered and the restriction mappings form a direct system. Therefore $O(K) = \varinjlim O(U)$ is defined.

However, as we have seen before, holomorphic functions are locally finitely determined, so the same happens to holomorphic functions over a sufficiently small neighborhood of a compact set. We have, in fact:

PROPOSITION. *Let* K *be a compact* β*-convex subset of* A'. *Then* $O(K) = \varinjlim F(W)$, *where* W *are open, finitely determined,* β*-convex neighborhoods of* K.

Proof. Say $W' \subset W$, $f \in F(W)$, and u determines f and W. Then there is a $v \geqslant u$ determining both W and W'.



If $f = Fu$, let $F' = F\pi_{mn}$. Then $F'v = f$ over W, and therefore over W'. This defines maps

$$F(W) \longrightarrow F(W')$$

which form a direct system, so $\varinjlim_{W} F(W)$ is defined. The maps

$$F(W) \longrightarrow O(W) \longrightarrow O(K)$$

induce a morphism

$$\varinjlim_{W} F(W) \longrightarrow O(K)$$

which is easily seen to be an isomorphism.

So if K is a compact β-convex subset of A', O(K) may be thought of as a direct limit of algebras O(u(W)), where u(W) are open polynomially convex subsets of $\mathbf{C}^n$. We consider O(K) with the direct limit topology.

§5. We are now ready for our main theorem.

THEOREM. *Let* A *be a commutative Banach algebra. There is a unique continuous unitary algebra homomorphism*

$$E: O(X(A)) \longrightarrow A$$

with $E(\hat{a}) = a$.

Proof. The spectrum X(A) is compact and β-convex, so we have $O(X(A)) = \varinjlim_{W}(\varinjlim_{u} O(u(W)))$, where u(W) are open polynomially convex neighborhoods of $u(X(A)) = sp(a_1,\ldots,a_n)$, if $u = \hat{a}_1 \times \ldots \times \hat{a}_n$. Therefore all holomorphic functions over u(W) are uniformly approximable by polynomials on the compact subsets of u(W) [4]. Also, O(u(W)) induces a topology on $\mathbf{C}[X_1,\ldots,X_n]$ for which the unitary algebra homomorphism defined by $X_i \longmapsto a_i$ is continuous. We then have continuous unitary algebra homomorphisms

$$O(u(W)) \longrightarrow A$$

It is a purely technical matter to verify that these maps induce a map

$$O(X(A)) \longrightarrow A$$

with the required properties.

It is also easy to see that for every $f \in O(X(A))$, f and $\widehat{E(f)}$ coincide over X(A), though not, in general, as elements of O(X(A)). We have found the following proposition useful.

PROPOSITION. *Let* $N = \{f \in O(X(A)) : f|X(A) = 0\}$. *Then* $E(N) = Rad(A)$ *and* $N = E^{-1}(Rad(A))$.

Proof. If $f \in N$, $\widehat{E(f)}|X(A) = f|X(A) = 0$, so $E(f) \in Rad(A)$. If $a \in Rad(A)$, $\hat{a} \in N$, and $E(\hat{a}) = a$. For the other equality, we already have $N \subset E^{-1}(Rad(A))$, and if $E(f)$ belongs to the radical, $f|X(A) = \widehat{E(f)}|X(A) = 0$, so $f \in N$.

Note that when A is semisimple, the proposition says that N is the kernel of E.

The homomorphism defined by the theorem is, of course, the same as Craw's [3], the only possible difference being in the topologies of O(X(A)). In [3] O(X(A)) is presented as $\varinjlim_U H^\infty(U)$, where U are open neighborhoods of the spectrum and $H^\infty(U)$ the set of holomorphic functions over U which are bounded, with the supremum norm. Actually, the topologies are the same:

PROPOSITION. $\varinjlim_U H^\infty(U) \cong \varinjlim_{W,u} O(u(W))$.

Proof. As we have shown before, the open neighborhoods U of X(A) may be taken to be finitely determined and β-convex, for these form a basis for neighborhoods of X(A).

All maps $H^\infty(W) \longrightarrow \varinjlim O(u(W))$ are continuous, for if $f_n \longrightarrow 0$ in $H^\infty(W)$, all may be written as $F_n u$ (the same u, since the f_n are all bounded) and $F_n \longrightarrow 0$ uniformly over all of u(W), not just compact subsets. Therefore

$$\varinjlim H^\infty(U) \longrightarrow \varinjlim O(u(W))$$

is continuous.

Now fix u, W, and suppose $F_n \longrightarrow 0$ in O(u(W)). Let Q be a compact neighborhood of u(X(A)), contained in u(W). Then $F_n \longrightarrow 0$ uniformly over Q, hence over Q^0. Let $U = u^{-1}(Q^0)$. U is a neighborhood of X(A), and $f_n = F_n u \longrightarrow 0$ uniformly over U. So

$$\varinjlim O(u(W)) \longrightarrow \varinjlim H^\infty(U)$$

is also continuous.

The authors have found the presentation $O(K) = \varinjlim O(u(W))$ more manageable, for example in the following setting. Suppose F is a complex homogeneous space, not contained in $\mathbf{C}^n$. We want to define the set of germs of holomorphic functions defined near X(A), with images in F. This can be done considering for each finitely determined β-convex open neighborhood, W, of X(A) and each u that determines it, the set O(u(W),F) with the compact-open topology, and then taking

$$O(X(A),F) = \lim_{\to} O(u(W),F).$$

REFERENCES

[1] Allan, G.R.: *A form of local characterization of Gelfand transforms;* Jour. Lond. Math. Soc. 43 (1968), 623-625.

[2] Arens, R. and Calderón, A.P.: *Analytic functions of several Banach algebra elements;* Ann. Math. 62 (2) (1955), 204-216.

