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SUBGROUPS OF THE GALILEO GROUP AND MEASURABLE FAMILIES OF CURVES

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ABSTRACT. The one and two-parameter subgroups of the Galileo group of actions on space-time of spaces dimension one are fully determined. Then, the measurable families of curves having the Galileo group or one of its subgroups as maximal invariance group are found.

1. INTRODUCTION**

The search for the Lie subgroups of a Lie group was initiated by Sophus Lie when he determined all subgroups of the projective group P_n. We intend in this work to obtain all the subgroups of the Galileo group of transformations of space-time of space dimension one. The one, two and three parameters families of measurable submanifolds (curves) of space-time will be also determined. The following result ([5]) will be used throughout:

Theorem 1.1. *Let $Y_1, \dots, Y_r$ be vectors fields on a manifold M, such that*

$$[Y_i,\ Y_j] = \sum_{k=1}^{r} C_{ij}^{k}\ Y_k; \qquad i,\ j = 1, \dots, r \tag{1.1}$$

where the C_{ij}^k. are constants. Then, there is a Lie group G whose Lie algebra has the C_{ij}^k as structure constants for some basis $X_1, \dots, X_r$, and a local action ϕ of G on M such that $X_{i_M} = Y_i$, $i = 1, \dots, r$

We also mention ([4]) that

Theorem 1.2. *Let G be a Lie group. If H is a Lie subgroup of G then the Lie algebra $\mathbb{H}$ of H is a subalgebra of $\mathbb{G}$, the Lie algebra of G. Each subalgebra of $\mathbb{G}$ is the Lie algebra of exactly one connected Lie subgroup of G.*

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2. The Galileo Group and its Subgroups

2.1 The Galileo Group.

As mentioned above, we restrict overselves to the simplest case of the Galileo group G of actions on space-time of space dimension one.
This group is determined by the equations

$$\begin{cases} r^* = r + vt + c \\ t^* = t \ + \ s \end{cases} \tag{2.1}$$

Where v, c, s are the group parameters. Thus G is a Lie group of dimension three. Its infinitesimal transformations are

$$X_1 = t\,\frac{\partial}{\partial r}, \qquad X_2 = \frac{\partial}{\partial r}, \qquad X_3 = \frac{\partial}{\partial t} \tag{2.2}$$

with structure equations

$$\begin{aligned} [X_1, X_1] &= [X_1, X_2] = [X_2, X_2] \\ &= [X_2, X_3] = [X_3, X_3] = 0, \\ [X_1, X_3] &= -X_2 \end{aligned} \tag{2.3}$$

2.2 Two-parameter subgroups of G.

These are the subgroups determined by two linearly independent vector fields Y_1, Y_2 in the linear span of X_1, X_2, X_3 such that their Lie bracket is a linear combination of Y_1, Y_2. Thus, equations

$$\begin{aligned} Y_1 &= \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \\ [Y_1, Y_2] &= (\alpha_3\beta_1 - \alpha_1\beta_3)\, X_2 \end{aligned} \tag{2.4}$$

with X_i, $i = 1, 2, 3$ as in (2.2), fully determine the two-parameters subgroups of the Lie group G. The following possibilities arise:

(i) $\alpha_1 \neq 0$. We may assume $\alpha_1 = 1$, $\beta_1 = 0$, to get

$$\begin{aligned} Y_1 &= X_1 + \alpha_2 X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_2 X_2 + \beta_3 X_3 \\ [Y_1, Y_2] &= -\beta_3 X_2 = \theta Y_1 + \phi Y_2 \\ &= \ \theta X_1 + (\theta\alpha_2 + \phi\beta_2)\, X_2 + (\theta\alpha_3 + \phi\beta_3)\, X_3 \end{aligned} \tag{2.5}$$

which together with (2.4) ensures that

$$\theta \ = \ 0, \qquad \phi\beta_2 \ = -\beta_3, \qquad \phi\beta_3 \ = 0$$

Now, condition $\phi\,\beta_3 = 0$ opens the alternative

$$\begin{cases} \phi \neq 0 & \text{This lead to } \beta_3 = \beta_2 = 0 \text{ which is absurd} \\ \phi = 0 & \text{Then } \beta_3 = 0 \text{ and } \beta_2 \neq 0 \end{cases}$$

So, we may assume $\beta_2 = 1,\ \alpha_2 = 0$ Then

$$\begin{aligned} Y_1 &= X_1 + \alpha_3 X_3 = t\frac{\partial}{\partial r} + \alpha_3\frac{\partial}{\partial t} \\ Y_2 &= X_2 = \frac{\partial}{\partial r} \end{aligned} \tag{2.6}$$

Now if $Y = aY_1 + bY_2$ is in the span of Y_1 y Y_2 its integral curves are determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aY_1 r + bY_2 r = at + b \\ \dfrac{dt}{d\eta} = aY_1 t + bY_2 t = a\alpha_3 \end{cases} \tag{2.7}$$

Integrating this system and determining the transformations sending $(r(0), t(0))$ into $(r(1), t(1))$, we obtain (see [2]) the subgroup H_2^1 given by the system of equations

$$\begin{cases} r^* = r + vt + c \\ t^* = t + kv \qquad k \text{ a constant} \end{cases} \tag{2.8}$$

and having

$$t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t}\,, \qquad \frac{\partial}{\partial r} \tag{2.9}$$

as infinitesimal trasformations

(ii) $\alpha_2 \neq 0$ We may take $\alpha_2 = 1,\ \beta_2 = \alpha_1 = 0$. Then:

$$\begin{aligned} Y_1 &= X_2 + \alpha_3 X_3 \\ Y_2 &= \beta_1 X_1 + \beta_3 X_3 \\ [Y_1, Y_2] &= \alpha_3\beta_1 X_2 = \theta Y_1 + \phi Y_2 \\ &= \phi\beta_1 X_1 + \theta X_2 + (\theta\alpha_3 + \phi\beta_3) X_3 \end{aligned} \tag{2.10}$$

which together with (2.4) and (2.5) yields

$$\phi\,\beta_1 = 0, \qquad \theta = \alpha_3\,\beta_1, \qquad \theta\,\alpha_3 + \phi\,\beta_3 = 0$$

so that

$$\alpha_3^2\,\beta_1 + \phi\,\beta_3 = 0, \quad \phi\,\beta_1 = 0$$

Condition $\phi\beta_1 = 0$ raises the alternative

$$\begin{cases} \phi \neq 0 \text{ then } \beta_1 = \beta_3 = 0, \text{ This is absurd} \\ \phi = 0, \text{ so that } \alpha_3^2\,\beta_1 = 0. \text{ Then } \begin{cases} \alpha_3 \neq 0 \text{ and } \beta_1 = 0\,,\ \beta_3 \neq 0 \\ \text{or} \\ \alpha_3 = 0 \end{cases} \end{cases}$$