[3] Craw, I.G.: *A condition equivalent to the continuity of characters on a Fréchet algebra;* Proc. Lond. Math. Soc. (3) 22 (1971), 452-464.

[4] Gunning, R. and Rossi, H.: *Analytic functions of several complex variables;* Prentice-Hall, Englewood Cliffs, N.J., 1965.

[5] Taylor, J.L.: *Topological invariants of the maximal ideal space of a Banach algebra;* Adv. in Math. 19 (1976), 149-206.

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THE GENERAL FORM OF ISOTROPIC TENSORS

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1. INTRODUCTION.

In this paper we find the most general form of isotropic tensors; we show that they are linear combinations, with differentiable functions as coefficients, of tensor products of the Kronecker delta. We also find the most general form of isotropic tensorial densities of any (integer) weight.

2. ISOTROPIC TENSORS.

An isotropic tensor is defined as a tensor whose components are the same in all coordinate systems. Let $L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$ be the components of an isotropic tensor of type (r,s); then for a change in the coordinates:

$$\bar{x}^i = \bar{x}^i(x^j) \tag{2.1}$$

it must be:

$$L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = B^{j_1}_{k_1} \ldots B^{j_s}_{k_s} A^{h_1}_{i_1} \ldots A^{h_r}_{i_r} L^{i_1 \ldots i_r}_{j_1 \ldots j_s} \tag{2.2}$$

where $B^i_j = \dfrac{\partial x^i}{\partial \bar{x}^j}$, $A^i_j = \dfrac{\partial \bar{x}^i}{\partial x^j}$ and use is made of the summation convention. Making the particular change $\bar{x}^i = \lambda x^i$ ($\lambda \neq 0$), we obtain from (2.2):

$$L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = \lambda^{r-s} L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$$

and so we see that, if $r \neq s$, the tensor must be the null tensor. Let us consider then the case $r = s$; then (2.2) is written as:

$$L^{h_1 \ldots h_s}_{k_1 \ldots k_s} = B^{j_1}_{k_1} \ldots B^{j_s}_{k_s} A^{h_1}_{i_1} \ldots A^{h_s}_{i_s} L^{i_1 \ldots i_s}_{j_1 \ldots j_s} \tag{2.3}$$

Following Rund [3], we consider all posible changes of the type (2.1); then, for a fixed point in the manifold, the B^i_j may be considered as points in GL(n,R). We differentiate with respect to B^a_b and evaluate at $B^a_b = \delta^a_b$ to obtain:

$$\begin{aligned} 0 = {} & \delta^b_{k_1} L^{h_1\ldots h_s}_{ak_2\ldots k_s} + \delta^b_{k_2} L^{h_1\ldots h_s}_{k_1 a\ldots k_s} + \ldots + \delta^b_{k_s} L^{h_1\ldots h_s}_{k_1\ldots k_{s-1}a} - \\ & - \delta^{h_1}_a L^{bh_2\ldots h_s}_{k_1\ldots k_s} - \delta^{h_2}_a L^{h_1 b\ldots h_s}_{k_1\ldots k_s} - \ldots - \delta^{h_s}_a L^{h_1 h_2\ldots b}_{k_1\ldots k_s} \end{aligned} \tag{2.4}$$

As a preliminary step for our final result, we prove:

LEMMA 1. *If* $L^{h_1\ldots h_s}_{k_1\ldots k_s}$ *are the components of an isotropic tensor and if* $(h_1,\ldots,h_s)$ *is not a permutation of* $(k_1,\ldots,k_s)$, *then* $L^{h_1\ldots h_s}_{k_1\ldots k_s} = 0$.

Proof. First we observe that the sets $\{h_1,\ldots,h_s\}$ and $\{k_1,\ldots,k_s\}$ must be the same if the corresponding component is different from zero. For if $h_1 \notin \{k_1,\ldots,k_s\}$, we take $b = h_1 = a$ in (2.4); all the first terms are null being b different from $k_1,\ldots,k_s$ and we obtain $-m\, L^{h_1\ldots h_s}_{k_1\ldots k_s} = 0$, where m is the number of times h_1 is repeated in $(h_1,\ldots,h_s)$. Now, if h_1 is repeated m times in $(h_1,\ldots,h_s)$ and n times in $(k_1,\ldots,k_s)$, taking $b = a = h_1$ we obtain $(n-m)L^{h_1\ldots h_s}_{k_1\ldots k_s} = 0$, and so the lemma is proved.

As a second step, we prove:

LEMMA 2. *Let* σ *be a permutation of* $\{1,2,\ldots,s\}$. *Then:*

$$L^{h_{\sigma(1)}\ldots h_{\sigma(s)}}_{k_{\sigma(1)}\ldots k_{\sigma(s)}} = L^{h_1\ldots h_s}_{k_1\ldots k_s}$$

Proof. Make the change $\bar{x}^i = x^{\sigma(i)}$ and use (2.3).