Assuming α_3, $\beta_3 \neq 0$, $\beta_1 = 0$ and letting, as we may, $\beta_3 = 1$, produces the two fields

$$\begin{aligned} Y_1 &= X_2 + \alpha_3 X_3 = \frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \\ Y_2 &= X_3 \;=\; \frac{\partial}{\partial t} \end{aligned} \tag{2.11}$$

and if $Y = aY_1 + bY_2$ its integral curves are determined by

$$\frac{dr}{d\eta} = a, \qquad \frac{dt}{d\eta} = a\alpha_3 + b \tag{2.12}$$

Integrating this system and determining the transformations sending $(r(0),\ t(0))$ into $(r(1),\ t(1))$, we obtain the subgroup H_2^2 of equations

$$\begin{cases} r^* &= r + c \\ t^* &= t + s \end{cases} \tag{2.13}$$

with infinitesimal transformation

$$\frac{\partial}{\partial r} \quad , \quad \frac{\partial}{\partial t} \tag{2.14}$$

If $\alpha_3 = 0$ and $\beta_1 = 0$ the group H_2^2 above is obtained. However, if $\beta_1 \neq 0$, and we assume as we may $\beta_1 = 1$, the vectors fields are

$$\begin{aligned} Y_1 &= \; X_2 \; = \frac{\partial}{\partial r} \\ Y_2 &= X_1 + \beta_3 X_3 = t\frac{\partial}{\partial r} + \beta_3 \frac{\partial}{\partial t} \end{aligned} \tag{2.15}$$

and the corresponding group is H_2^1

(iii) $\alpha_3 \neq 0$. We take $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 1$, $\beta_3 = 0$, to get

$$\begin{aligned} Y_1 &= X_3 \\ Y_2 &= \beta_1 X_1 + \beta_2 X_2 \\ [Y_1, Y_2] &= \beta_1 X_2 = \theta Y_1 + \phi Y_2 \\ &= \phi\beta_1 X_1 + \phi\beta_2 X_2 + \theta X_3 \end{aligned} \tag{2.16}$$

Comparing with (2.4) and (2.5), we get

$$\begin{cases} \theta = 0 \\ \phi\beta_2 = \beta_1 \\ \phi\beta_1 = 0, \text{so that } \begin{cases} \beta_1 = \beta_2 = 0 \text{ if } \phi \neq 0 \text{ Absurd} \\ \beta_1 = 0 \text{ if } \phi = 0 \end{cases} \end{cases}$$

Thus

$$\begin{aligned} Y_1 &= \; X_3 \; = \frac{\partial}{\partial t} \\ Y_2 &= \; \beta_2 X_2 \; = \; \beta_2 \frac{\partial}{\partial r} \end{aligned} \tag{2.17}$$

and the group is H_2^2.

We have proved that

Theorem 2.1. . *The Galileo group G determined by the infinitesimal transformations (2.2) with structure equation (2.3) has the two two-parameter subgroups*

$$\begin{aligned} H_2^1 &= \left\{ t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \, , \, \frac{\partial}{\partial r} \right\} \\ H_2^2 &= \left\{ \frac{\partial}{\partial r} \, , \frac{\partial}{\partial t} \right\} \end{aligned} \tag{2.18}$$

2.3 One-parameter subgroups.

These are determined by the fields

$$Y = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3,$$

where X_i, $i = 1, 2, 3$ are the infinitesimal transformations of the Galileo group and α_1, α_2, α_3 are constants.
The different possibilities are:

(i) $\alpha_1 \neq 0$. Changing variables , if necessary , we may assume $\alpha_1 = 1$, so that

$$Y = X_1 + \alpha_2 X_2 + \alpha_3 X_3 = (t + \alpha_2) \frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t},$$

i.e, with t instead of $t + \alpha_2$,

$$Y = t\frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \tag{2.19}$$

Its integral curve is determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = at \\ \dfrac{dt}{d\eta} = aYt = a\alpha_3 \end{cases} \tag{2.20}$$

Upon integration of this system, the group is defined by the transformations sending $(r(0), t(0))$ into $(r(1), t(1))$ and thus we obtain the group H_1^1 determined by

$$\begin{cases} r^* = r + vt + v^2 k \\ t^* = t + 2vk \end{cases} \tag{2.21}$$

or by infinitesimal transformation

$$t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t} \tag{2.22}$$

(ii) $\alpha_2 \neq 0$. We may assume $\alpha_2 = 1$ and $\alpha_1 = 0$, so that

$$Y = X_2 + \alpha_3 X_3 = \frac{\partial}{\partial r} + \alpha_3 \frac{\partial}{\partial t} \tag{2.23}$$

with integral curve determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = a, \\ \dfrac{dt}{d\eta} = aYt = a\alpha_3, \end{cases} \tag{2.24}$$

which yields the group H_1^2 defined by

$$\begin{cases} r^* = r + ca \\ t^* = t + ck, \quad k \text{ a constant} \end{cases} \tag{2.25}$$

or by infinitesimal transformatcion

$$\frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \tag{2.26}$$

(iii) $\alpha_3 \neq 0$. We may assume $\alpha_1 = \alpha_2 = 0$, so that

$$Y = X_3 = \frac{\partial}{\partial t}, \tag{2.27}$$

and its integral curve is determined by

$$\begin{cases} \dfrac{dr}{d\eta} = aYr = 0, \\ \dfrac{dt}{d\eta} = aYt = a \end{cases} \tag{2.28}$$

which upon integration yields the group H_1^3 defined by

$$\begin{cases} r* = r \\ t^* = t + c \end{cases} \tag{2.29}$$

or by infinitesimal transformation

$$\frac{\partial}{\partial t} \tag{2.30}$$

Hence

Theorem 2.2.. *The Galileo group G determined by the infinitesimal transformations (2.2) with structure equations (2.3) has the three one-parameter subgroups*

$$\begin{aligned} H_1^1 &= \left\{ t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t} \right\} \\ H_1^2 &= \left\{ \frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \right\} \\ H_1^3 &= \left\{ \frac{\partial}{\partial t} \right\} \end{aligned} \tag{2.31}$$

3. Some Basic Notions

3.1 Maximal Invariance Subgroup of a family of manifolds.

Let G be a group acting on a manifold M and let F be a q-parameter family of p-dimensional submanifolds of M. If G^* is the subgroup of G leaving globally invariant the family F, (i.e. $s \in G^*$ *and* $v \in F$, implies $s(v) \in F$) and H^* is the subgroup of G^* fixing every submanifold in F, (i.e. $s \in H^*$, and$v \in F$ grants $s(v) = v$), the quotient group $K = G^* \,/\, H^*$ is called the **maximal invariance subgroup of F**. The subgroups of K are called the invariance subgroups of F. The group K leaves invariant the family F and has no other transformations but the identity fixing all submanifolds in F.

3.2 Associated Group.

If to each s in an invariance group G of F we associate a transformation β on the parameter of F, the set H of all such transformations is a group isomorphic to G which acts on the parameter space of F. The group H is called the **associated group of G relative to F.**

In [6] it is shown that

Teorem 3.1. *Let G, and H, be isomorphic groups. A necessary and sufficient condition for the existence of a q-parameter family F_q of p-dimensional submanifolds V_p having G as an invariance subgroup is that the matrix*

$$(\xi_h^1(x), \ldots, \xi_h^n(x),\ \eta_h^1(\alpha), \ldots, \eta_h^q(\alpha)), \quad h = 1, \ldots, r \tag{3.1}$$

be of range $r_1 < n + q$. Where $x = (x^1, \ldots, x^n)$, $\alpha = (\alpha^1, \ldots, \alpha^q)$, $r \geq 1$ and , $\xi_h^i(x)$, $\eta_k^j(\alpha)$ are respectively the coefficients in the infinitesimal transformations of the groups G and H.

Under the above circumstances, the family F_q is determined by the equations

$$\Phi^\lambda \left(\phi^1(x, \alpha), \ldots, \phi^{n+q-r_1}(x, \alpha)\right) = 0, \qquad \lambda = 1, \ldots, n - p \tag{3.2}$$

where $\phi^k(x, \alpha)$, $k = 1, \ldots, n + q - r_1$, are the independent integrals of the system

$$\xi_k^i(x)\, \frac{\partial F^\lambda(x, \alpha)}{\partial x^i} + \eta_k^j(\alpha)\, \frac{\partial F^\lambda(x, \alpha)}{\partial \alpha^j} = 0 \quad k = 1, \ldots, r_1 \tag{3.3}$$

and

$$F^\lambda(x, \alpha) = \Phi^\lambda \left(\phi^1(x, \alpha), \ldots, \phi^{n+q-r_1}(x, \alpha)\right)$$

Remark 3.1. For our purposes, $\lambda = 1$ and $r_1 = 1, 2$

3.3 Integral invariants of a Lie group.

Let G be an r-parameter group of transformations $\phi(x^1, \ldots, x^n, \alpha^1, \ldots, \alpha^r)$ of $\mathbb{R}^n$. A differentiable function $\psi : \mathbb{R}^n \to \mathbb{R}$ is an **integral invariant** of G if

$$\int_{\phi(U)} \psi(x^1, \ldots, x^n) dx^1 \ldots dx^n = \int_U \psi(y^1, \ldots, y^n) dy^1 \ldots dy^n \tag{3.4}$$

for any transformation ϕ of G where $y^k = \phi^k(x^1, \ldots, x^n; \alpha^1, \ldots, \alpha^r)$ and U is any subset of $\mathbb{R}^n$ where the right hand integral exists.

3.4 Families of measurable submanifolds.

Definition 3.1. A Lie group of transformations of $\mathbb{R}^n$ is measurable if it has a unique integral invariant, except for constant multiples.
A necessary condition for the measurability of a Lie group G is that G be transitive (see [6]).
Let F be a q-parameter family of p-dimensional submanifolds of $\mathbb{R}^n$ and let G be an invariance subgroup of F. Let H be the associated group of G relative to F. If H is measurable, F is said to be **measurable relative to** H **(or** G **)**; and if ψ is the essentially unique integral invariant of H,

$$\int_F \psi(\alpha^1, \ldots, \alpha^q) d\alpha^1 \ldots d\alpha^q \tag{3.5}$$

is called the **measure of the family** F **for** H **(or** G**)**, and the q-form

$$\psi(\alpha^1, \ldots, \alpha^q) d\alpha^1 \ldots d\alpha^q$$

is called the **invariant density** of F for H

Deltheil [3] has shown that

Theorem 3.2. *The integral invariants of a Lie transformation group are the solutions of the system of partial differential equations*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x^i} [\xi_h^i(x)\, \psi(x)] = 0, \qquad h = 1, \ldots, r \tag{3.6}$$

where the ξ_h^i are the coefficients in the infinitesimal transformations of the group.

A sufficient condition for the measurability of a family F of submanifolds of $\mathbb{R}^n$ is the measurability of the associated group of the maximal invariance group of F. This condition is also necessary for one, two and three-parameter families.

4. Measurable Families of Curves for the Galileo Group

4.1 Families of one-parameter curves which are measurable for the action of the Galileo group or of one of its subgroups.