As a result of the previous lemmas, we see that each of the components of an isotropic tensor is equal to one of the quantities:

$$L^{h_{\sigma(1)}\ldots h_{\sigma(s)}}_{h_1\ldots h_s} \quad , \quad h_1 \leqslant h_2 \leqslant \ldots \leqslant h_s \ , \quad \sigma \in S_s \tag{2.5}$$

Next we prove that we can replace a given index in (2.5) by any other index k not belonging to $\{h_1,\ldots,h_s\}$. For the sake of simplicity, we prove $L^{h_1\ldots h_s}_{h_1\ldots h_s} = L^{kh_2\ldots h_s}_{kh_2\ldots h_s}$ for $k \notin \{h_1,\ldots,h_s\}$. Changing indices in (2.4), we obtain:

$$0 = \delta^b_{k_1} L^{h_1\ldots h_s}_{ah_2\ldots h_s} + \delta^b_{h_2} L^{h_1\ldots h_s}_{k_1ah_2\ldots h_s} + \ldots + \delta^b_{h_s} L^{h_1\ldots h_s}_{k_1h_2\ldots h_{s-1}a} - \quad (2.6)$$
$$- \delta^{h_1}_a L^{bh_2\ldots h_s}_{kh_2\ldots h_s} - \delta^{h_2}_a L^{h_1bh_3\ldots h_s}_{kh_2\ldots h_s} - \ldots - \delta^{h_s}_a L^{h_1\ldots h_{s-1}b}_{kh_2\ldots h_s}$$

Taking $b = k_1$ and $a = h_1$ in (2.6) (no summation convention here) we obtain $L^{h_1\ldots h_s}_{h_1\ldots h_s} = L^{kh_2\ldots h_s}_{kh_2\ldots h_s}$. Similarly it follows:

$$L^{h_{\sigma(1)}\ldots h_{\sigma(i)}\ldots h_{\sigma(s)}}_{h_1\ldots h_{\sigma^{-1}(i)}\ldots h_s} = L^{h_{\sigma(1)}\ldots k\ldots h_{\sigma(s)}}_{h_1\ldots k\ldots h_s} \quad (2.7)$$

for $k \notin \{h_1,\ldots,h_s\}$.

According to (2.7), any quantity in (2.5) is equal to a quantity of the form

$$L^{\sigma(1)\ \sigma(2)\ldots\sigma(s)}_{1\ 2\ldots s}$$

for $s \leqslant$ dimension of the manifold. Then the dimension of isotropic tensors at a fixed point of the manifold is s! if $s \leqslant d$ = dimension of the manifold. Similarly, it can be proved that this dimension is d! if $s > d$, but the proof is too involved to be written out in full detail here. The problem is that we have to replace indices already appearing in $\{h_1,\ldots,h_s\}$. We give an example from which the idea of the proof will be clear. Suppose we want to replace h_1 by h_2 in the quantity

$$L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_1\ldots h_1h_2\ldots h_2h_3} \quad (2.8)$$

(no summation convention here). Changing indices in (2.4), we obtain:

$$0 = \delta^b_{h_2} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{ah_1\ldots h_1h_2\ldots h_2h_3} + \delta^b_{h_1} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_2a\ldots h_1h_2\ldots h_2h_3} + \ldots +$$
$$+ \delta^b_{h_1} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_2\ldots h_1ah_2\ldots h_2h_3} + \delta^b_{h_2} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_2\ldots h_1ah_2\ldots h_2h_3} + \ldots +$$
$$+ \delta^b_{h_2} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_2\ldots h_1h_2\ldots ah_3} + \delta^b_{h_3} L^{h_1\ldots h_1h_2\ldots h_2h_3}_{h_2\ldots h_1h_2\ldots h_2a} -$$
$$- \delta^{h_1}_a L^{bh_1\ldots h_1h_2\ldots h_2h_3}_{h_2h_1\ldots h_1h_2\ldots h_2h_3} - \delta^{h_1}_a L^{h_1b\ldots h_1h_2\ldots h_2h_3}_{h_2h_1\ldots h_1h_2\ldots h_2h_3} - \ldots -$$

$$- \delta_a^{h_1} L^{h_1\dots h_1 b h_2\dots h_2 h_3}_{h_2 h_1\dots h_1 h_2\dots h_2 h_3} - \delta_a^{h_2} L^{h_1\dots h_1 b h_2\dots h_2 h_3}_{h_2\dots h_1 h_2 h_2\dots h_2 h_3} - \dots -$$

$$- \delta_a^{h_2} L^{h_1\dots h_1 h_2\dots h_2 b h_3}_{h_2 h_1\dots h_1 h_2\dots h_2 h_3} - \delta_a^{h_3} L^{h_1\dots h_1 h_2\dots h_2 b}_{h_2\dots h_1 h_2\dots h_2 h_3}$$

Making $b = h_2$ and $a = h_1$, we find that $L^{h_1\dots h_1 h_2\dots h_2 h_3}_{h_1\dots h_1 h_2\dots h_2 h_3}$ can be written as a sum of terms of the form (2.8) with the first h_1 replaced by h_2 in the upper indices and the lower ones and with permutations on the upper indices. Generalizing this procedure, we see that the components of an isotropic tensor for $d < s$ are linear combinations of quantities of the form:

$$L^{\sigma(1)\ \sigma(2)\dots\sigma(d)\ d\dots d}_{1\ 2\dots d\ d\dots d}$$

and so the dimension of isotropic tensors is $d!$ for $d < s$.

Now we observe that we have at our disposal $s!$ linearly independent isotropic tensors for $s \leqslant d$, namely

$$\delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \dots \delta_{k_s}^{h_{\sigma(s)}} , \quad \sigma \in S_s \tag{2.9}$$

and $d!$ linearly independent isotropic tensors for $d < s$, namely

$$\delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \dots \delta_{k_d}^{h_{\sigma(d)}} \delta_{k_{d+1}}^{h_{d+1}} \dots \delta_{k_s}^{h_s} , \quad \sigma \in S_d \tag{2.10}$$

Since (2.10) is a subset of (1.9) for $d < s$, we obtain (2.9) as a set of generators in any case. Thus we conclude:

THEOREM 1. *Let* $L^{h_1\dots h_r}_{k_1\dots k_s}$ *be the components of an isotropic tensor. Then:*

a) $L^{h_1\dots h_r}_{k_1\dots k_s} = 0$ *for* $r \neq s$

b) *If* $r = s$, *then*

$$L^{h_1\dots h_s}_{k_1\dots k_s} = \sum_{\sigma\epsilon S_s} f_\sigma\, \delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \dots \delta_{k_s}^{h_{\sigma(s)}}$$

where f_σ *is a differentiable function for each permutation* σ.

As a particular case, if $f_\sigma = \varepsilon(\sigma)$ (sign of the permutation σ), we obtain the generalized Kronecker delta $\delta^{h_1\dots h_s}_{k_1\dots k_s}$ (see [1]).

3. ISOTROPIC TENSORIAL DENSITIES.

Let now $L^{h_1\ldots h_r}_{k_1\ldots k_s}$ be the components of an isotropic density of weight M. The transformation rule is:

$$L^{h_1\ldots h_r}_{k_1\ldots k_s} = J^M\, B^{j_1}_{k_1}\ldots B^{j_s}_{k_s}\, A^{h_1}_{i_1}\ldots A^{h_r}_{i_r}\, L^{i_1\ldots i_r}_{j_1\ldots j_s} \tag{3.1}$$

We differentiate respect to B^a_b and evaluate at $B^a_b = \delta^a_b$. Contracting $b = a$ it is easy to obtain:

$$Md = s - r$$

and so, for $M > 0$ (integer), the components of the isotropic density are of the form:

$$L^{h_1\ldots h_r}_{k_1\ldots k_r\ k_{r+1}\ldots k_{r+Md}}$$

We obtain an isotropic tensor by multiplying this with the Levi-Civita symbols $\varepsilon^{\cdots}$; from theorem 1 it must be:

$$L^{h_1\ldots h_r}_{k_1\ldots k_r\ k_{r+1}\ldots k_{r+Md}}\ \varepsilon^{h_{r+1}\ldots h_{r+d}}\ldots\varepsilon^{h_{r+(m-1)d+1}\ldots h_{r+Md}} =$$

$$= \sum_{\sigma\in S_s} f_\sigma\ \delta^{h_{\sigma(1)}}_{k_1}\ldots\delta^{h_{\sigma(s)}}_{k_s}\quad , \tag{3.2}$$

and since $\varepsilon^{i_1\ldots i_d}\ \varepsilon_{i_1\ldots i_d} = n!$, we obtain $L^{h_1\ldots h_r}_{k_1\ldots k_s}$ from (3.2). Following a similar procedure for $M < 0$, we obtain:

THEOREM 2. *Let* $L^{h_1\ldots h_r}_{k_1\ldots k_s}$ *be the components of an isotropic tensorial density of weight* M.

Then $Md = s-r$ *and:*

a) $$L^{h_1\ldots h_r}_{k_1\ldots k_s} = \sum_{\sigma\in S_s} f_\sigma\ \delta^{h_{\sigma(1)}}_{k_1}\ldots\ \delta^{h_{\sigma(s)}}_{k_s}\ .$$

$$.\ \varepsilon_{h_{r+1}\ldots h_{r+1}}\ \ldots\ \varepsilon_{h_{s-d+1}\ldots h_s} \tag{3.3}$$

if $M > 0$, *where* $\varepsilon_{\ldots}$ *are the Levi-Civita symbols*

b) $$L^{h_1\ldots h_r}_{k_1\ldots k_s} = \sum_{\sigma\in S_s} f_\sigma\ \delta^{h_{\sigma(1)}}_{k_1}\ \ldots\ \delta^{h_{\sigma(r)}}_{k_r}\ .$$

$$.\ \varepsilon^{k_{s+1}\ldots k_{s+d}}\ \ldots\ \varepsilon^{k_{r-d+1}\ldots k_r} \tag{3.4}$$

if $M < 0$.

As a particular case, if $L_{k_1 \ldots k_d}$ is a tensorial isotropic density of weight 1, then it follows from (3.3) that:

$$L_{k_1 \ldots k_d} = \sum_{\sigma \in S_d} f_\sigma \, \delta_{k_1 \ldots k_d}^{h_{\sigma(1)} \ldots h_{\sigma(d)}} . \varepsilon_{h_1 \ldots h_d} = \sum_{\sigma \in S_d} f_\sigma \, \varepsilon_{h_{\sigma(1)} \ldots h_{\sigma(d)}} =$$

$$= \sum_{\sigma \in S_d} \varepsilon(\sigma) \, f_\sigma \, \varepsilon_{h_1 \ldots h_d} = g . \varepsilon_{h_1 \ldots h_d} ,$$

where g is a differentiable function.

REMARK. The knowledge of isotropic tensors is useful in concomitant theory if null tensors are included into the domain of concomitance. See [2].

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REFERENCES

[1] Lovelock, D. *The Einstein tensor and its generalizations*, J. Math. Phys., 12 (1971), 498-501.

[2] Noriega, R.J. *The uniqueness of the covariant derivative*, (to appear).

[3] Rund, H. *Variational problems involving combined tensor fields*, Abh. Math. Sem. Univ. Hamburg, 29 (1966), 243-262.

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Dr. MIGUEL E.M. HERRERA

El prematuro fallecimiento de Miguel Herrera ha causado consternación en todos los círculos matemáticos de nuestro país y en aquellos del exterior que lo tuvieron como miembro, y una íntima y persistente pena entre quienes tuvieron el privilegio de ser sus amigos. La causa de esta consternación reside en las cualidades de su singular personalidad que le valieron la espontánea simpatía de quienes lo trataron y el intenso afecto de las muchas personas sobre las que ejerció su benéfico influjo.

Vástago de una familia con antiguas raíces en Catamarca, Miguel Emilio Marcos había nacido en Buenos Aires el 25 de abril de 1938 y murió en la misma ciudad el 20 de enero de 1984.

Después de cursar estudios secundarios en el Liceo Militar de la Nación ingresó a la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires, graduándose de Licenciado en Ciencias Matemáticas en el año 1960.