Let F be a family of one-parameter curves determined in space-time (r,t) by the equation

$$\psi(r,t,\alpha) = 0, \tag{4.1}$$

α a parameter. If G is the maximal invariance group of F, its associated group H acts on $\mathbb{R}$, and F is measurable if and only if H is.
As Lie proved, the groups acting on $\mathbb{R}$ are the translation, the afin and the projective group, and only the first of these groups is measurable. Thus, H above is $\{\partial/\partial\alpha\}$, the translation group, and G is one of the group H_1^1, H_1^2, H_1^3. We examine the three posibilities:

(1) If the maximal invariance group G of F is

$$H_1^1 = \left\{t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t}\right\} \text{ and its associated group is } H = \left\{\frac{\partial}{\partial \alpha}\right\}$$

then, in (3.6),

$$\xi^1 = t,\ \xi^2 = 2k,\ \eta^1 = 1$$

i.e. ψ must satisfy

$$t\frac{\partial\psi}{\partial r} + 2k\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\alpha} = 0 \tag{4.2}$$

whose solutions are of the form

$$\psi(r - t\alpha + k\alpha^2,\ t - 2k\alpha) = 0, \quad \text{or,} \quad r = t\alpha - k\alpha^2 + \phi(t - 2k\alpha) \tag{4.3}$$

(2) If $G = H_1^2$,

$$H_1^2 = \left\{\frac{\partial}{\partial r} + k\frac{\partial}{\partial t}\right\}, \quad \text{and} \quad H = \left\{\frac{\partial}{\partial\alpha}\right\},$$

then $\xi^1 = 1,\ \xi^2 = k,\ \eta^1 = 1$ in (3.3), and $\psi(r,t,\alpha)$ must satisfy

$$\frac{\partial\psi}{\partial r} + k\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial\alpha} = 0, \tag{4.4}$$

whose solutions have the form

$$\psi(r - \alpha, t - k\alpha) = 0, \quad \text{or,} \quad r = \alpha + \phi(t - k\alpha) \tag{4.5}$$

(3) If $G = H_1^3 = \left\{ \dfrac{\partial}{\partial t} \right\}$ and $H = \left\{ \dfrac{\partial}{\partial \alpha} \right\}$, then $\xi^1 = 0$ $\xi^2 = 1$ $\eta^1 = 1$ so that

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \alpha} = 0 \tag{4.6}$$

whose solutions are of the form

$$\psi(r,\ t-\alpha) \ = \ 0, \qquad \text{or,} \quad r \ = \ \phi(t-\alpha) \tag{4.7}$$

summing up, we have shown that

Theorem 4.1. *The one-parameter families of curves*

$$\begin{aligned} \psi(r - t\alpha + k\alpha^2,\ t - 2k\alpha) &= 0, \\ \psi(r-\alpha,\ t-k\alpha) &= 0, \\ \psi(r,\ t-\alpha) &= 0 \end{aligned} \tag{4.8}$$

have, respectively, as their maximal invariance groups, the one-parameter subgroups

$$\begin{aligned} H_1^1 &= \left\{ t\frac{\partial}{\partial r} + 2k\frac{\partial}{\partial t} \right\} \\ H_1^2 &= \left\{ \frac{\partial}{\partial r} + k\frac{\partial}{\partial t} \right\} \\ H_1^3 &= \left\{ \frac{\partial}{\partial t} \right\} \end{aligned}$$

of the Galileo group in space-time of space of dimension one.

4.2 Two- parameters families of curves which are measurable for the action of the Galileo group or one of its subgroups.

Let F be the two-parameters family of curves

$$\psi(r,t,\alpha,\beta) \ = \ 0 \tag{4.9}$$

where α, β are the parameters and (r,t) the space-time coordinates.
Let G be the maximal invariance group of F and H its associated group. Since F is measurable, also H is measurable, and therefore transitive. Then dim $H =$ dim $G \geq 2$ and G has to be one of the groups H_2^1, H_2^2, or the full three-parameters Galileo group.

(1) If $G = H_2^1 = \{Y_1, Y_2\}$ where

$$\begin{cases} Y_1 = \ t\dfrac{\partial}{\partial r} + k\dfrac{\partial}{\partial t}, \\ Y_2 = \ \dfrac{\partial}{\partial r} \end{cases} \tag{4.10}$$

with structure determined by $[Y_1,\ Y_2]=0$, then H is a two parameters group with infinitesimal transformations

$$\begin{aligned} A_1 &= \eta_1^1(\alpha,\beta)\frac{\partial}{\partial\alpha}+\eta_1^2(\alpha,\beta)\frac{\partial}{\partial\beta} \\ A_2 &= \eta_2^1(\alpha,\beta)\frac{\partial}{\partial\alpha}+\eta_2^2(\alpha,\beta)\frac{\partial}{\partial\beta} \end{aligned} \tag{4.11}$$

and structure equation $[A_1,\ A_2] = 0.$ Then, the corresponding coefficients for equation (3.6) are

$$\begin{cases} \xi_1^1=t,\ \xi_1^2=k,\ \xi_2^1=1,\ \xi_2^2=0 \\ \eta_1^1=\alpha,\ \eta_1^2=0,\ \eta_2^1=0,\ \eta_2^2=\beta \end{cases}$$

and F is determined by the equations

$$\begin{cases} (i)\ t\dfrac{\partial\psi}{\partial r}+k\dfrac{\partial\psi}{\partial t}+\alpha\dfrac{\partial\psi}{\partial\alpha} = 0 \\ (\text{ii})\ \dfrac{\partial\psi}{\partial r} + \beta\dfrac{\partial\psi}{\partial\beta} = 0 \end{cases} \tag{4.12}$$

The solution to $(4.12),(i))$ is the family of curves implicitly given by

$$\psi(\alpha e^{-t/k},\ \beta,\ r-t^2/2k) = 0 \tag{4.13}$$

i.e by $\psi(r,\ t,\ \alpha,\ \beta)=0$ where

$$\psi(r,\ t,\ \alpha,\ \beta)= r-t^2/2k-\phi(\alpha e^{-t/k},\ \beta) \tag{4.14}$$

since $\psi(r,t,\alpha)$ must verify $(4.12),(ii)$, and

$$\frac{\partial\psi}{\partial r}=1;\qquad \frac{\partial\psi}{\partial\beta}=-\frac{\partial\phi}{\partial\beta} \tag{4.15}$$

we get

$$\phi = \ln\beta + f(\alpha e^{-t/k}) \tag{4.16}$$

and

$$\psi(\ r,\ t,\ \alpha,\ \beta) = r-t^2/2k-\ln\beta-f(\alpha\ e^{-t/k})=0 \tag{4.17}$$

is the measurable family of two-parameter curves having H_2^1 as maximal invariance group.