Por aquella época inició su carrera docente universitaria: como alumno, su brillante inteligencia había llamado la atención de los profesores de un Departamento que tenía entre sus miembros a matemáticos como González Domínguez, Santaló y Mischa Cotlar, y en virtud de ello fue designado Jefe de Trabajos Prácticos, iniciando su labor docente en la materia Funciones Reales que dictaba Mischa Cotlar.

Se ha dicho muchas veces, y sin embargo vale la pena repetirlo, que un estudioso de las matemáticas no es un matemático mientras no realice aportes personales susceptibles de ser reconocidos como originales y valiosos por la comunidad matemática internacional. Si como estudiante Herrera había demostrado su inteligencia excepcional, ahora, al encarar su trabajo de tesis, demostraría sin lugar a dudas que tenía talento matemático.

En un trabajo publicado en 1962 el matemático S. Lojasiewicz había introducido la noción de conjunto semianalítico. La teoría se hallaba en su comienzo y varios de los resultados obtenidos por Lojasiewicz no estaban todavía impresos.

A pesar de ello y de la distancia que lo separaba de los centros donde se estaban desarrollando estas investigaciones, Herrera volcó sus esfuerzos en esa dirección y vió la posibilidad de desarrollar una teoría de la integración sobre conjuntos semianalíticos, idea que logró concretar en su trabajo de tesis, donde extiende a dichos conjuntos los resultados que había obtenido P. Lelong para conjuntos analíticos complejos. Los resultados obtenidos en esa tesis fueron publicados en el Boletín de la Sociedad Matemática Francesa.

En esta primera etapa de su fecunda labor matemática tuvo que vencer algunos obstáculos que hoy asombrarían a muchos jóvenes. Tal vez por el escaso número de experiencias anteriores, las condiciones y requisitos normales para la carrera del doctorado no estaban todavía establecidas y los criterios sobre este tema distaban mucho de ser uniformes. Sin embargo, a pesar de esta circunstancia adversa, prevalecía una atmósfera estimulante que habían sabido crear los profesores antes mencionados y algunos condiscípulos de Herrera pertenecientes a una generación en la que brillaban varios jóvenes con talento, a la que Santaló ha designado con el nombre de "la generación del 61".

Igualmente importante es el hecho de que por esa época a la vez difícil y feliz de su vida Herrera había conseguido un valiosísimo apoyo: en 1963 se había casado con su gran compañera María Marta Schiaffino Carreño, quien desde entonces y a lo largo de todos los años que siguieron lo acompañó con decisión y alegría y nunca vaciló en realizar los sacrificios que exigía la vocación científica de su esposo.

En el año 1965 se trasladó con su familia a los Estados Unidos para incorporarse, en carácter de miembro visitante, al Instituto de Estudios Avanzados de Princeton. Allí trabajó intensamente bajo la supervisión del gran matemático Armand Borel, quien se había interesado por las ideas y proyectos de investigación de Herrera, e inició un activo intercambio con otros jóvenes que estaban trabajando en temas afines, particularmente D. Lieberman y C.Kiselman.

De esa época data su segunda contribución importante: el estudio de la cohomología de De Rham de un espacio analítico, donde utiliza técnicas de integración sobre cadenas semianalíticas, que él mismo había desarrollado en su trabajo de tesis.

En 1967 fue designado profesor en la "Washington University" en Seattle, y al cabo del semestre lectivo regresó a la Argentina, incorporándose como Profesor Titular de la Facultad de Ciencias de la Universidad Nacional de La Plata. En esa Facultad desarrolla una intensa labor docente, contribuyendo decisivamente a la formación de un destacado grupo de jóvenes matemáticos que completan estudios de doctorado bajo su dirección. Nicolás Coleff y Jorge Solomín son sus primeros discípulos.

El carisma de Herrera es notable y su simpatía allana obstáculos que se presentaban como insalvables a la vista de los demás. Como pocos, sabe estimular y organizar el trabajo, siendo él mismo un trabajador tesonero, y extraer de cada uno lo mejor que puede dar. En 1970 fue elegido como el Jefe del Departamento de Matemáticas: una de las más felices decisiones, que habría de dar a ese Departamento un impulso extraordinario.

Había instalado su hogar en la localidad de City Bell, cercana a La Plata, en una casa que solía llenarse de sol, al que amaba, rodeado por el cariño de su familia y el afecto de sus numerosos amigos. Allí vivió hasta 1975, año en que se trasladó con su familia a la ciudad de Buenos Aires, en cuya Universidad fue designado Profesor Titular.

A los primeros tiempos de trabajo constructivo y fecundo en La Plata habían seguido los años difíciles en que nuestra sociedad parece buscar el rumbo y lo encuentra en el camino equivocado de la destrucción. En ámbitos universitarios, voces demasiado exaltadas para ser efectivas proclamaban la necesidad de promover los estudios de Matemática Aplicada, una necesidad que precisamente Herrera, sin estridencias, había sido de los primeros en advertir: Herrera estudiaba con ahínco Optimización y llegó a ofrecer en esos mismos años, en la Universidad de La Plata, el primer curso de Programación Dinámica.

Paralelamente continuaba desarrollando sus investigaciones en la teoría de funciones de varias variables complejas. Motivado en conversaciones con L. Bungart, durante una de sus visitas a los Estados Unidos, su interés se volvió hacia la teoría de los residuos en espacios analíticos.

Un primer resultado en esta dirección fue la demostración de la existencia de la corriente residuo simple asociada a una forma meromorfa. Posteriormente logró extender estos resultados a corrientes residuales múltiples, lo que permitió establecer un correlato analítico con la teoría algebraica de residuos, delineada por A. Grothendieck. Estas investigaciones habrían de continuar con la valiosa colaboración de su primer discípulo y entrañable amigo Nicolás Coleff.