(2) If $G=H_2^2$

$$H_2^2=\left\{\frac{\partial}{\partial r},\ \frac{\partial}{\partial t}\right\} \qquad \text{then} \qquad H=\left\{\alpha\frac{\partial}{\partial\alpha},\ \beta\frac{\partial}{\partial\beta}\right\}$$

is as before the associated group of G, and the corresponding coefficients for (3.6) are

$$\begin{cases} \xi_1^1 = 1, & \xi_1^2 = 0, \quad \eta_1^1 = \alpha, \quad \eta_1^2 = 0 \\ \xi_2^1 = 0, & \xi_2^2 = 1, \quad \eta_2^1 = 0, \quad \eta_2^2 = \beta \end{cases}$$

so that (3.3) becomes

$$\begin{cases} \dfrac{\partial \psi}{\partial r} + \alpha \dfrac{\partial \psi}{\partial \alpha} = 0 \\ \dfrac{\partial \psi}{\partial t} + \beta \dfrac{\partial \psi}{\partial \beta} = 0 \end{cases} \tag{4.18}$$

whose solutions are implicitly given by

$$\psi(\alpha\, e^{-r},\ \beta\, e^{-t}) = 0 \tag{4.19}$$

i.e, F is the family of curves determined by

$$\psi(r, t, \alpha, \beta) = r - \ln \alpha - \phi(t - \ln \beta) \tag{4.20}$$

So far, we have proved that

Theorem 4.2. *The two parameters families of curves :*

$$\begin{aligned} r &= t^2/2k + \ln \beta + \phi(\alpha e^{-t/k}) \\ r &= \ln \alpha + \phi(t - \ln \beta) \end{aligned} \tag{4.21}$$

have, repectively as their maximal invariance groups, the two parameter subgroups

$$\begin{aligned} H_2^1 &= \left\{ t\frac{\partial}{\partial r} + k\frac{\partial}{\partial t},\ \frac{\partial}{\partial r} \right\} \\ H_2^2 &= \left\{ \frac{\partial}{\partial r},\ \frac{\partial}{\partial t} \right\} \end{aligned} \tag{4.22}$$

of the Galileo group in space-time of space dimension one.

M. Stoka [6] shows that the two-parameter family of measurable curves having as invariance group G the full Galileo group (2.1), are family of straight lines. He also proves that the measurable three-parameter family having G as maximal invariance group is given by

$$\psi(t - \gamma r - \beta, r - \alpha) = 0 \tag{4.23}$$

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ON THE STRUCTURE OF THE CLASSIFYING RING OF SO(n,1) AND SU(n,1)

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Presentado por R. Panzone

ABSTRACT. Let G_o be a non compact real semisimple Lie group with finite center, and let $U(\mathfrak{g})^K$ denote the centralizer in $U(\mathfrak{g})$ of a maximal compact subgroup K_o of G_o. By the fundamental work of Harish-Chandra it is known that many deep questions concerning the infinite dimensional representation theory of G_o reduce to questions about the structure and finite dimensional representation theory of the algebra $U(\mathfrak{g})^K$, called the classifying ring of G_o. To study the algebra $U(\mathfrak{g})^K$, B. Kostant suggested to consider the projection map $P\colon U(\mathfrak{g}) \to U(\mathfrak{k})\otimes U(\mathfrak{a})$, associated to an Iwasawa decomposition $G_o = K_oA_oN_o$ of G_o, adapted to K_o. When P is restricted to $U(\mathfrak{g})^K$ P becomes an injective anti-homomorphism of algebras. In this paper we use the characterization of the image of $U(\mathfrak{g})^K$, when G_o =SO(n,1) or SU(n,1) obtained in Tirao [11], to prove that $U(\mathfrak{g})^K \simeq Z(\mathfrak{g})\otimes Z(\mathfrak{k})$, where $Z(\mathfrak{g})$ and $Z(\mathfrak{k})$ denote respectively the centers of $U(\mathfrak{g})$ and of $U(\mathfrak{k})$. By a well known theorem of Harish-Chandra these two centers are polynomial rings in rank($\mathfrak{g}$) and rank($\mathfrak{k}$) indeterminates, respectively. Thus the algebraic structure of $U(\mathfrak{g})^K$ is completely determined in this two cases.

1. INTRODUCTION

Let G_o be a non compact real semisimple Lie group with finite center, and let K_o denote a maximal compact subgroup of G_o. If $\mathfrak{k} \subset \mathfrak{g}$ denote the respective complexified Lie algebras, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^K$ denote the centralizer of K_o in $U(\mathfrak{g})$.

By the fundamental work of Harish-Chandra it is known that many deep questions concerning the infinite dimensional representation theory of G_o reduce to questions about the structure and finite dimensional representaton theory of the algebra $U(\mathfrak{g})^K$, called the classifying ring of G_o (cf. Cooper [2]). Briefly, the reason for this is as follows: To any quasi-simple irreducible Banach space representation π of G_o there is associated an algebraically irreducible $U(\mathfrak{g})$-module V which is locally finite for K_o and which determines π up to infinitesimal equivalence. In fact one has a primary decomposition $V = \bigoplus V_\delta$, where the sum is taken over the set $\hat{K}_o$ of all equivalence classes δ of finite dimensional irreducible representations of K_o, and the multiplicity of δ is finite for any $\delta \in \hat{K}_o$. Then, in particular, any V_δ is finite dimensional and hence, a finite dimensional $U(\mathfrak{g})^K$-module. The point is that V itself as a $U(\mathfrak{g})$-module is completely determined by V_δ as a $U(\mathfrak{g})^K$-module

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for any fixed δ when $V_\delta \neq 0$. See Lepowsky and McCollum [10] and Lepowsky [9] for a nice exposition of this. See also Dixmier [3] and Wallach [12].

When $V_{\delta_o} \neq 0$, where δ_o is the class of the trivial representation of K_o, then π is called spherical. The approach above has been quite successful in dealing with spherical irreducible representations of G_o (see e.g. Kostant [7]). Indeed, we may take $\delta = \delta_o$ and thus we have only to consider a quotient $U(\mathfrak{g})^K/I$ instead of $U(\mathfrak{g})^K$. Here I is the intersection of $U(\mathfrak{g})^K$ with the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{k}$. Now by a theorem of Harish-Chandra, $U(\mathfrak{g})^K/I$ is not only commutative but also isomorphic to a polynomial ring in r variables, where r is the split rank of G_o. More precisely one has an algebra exact sequence

$$0 \to I \to U(\mathfrak{g})^K/I \xrightarrow{\gamma} U(\mathfrak{a})^{\widetilde{W}} \to 0 \tag{1}$$

where $\mathfrak{a}$ is the complex abelian Lie algebra associated to an Iwasawa decomposition $G_o = K_oA_oN_o$ of G_o adapted to K_o, and $U(\mathfrak{a})^{\widetilde{W}}$ is the ring of $\widetilde{W}$-invariants in $U(\mathfrak{a})$, $\widetilde{W}$ being the translated Weyl group.