J.P. Ramis y G. Ruget utilizaron estos residuos para construir una inmersión del complejo dualizante en las corrientes, que era indispensable para estudiar la coherencia de ciertas imágenes directas de haces coherentes.

En el Departamento de Matemáticas de la Facultad de Ciencias Exactas en Buenos Aires, Herrera realizó una labor de formación tan sobresaliente como la que había desarrollado en La Plata. Dictando cursos y organizando seminarios formó grupos de investigación y dirigió seis trabajos de tesis. En orden cronológico, Adrián Paenza, Néstor Bucari, Luis A. Romero Grados, Alicia Dickenstein, Carmen Sessa y Carlos Cabrelli, completaron sus respectivos trabajos de tesis bajo la supervisión de Herrera. Varios de ellos continuaron trabajando con Herrera en temas de investigación de interés común.

El interés de Herrera por los Algoritmos de Optimización fue creciendo con el tiempo y se había intensificado notablemente en los últimos años de su vida. En este campo había ideado un modelo para la utilización óptima de recursos financieros y dirigía en La Plata, con el auspicio de la Dirección de Investigaciones de la Provincia de Buenos Aires, un grupo de investigación que estaba trabajando en un plan destinado a mejorar el sistema de transportes de aquella ciudad.

En 1982 Herrera fue elegido por el voto de sus colegas para dirigir el Departamento de Matemáticas de la Facultad de Ciencias Exactas de la Universidad de Buenos Aires, tarea que desempeñó con ejemplar eficiencia hasta el día de su muerte.

Con la desaparición de Miguel Herrera la comunidad matemática ha perdido a un matemático de excepcional talento y genio organizador; una generación de jóvenes a su maestro y guía, y todos sus compañeros a un amigo leal.

La viril entereza y la dignidad con las que supo afrontar la suprema adversidad representan para todos nosotros su última e inolvidable lección.

N.A. Fava

BIBLIOGRAFIA

1. On the extension of currents, Revista de la Unión Matemática Argentina, t.22 (1965).

2. Integration on a semianalytic set, Bulletin de la Société Mathématique Française, 94 (1966) pp.141-180.

3. De Rham theorems on semianalytic sets, Bulletin of the American Mathematical Society, t.73, N°3, Mayo de 1967.

4. De Rham cohomology of an analytic space (con T. Bloom), Inventiones Mathematicae 7 (1969), pp.265-296.

5. Residues and Principal Values on Complex Spaces (con D. Lieberman), Mathematische Annalen 194 (1971) pp.259-294.

6. Duality and De Rham Cohomology of Infinitesimal Neighborhoods, (con D. Lieberman), Inventiones Mathematicae 13, pp.97-124.

7. Residues of forms with logarithmic singularities, Revista de la Unión Matemática Argentina 25 (1971).

8. Les courants résidus multiples, Journées de Géométrie Analytique, Poitiers (1972), Bulletin de la Soc. Mathém. de France, Mémoire 38, pp. 27-30.

9. Résidus multiples sur les espaces complexes, Journées Complexes de Metz (1972), Institut de Recherche Mathématique, Université L. Pasteur, Strasbourg, France.

10. Fibering of residual currents (con N. Coleff) Symposia in Pure and Applied Mathematics XXX, American Mathematical Society (1976).

11. Les courants residuels associés à une forme méromorphe (con N. Coleff) Springer Lecture Notes 633 (1978).

12. On the construction of the trace in Serre duality, Geometry an Topology, Río de Janeiro (1976) Springer Lecture Notes 597.

13. Algebraic cycles as residues of meromorphic forms (con N. Coleff y D. Lieberman) Mathematische Annalen (1980).

14. On the global lifting of meromorphic forms (con A. Dickenstein y C. Sessa). Manuscripta Mathematica 47 (1984).

Ing. JOSE BABINI

El 18 de mayo de 1984 falleció, a los 87 años de edad, el Ing. José Babini; su nombre está ligado al desarrollo, en este siglo, de la matemática argentina en general, y en particular al de la Unión Matemática Argentina.

Babini se diplomó como Profesor de Matemáticas en 1918, cuando ya había comenzado la carrera de Ingeniería en la Facultad de Buenos Aires, donde obtuvo en 1922 el título de Ingeniero Civil. En 1917 llegó al país el matemático español Julio Rey Pastor, invitado para dar conferencias por la Institución Cultural Española. Las exposiciones de Rey Pastor deslumbran al inquieto Babini que quedó atrapado para siempre en las redes de la matemática y subyugado por la agudeza de Rey Pastor que desde entonces fue su maestro y consejero. Babini fue también, y hasta la muerte de Rey Pastor, su amigo dilecto; esta entrañable amistad comenzó en 1917 cuando Rey encargó a Babini la recopilación del curso que había dictado sobre funciones de variable compleja, publicado ese mismo año en la Revista del Centro de Estudiantes de Ingeniería, n°183.

Un año después Babini realiza gestiones, junto con otros estudiantes de la Facultad, para la contratación en ella de Rey Pastor por un período largo. La gestión tuvo éxito y condujo a la radicación definitiva en la Argentina de Rey Pastor.

Babini no ejerció nunca la profesión de ingeniero; fue siempre un profesor de matemáticas, enamorado de su disciplina y de su historia. Hasta 1940 publicó una veintena de trabajos de investigación en matemática, más tarde sus publicaciones se hicieron de preferencia en el dominio de la historia de la ciencia. Los trabajos de matemática aparecieron en el Boletín del Seminario Matemático Argentino y en la Revista Matemática Hispano-Americana. También presentó un trabajo al Congreso Internacional de Matemáticas celebrado en Bolonia en 1928; fue publicado en las Actas del Congreso y se titulaba: "Sobre la integración aproximada de las ecuaciones diferenciales de segundo orden" (vol.III pp.103-107). Junto con otras dos comunicaciones fueron los primeros trabajos de matemáticos argentinos presentados a un congreso internacional.

La carrera universitaria de Babini se inicia en 1920 al ser incorporado como profesor en la Facultad de Ingeniería Química de Santa Fe, que contribuyó a fundar, y donde desarrolló una muy valiosa actuación; dió a sus cursos un nivel elevado y también adecuado a las necesidades de los futuros ingenieros. Por su temperamento, y seguramente incentivado por sus obligaciones como profesor de una facultad tecnológica, Babini dedicó muchos esfuerzos al cálculo numérico y a la matemática aplicada; aparte de los trabajos originales publicó un libro sobre Nomografía, muy útil y difundido como texto en varias universidades. En Santa Fe fue Decano de la Facultad en varias oportunidades; tuvo activa participación en la creación del Instituto de Matemática de Rosario, al que se incorporaron, traídos de Europa, Beppo Levi y Santaló y fue el "alma mater" de la creación en Santa Fe del Instituto de Filosofía e Historia de la Ciencia que se puso bajo la dirección del historiador Aldo Mieli. El contacto con éste fue de gran importancia y marca el comienzo del período en que Babini se interesa primordialmente en la historia de la ciencia.

En 1946, en una de las periódicas y destructivas crisis universitarias argentinas, fue separado de sus cargos en la Universidad Nacional del Litoral. La reparación de esta injusticia vendría nueve años más tarde.

En 1955 fue Decano Reorganizador de la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires. En 1956 fue Rector Fundador de la Universidad Nacional del Nordeste y en 1957 Vicerrector de la Universidad de Buenos Aires. En la Facultad de Ciencias Exactas y Naturales de la Universidad de Buenos Aires fue de 1958 a 1968 profesor titular de la cátedra de Historia de la Ciencia. En los últimos años de su vida se desempeñó como profesor plenario en el Centro de Altos Estudios de Ciencias Exactas, donde dictó cursos y conferencias que tuvieron amplia repercusión en el ambiente científico argentino.

Hasta sus últimos momentos conservó su extraordinaria lucidez mental; a los ochenta y tantos años sus conferencias eran tan claras y brillantes como las de cincuenta años atrás.

Babini fue también un distinguido hombre de letras. Como dijimos antes, a partir de 1940 la primordial preocupación intelectual de Babini fue la historia de la ciencia en la que desarrolló una brillante tarea publicando más de doce libros en Argentina, México, España y Estados Unidos; colaboró en varios diarios y revistas (Sur, La Nación, Clarín y Gaceta de Tucumán, entre otros). Entre los libros publicados, destacaremos por su mayor interés para el ambiente matemático, la excelente "Historia de la Matemática", en colaboración con Rey Pastor.

Babini era miembro correspondiente de la Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid, desde 1936, y miembro efectivo, desde de 1957, de la Academia Internacional de Historia de la Ciencia. Fue miembro del primer Directorio del CONICET, Director Nacional de Cultura, primer presidente de EUDEBA y también primer presidente de la Comisión Nacional de Enseñanza de la Matemática.

Por sus brillantes condiciones de escritor la SADE (Sociedad Argentina de Escritores) le confirió en 1980 el Gran Premio de Honor.

Aparte de sus contribuciones originales, Babini tuvo mucha influencia sobre la matemática argentina por haber sido uno de los fundadores (1936) y luego uno de los miembros más activos y eficientes, de la Unión Matemática Argentina. Desde 1941 y durante varios años fue el Director de la Revista, que consiguió fuera impresa en la Imprenta de la Universidad Nacional del Litoral. Publicar trabajos de matemáticas, con su simbolismo especial y fórmulas complicadas, no ha sido nunca fácil y Babini consiguió, pacientemente y con mucho esfuerzo, que dicha imprenta, al principio con escasos medios, fuera poco a poco adquiriendo los elementos necesarios para una impresión muy presentable y cuidada.

Más tarde, entre 1957 y 1967, sin discontinuidad, Babini fue Presidente de la Unión Matemática Argentina. Su acción durante este período fue hábil y exitosa, sorteando inconvenientes de toda índole, a veces por discrepancias entre los miembros y otras veces por dificultades ambientales. Durante los diez años de presidencia de Babini la Unión Matemática Argentina se fue fortaleciendo y sus actividades fueron extendiéndose, adquiriendo mayor significación interna y externa. Se celebraron casi anualmente Jornadas Matemáticas en distintos lugares del país, algunas de ellas de carácter internacional.

Importantes fueron las Sesiones Matemáticas del Sesquicentenario, celebradas en 1960, en conmemoración de la revolución de Mayo, para las cuales y gracias a una gestión de Babini, la UMA obtuvo una subvención especial del gobierno nacional para invitar a matemáticos extranjeros, que efectivamente asistieron a las Sesiones en número abundante y calificado (Ehresmann, Eilenberg, Zygmund, Guido Weiss, Lefschetz, Nachbin,...).

En 1967, al retirarse como presidente, Babini fue nombrado Miembro Honorario de la UMA, como reconocimiento de su acción eficaz y perdurable, siendo el primer argentino designado en tal carácter.