To investigate the general (not necessarily spherical) case along these lines one must look at $U(\mathfrak{g})^K$ itself, not just $U(\mathfrak{g})^K/I$. It is known (see e.g. Kostant and Tirao [8]) that the map (1) may be replaced by an exact sequence

$$0 \to U(\mathfrak{g})^K \xrightarrow{P} U(\mathfrak{k})^M \otimes U(\mathfrak{a})$$

where $U(\mathfrak{k})^M$ denote the centralizer of M_o in $U(\mathfrak{k})$, M_o being the centralizer of A_o in K_o and $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ is given the tensor product algebra structure. Moreover P is an antihomomorphism of algebras. In order to generalize (1) it is necessary to determine the image of P. Towards this end we introduced in Tirao [11] a subalgebra B of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ defined by a set of equations derived from certain imbeddings among Verma modules and the subalgebra $B^{\widetilde{W}}$ of all elements in B which commute with certain intertwining operators. Such operators are in a one to one correspondence with the elements of the Weyl group W and are rather closely related to the Kunze-Stein intertwining operators. In fact the relation of $B^{\widetilde{W}}$ to B may be taken as the generalization of the relation of $U(\mathfrak{a})^{\widetilde{W}}$ to $U(\mathfrak{a})$. In Tirao [11] it is proved that the image of P lies always in $B^{\widetilde{W}}$, and that when G_o =SO(n,1) or SU(n,1) we have $P(U(\mathfrak{g})^K) = B^{\widetilde{W}}$.

In this paper we use this result to exhibit the structure of $U(\mathfrak{g})^K$ in this two cases. In fact we shall prove that $U(\mathfrak{g})^K \simeq Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$, where $Z(\mathfrak{g})$ and $Z(\mathfrak{k})$ denote respectively the centers of $U(\mathfrak{g})$ and of $U(\mathfrak{k})$. By a well known theorem of Harish-Chandra these two centers are polynomial rings in rank($\mathfrak{g}$) and rank($\mathfrak{k}$) indeterminates, respectively. Thus our work is finished.

Nowadays there are several proofs that $U(\mathfrak{g})^K$ is a polynomial ring (Cooper [2], Benabdallah [1], Knop [6]), nevertheless our approach should prove to be useful to attack the general case, or at least the case when G_o is any real rank one group.

2. The algebra B

Let $\mathfrak{t}_o$ be a Cartan subalgebra of the Lie algebra $\mathfrak{m}_o$ of M_o. Set $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the corresponding complexification. Then $\mathfrak{h}_o$ and $\mathfrak{h}$ are Cartan subalgebras of $\mathfrak{g}_o$ and $\mathfrak{g}$, respectively. Now we choose a Borel subalgebra $\mathfrak{t} \oplus \mathfrak{m}^+$ of the complexification $\mathfrak{m}$ of $\mathfrak{m}_o$ and take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{n}$ as a Borel subalgebra of $\mathfrak{g}$. Let Δ^+ be the corresponding set of positive roots, put $\mathfrak{g}^+ = \mathfrak{m}^+ \oplus \mathfrak{n}$ and $\mathfrak{g}^- = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. Also put $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. Set $\langle , \rangle$ denotes the Killing form of $\mathfrak{g}$ and $(\mu, \alpha) = 2\langle\mu, \alpha\rangle / \langle\alpha, \alpha\rangle$. For $\alpha \in \Delta^+$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $(\mu, \alpha) = \mu(H_\alpha)$ for all $\mu \in \mathfrak{h}^*$. Also set $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{a}$. Let $P^+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition, associated to K_o, and let θ denote the corresponding Cartan involution. Also let M'_o denote the normalizer of A_o in K_o. Let $\alpha \in P^+$ be a simple root such that $Y_\alpha \neq 0$. Set $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$ where $X_{-\alpha}$is a non zero root vector corrresponding to $-\alpha$.

When $G_o = \mathrm{SO}(n, 1)_e$ $(n \neq 3)$ there is only one simple root $\alpha_1 \in P^+$ (if $n = 3$ there are two simple roots $\alpha_1, \alpha_2 \in P^+$). When $G_o = \mathrm{SU}(n, 1)$ $(n \geq 2)$ there are exactly two simple roots α_1, α_n in P^+. Set $E_1 = E_{\alpha_1}$ $(n \neq 3)$ and $E_1 = E_{\alpha_1}, E_{\alpha_2}$ when $n = 3$ in the first case, and $E_2 = E_{\alpha_1}, E_3 = E_{\alpha_n}$ in the second case. We shall also use E to designate any one of the vectors E_1, E_2 or E_3 and α for α_1, (α_1 or α_2), α_1 or α_n, respectively. Moreover $Y_\alpha \neq 0$ if $G_o = \mathrm{SO}(n, 1)_e$ $n \geq 3$ or $G_o = \mathrm{SU}(n, 1)$ $n \geq 2$. From now on we shall take for granted that we are in one of these cases.

From (8) and (9) of Tirao [11] we know that the algebra B is the set of all $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ such that for all $n \in \mathbf{N}$

$$E^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E^n \qquad \mod (U(\mathfrak{k})m^+) \tag{2}$$

holds for $(E, \alpha) = (E_1, \alpha_1)$ and $(E, \alpha) = (E_2, \alpha_1), (E_3, \alpha_n)$, respectively. Also

$$B^{\widetilde{W}} = \{b \in B : \delta_w * b(\lambda - \rho) = b(w(\lambda) - \rho) * \delta_w \text{ for all } w \in M'_o, \lambda \in \mathfrak{a}^*\}. \tag{3}$$

The algebraic structure of $U(\mathfrak{g})^K$ when G_o =SO(n,1) or SU(n,1) $n \geq 2$ will be determined by induction on n. The case SO(2,1) is quite simple and will be considered later. Thus we shall take up now the case $G_o = \mathrm{SU}(2, 1)$. If $\mathfrak{u}$ is any Lie algebra $z(\mathfrak{u})$ will denote the center of $\mathfrak{u}$ and $Z(\mathfrak{u})$ will denote the center of $U(\mathfrak{u})$.

Lemma 1. *If $G_o = \mathrm{SU}(2, 1)$ set $Y = Y_{\alpha_1} = -Y_{\alpha_2}$. Also let $0 \neq D \in z(\mathfrak{k})$ and let ζ denote the Casimir element of $[\mathfrak{k}, \mathfrak{k}]$. Then $\{\zeta^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$. Moreover the canonical homomorphism $\mu : Z(\mathfrak{k}) \otimes Z(\mathfrak{m}) \to U(\mathfrak{k})^M$ is a surjective isomorphism.*

Proof. The set $\{E_2, E_3, D, Y\}$ is a basis of $\mathfrak{k}$. Therefore the monomials $E_2^i E_3^l D^j Y^k$ form a basis of $U(\mathfrak{k})$. Now $\mathfrak{m}$ is one-dimensional and $Y \in \mathfrak{m}$. From Lemma 29 of Tirao [11] it follows that $[Y, E_2] = -(3/2)E_2$ and $[Y, E_3] = (3/2)E_3$. Hence $\{E_2^i E_3^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$. Now $\zeta = aE_2E_3 + bY^2 + cD^2 + dYD + eY + fD$, $a, b, c, d, e, f \in \mathbf{C}$, $a \neq 0$. Thus $\{\zeta^i D^j Y^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$.

Since $\{\zeta^i D^j\}_{i,j\geq 0}$ is a basis of $Z(\mathfrak{k})$ and $\{Y^k\}_{k\geq 0}$ is a basis of $U(\mathfrak{m}) = Z(\mathfrak{m})$ the first assertion of the lemma implies the second.

Proposition 2. *For $j = 2, 3$ let*

$$B_j = \{b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a}) : E_j^t b(t - (-1)^j Y - 1) = b(-t - (-1)^j Y - 1)E_j^t, t \in \mathbf{N}\}.$$

Then B_j, as an algebra over $\mathbf{C}$, is generated by the algebraically independent elements $\zeta \otimes 1, D \otimes 1, (Y \otimes 1 + (-1)^j \otimes Z + (-1)^j)^2, Y \otimes 1 - 3(-1)^j \otimes Z$ and 1.

Proof. If $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ then by Lemma 1 b can be written uniquely as $b = \sum a_{i,j,k,l}\zeta^i D^j Y^k \otimes Z^l$, $a_{i,j,k,l} \in \mathbf{C}$. Since $[(-1)^j Y, E_j] = -\frac{3}{2}E_j$ ($j = 2,3$) from Lemma 18 (vi) of Tirao [11] we get

$$\begin{aligned} E_j^t b(t - (-1)^j Y - 1) &= \sum_{i,j,k,l} a_{i,j,k,l}\zeta^i D^j E_j^t Y^k (t - (-1)^j Y - 1)^l \\ &= \sum_{i,j,k,l} a_{i,j,k,l}\zeta^i D^j (Y + (-1)^j \tfrac{3}{2}t)^k (-\tfrac{t}{2} - (-1)^j Y - 1)^l E_j^t. \end{aligned}$$

Thus $b \in B_j$ if and only if for all $i, j, t \in \mathbf{N}$ we have

$$\sum_{k,l} a_{i,j,k,l}(Y + (-1)^j \tfrac{3}{2}t)^k (-\tfrac{t}{2} - (-1)^j Y - 1)^l = \sum_{k,l} a_{i,j,k,l} Y^k (-t - (-1)^j Y - 1)^l.$$

Hence the problem of characterizing all $b \in B_j$ is equivalent to determine all $f \in \mathbf{C}[x_1, x_2]$ such that

$$f(y + (-1)^j \tfrac{3}{2}t, -\tfrac{t}{2} - (-1)^j y - 1) = f(y, -t - (-1)^j y - 1) \tag{4}$$

for all $t, y \in \mathbf{C}$.

For $j = 2,3$ let $f_j \in \mathbf{C}[x_1, x_2]$ be defined by

$$f(x_1, x_2) = f_j(x_1 + (-1)^j(x_2 + 1), x_1 - 3(-1)^j(x_2 + 1)). \tag{5}$$

Then f satisfies (4) if and only if $f_j((-1)^j t, 4y + 3(-1)^j t) = f_j(-(-1)^j t, 4y + 3(-1)^j t)$ for all $t, y \in \mathbf{C}$. Equivalently if and only if

$$f = \sum_{k,l} a_{k,l}(x_1 + (-1)^j(x_2 + 1))^{2k}(x_1 - 3(-1)^j(x_2 + 1))^l. \tag{6}$$

From this it follows that B_j is generated by $\zeta \otimes 1, D \otimes 1, (Y \otimes 1 + (-1)^j \otimes Z + (-1)^j)^2, Y \otimes 1 - 3(-1)^j \otimes Z$ and 1. Clearly these elements are algebraically independent.

Now we want to determine the algebra $B = B_2 \cap B_3$. Given $f \in \mathbf{C}[x_1, x_2]$ let $a(f) \in \mathbf{C}[x_1, x_2]$ be defined by $a(f)(x_1, x_2) = f(\sqrt{3}x_1, x_2 - 1)$. Also let T_j ($j = 2,3$) be the automorphism of $\mathbf{C}[x_1, x_2]$ induced by the linear map: $T_j(x_1) = -\frac{1}{2}(x_1 + (-1)^j\sqrt{3}x_2)$, $T_j(x_2) = -\frac{1}{2}((-1)^j\sqrt{3}x_1 - x_2)$.