En definitiva, si se quisiera resumir a grandes rasgos la manera de ser de Babini durante toda su actuación, diríamos que fue un creador y organizador distinguido, que siempre apuntó hacia lo alto, deseoso de beber en las mismas fuentes de las altas cumbres en que brota el pensamiento, primero con Rey Pastor en Buenos Aires y luego con Aldo Mieli en Santa Fe. Fue exigente consigo mismo y también con los demás en toda su actuación como dirigente y organizador caracterizándose por su seriedad e impecable organización, procurando siempre la excelencia en vez de recostarse cómodamente en la mediocridad. Ello le ocasionó preocupaciones y desvelos, pero también lo ha hecho merecedor de la admiración de quienes lo conocieron y de la gratitud de quienes han tenido que proseguir su obra en los lugares que él transitó y dejó profunda huella.

Manuel Balanzat
Luis A. Santaló

Profesor Doctor JUAN CARLOS VIGNAUX
In Memoriam

El doctor Juan Carlos Vignaux falleció en Buenos Aires el día 26 de Junio de 1984, a la edad de 91 años. Sus restos mortales fueron trasladados a su provincia natal, Santiago del Estero.

El doctor Vignaux fue un distinguido matemático argentino que puede y debe considerarse como uno de los pioneros en el estudio de la matemática rigurosa y en cuanto a su formación matemática fue un verdadero autodidacta. En efecto, hacia el año 1917, cuando llegó al país el ilustre maestro doctor Julio Rey Pastor, en las escuelas de segunda enseñanza y en la universidad misma, las cátedras de matemática eran dictadas por ingenieros, cuyos conocimientos matemáticos eran totalmente intuitivos y sin demasiado rigor. Era una matemática que pertenecía al siglo XIX. En ninguna universidad argentina se dictaban cursos de matemática superior y moderna, tal vez con excepción de La Plata donde actuaba el italiano Hugo Broggi.

Cuando Rey Pastor dictó un brillantísimo curso que en forma panorámica comprendía toda la matemática del momento (1917), Vignaux ya estaba bastante familiarizado con la matemática moderna y se había especializado en el estudio de las series divergentes, siguiendo la orientación de Ernesto Cesaro y Emile Borel. Estaba igualmente familiarizado con la famosa colección de "Teoría de Funciones" de fama mundial, dirigida por el ilustre matemático francés Emile Borel.

Es interesante dejar constancia de que en virtud de la brillante actuación de Rey Pastor, un grupo de estudiantes de la Facultad de Ingeniería solicitó y obtuvo que Rey Pastor dictara cursos ordinarios en la misma. En la lista de solicitantes figuraban entre otros los nombres de Juan Blaquier, José Babini, Juan Carlos Vignaux y algunos otros.

La vida del doctor Vignaux fue muy activa. Fue profesor de Análisis Matemático en la Facultad de Ingeniería de la Universidad Nacional del Litoral; dictó varios cursos libres en la Facultad de Ingeniería de Buenos Aires sobre Ecuaciones Diferenciales, Ecuaciones Integrales y Cálculo de Variaciones. Fue también profesor de Análisis Matemático de la Facultad de Ingeniería de La Plata y en la Escuela Naval de Río Santiago, donde también ocupó el cargo de Director del Departamento de Matemática.

El doctor Vignaux se graduó en la Universidad de La Plata en el año 1925 con una tesis sobre Teoría de las Series Divergentes. Publicó luego diversas memorias en revistas nacionales, como ser las Contribuciones al Estudio de las Ciencias Fisicomatemáticas (Serie Matemática) de La Plata, los Anales de la Sociedad Científica Argentina de

Buenos Aires y las Publicaciones de la Facultad de Ciencias Fisicoquímicas y Naturales aplicadas a la Industria de la Universidad Nacional del Litoral.

Publicó también en revistas extranjeras, entre otros trabajos: "Un teorema sulle integrali doppi di Abel-Laplace" (Rendiconti R. Accad. Lincei (6), 17, 1933); "Sugli integrali de Laplace asintotici" (Rend. R. Acc. Lincei, 1939); "Sur l'extension du théorème de Dirichlet aux integrales doubles convergentes" (Bull. Soc. Math. Liège, 1933). Posteriormente, el doctor Vignaux reunió en un volumen varias de sus memorias bajo el título general de "Varias Contribuciones a la Teoría de Funciones", Talleres Gráficos Palumbo, 1938.

Hemos querido redactar esta breve nota en homenaje al querido amigo fallecido y en recuerdo de casi sesenta y cinco años de continua amistad y camaradería.

E.A. De Cesare

PROFESOR EMERITO AGR. EDUARDO GASPAR

13/X/1913 - 16/VII/1984

Fue el Profesor Gaspar un hombre que dedicó su atención principalmente a las tareas docentes y la investigación.

Brindó mucho entusiasmo a la enseñanza de la matemática y predicó a sus jóvenes alumnos para que siguieran su aprendizaje; otorgó su decidido apoyo a cuanta iniciativa se hiciera en ambos sentidos: enseñanza e investigación.

Tal accionar se hizo evidente en los distintos institutos de enseñanza en los que actuó y se acentuó en la Facultad de Ciencias Exactas e Ingeniería de la Universidad Nacional de Rosario (ex Facultad de Ciencias Matemáticas, Fisico-Químicas y Naturales Aplicadas a la Industria de la Universidad Nacional del Litoral) donde había obtenido su título de Agrimensor y en la que llegó a ser miembro del Consejo Directivo.

Su entusiasmo por las matemáticas lo demostró siendo miembro de la Unión Matemática Argentina, en la que repetidas veces fue electo Vice-Presidente. Su personalidad, hombría de bien y actuación descollante hicieron que durante varios períodos fuera presidente de la Asociación de Profesores de la Facultad a la que pertenecía.

Asimismo su conocimiento de los problemas universitarios hizo que ocasionalmente fuera designado Delegado Interventor en la Facultad de Química de la Universidad Nacional del Litoral.

Con la desaparición del Profesor Gaspar los ambientes que frecuentaba han tenido una sensible pérdida.

Enrique O. Ferrari

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