Lemma 3. *An element $f \in \mathbf{C}[x_1, x_2]$ satisfies (4) if and only if $T_j(a(f)) = a(f)$ ($j = 2,3$).*

Proof. First of all for $j = 2,3$ we compute $T_j(\sqrt{3}x_1 + (-1)^j x_2) = -(\sqrt{3}x_1 + (-1)^j x_2)$ and $T_j(\sqrt{3}x_1 - 3(-1)^j x_2) = \sqrt{3}x_1 - 3(-1)^j x_2$. If we use the notation introduced in (5) we get

$$\begin{aligned} a(f)(x_1, x_2) &= f_j(\sqrt{3}x_1 + (-1)^j x_2, \sqrt{3}x_1 - 3(-1)^j x_2), \\ T_j(a(f))(x_1, x_2) &= f_j(-(\sqrt{3}x_1 + (-1)^j x_2), \sqrt{3}x_1 - 3(-1)^j x_2). \end{aligned}$$

Therefore $T_j(a(f)) = a(f)$ if and only if f_j is even in the first variable. This is the same as saying that f has the form stated in (6), which was shown to be equivalent to (4).

Proposition 4. *Let W denote the group of automorphisms of $\mathbf{C}[x_1, x_2]$ generated by T_2, T_3. Then:*
(i) W is isomorphic to the Weyl group of $\mathfrak{su}(2,1)$.
(ii) The algebra $\mathbf{C}[x_1, x_2]^W$ of all W-invariants is generated by the algebraically independent polynomials $x_1^2 + x_2^2$, $x_1(x_1^2 - 3x_2^2)$ and 1.

Proof. Let us consider on $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ the inner product defined by requiring that x_1, x_2 be an orthonormal basis. Then the restriction of T_j to $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ is the reflection on the line generated by $\frac{1}{2}(x_1 - (-1)^j\sqrt{3}x_2)$ $(j = 2, 3)$. Moreover, if we identify $\mathfrak{h}_{\mathbf{R}}^*$ with $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ by the linear map $\iota\colon \mathfrak{h}_{\mathbf{R}}^* \to \mathbf{R}x_1 \oplus \mathbf{R}x_2$ defined by $\iota(\alpha_1) = \frac{1}{2}(\sqrt{3}x_1 + x_2)$, $\iota(\alpha_2) = \frac{1}{2}(-\sqrt{3}x_1 + x_2)$, then the simple reflections s_{α_1} and s_{α_2} correspond respectively to T_2 and T_3. This establishes (i).

To prove (ii) we just need to recall how one gets the Weyl group invariants on $\mathfrak{h}_{\mathbf{R}}$. Let e_1, e_2, e_3 be the canonical basis of $\mathbf{R}^3$ and let H be the orthogonal complement of $\mathbf{R}(e_1 + e_2 + e_3)$. Then the inclusion map $j\colon \mathfrak{h}_{\mathbf{R}}^* \to \mathbf{R}^3$ defined by $j(\alpha_1) = e_1 - e_2$, $j(\alpha_2) = e_2 - e_3$ identifies $h_{\mathbf{R}}^*$ with H. Also the action of the Weyl group on $\mathfrak{h}_{\mathbf{R}}^*$ corresponds to the restriction to H of the action of the symmetric group S_3 on $\mathbf{R}^3$ defined by $\sigma(e_i) = e_{\sigma(i)}$, $\sigma \in S_3$; $i = 1, 2, 3$. If y_1, y_2, y_3 denote the coordinate functions on $\mathbf{R}^3$ then it is well known that the S_3-invariants on $\mathbf{R}^3$ are generated by the elementary symmetric polynomials $p_1 = y_1 + y_2 + y_3$, $p_2 = y_1^2 + y_2^2 + y_3^2$, $p_3 = y_1^3 + y_2^3 + y_3^3$ and 1. Moreover the restrictions of p_2 and p_3 to H together with 1 generates all S_3-invariants on H. Since $j(x_1) = (e_1 - 2e_2 + e_3)/\sqrt{3}$ and $j(x_2) = e_1 - e_3$ we get

$$(p_2 \circ j)(ux_1 + vx_2) = 2(u^2 + v^2), \quad (p_3 \circ j)(ux_1 + vx_2) = -2u(u^2 - 3v^2)/\sqrt{3}.$$

But W is contained in the orthogonal group of $\mathbf{R}x_1 \oplus \mathbf{R}x_2$ therefore $x_1^2 + x_2^2$, $x_1(x_1^2 - 3x_2^2)$ and 1 generate $\mathbf{C}[x_1, x_2]^W$.

Theorem 5. *If $G_o = \mathrm{SU}(2,1)$ then the algebra B is generated by the algebraically independent elements $\zeta \otimes 1, D \otimes 1, Y^2 \otimes 1 + 3 \otimes (Z+1)^2, Y^3 \otimes 1 - 9Y \otimes (Z+1)^2$ and 1. Moreover $B^{\widetilde{W}} = B$.*

Proof. From Proposition 2 and Lemma 3 we know that all elements b of B are precisely of the form $b = \sum_{i,j}(\zeta^i D^j \otimes 1) f_{i,j}(Y \otimes 1, 1 \otimes Z)$ where $a(f_{i,j}) \in \mathbf{C}[x_1, x_2]^W$. Now Proposition 4 tells us that $a(x_1^2 + 3(x_2+1)^2) = 3(x_1^2 + x_2^2)$, $a(x_1^3 - 9x_1(x_2+1)^2) = 3\sqrt{3}x_1(x_1^2 - 3x_2^2)$ and 1 generates $\mathbf{C}[x_1, x_2]^W$. The first assertion is proved.

It is well known that there is an element w in the center of K_o such that $Ad(w)|_{\mathfrak{a}} = -I$. Then (3) implies that $B^{\widetilde{W}} = \{b \in B : b(\lambda - \rho) = b(-\lambda - \rho) \text{ for all } \lambda \in \mathfrak{a}^*\}$. Using Lemma 29 of Tirao [11] we obtain: $\alpha_1(Z_{\alpha_1}) = \alpha_1(H_{\alpha_1}) - \alpha_1(Y_{\alpha_1}) = 2 - 3/2 = 1/2$, thus $\rho(Z) = 2\alpha_1(Z_{\alpha_1}) = 1$. If $b = \sum b_j \otimes Z^j \in U(\mathfrak{k}) \otimes U(\mathfrak{a})$ let $\tilde{b} = \sum b_j \otimes (Z-1)^j$. Then $b(\lambda - \rho) = b(-\lambda - \rho)$ if and only if $\tilde{b}(\lambda) = \tilde{b}(-\lambda)$ $(\lambda \in \mathfrak{a}^*)$. Now $B = B^{\widetilde{W}}$ is a direct consequence of the first assertion. The theorem is proved.

3. The structure of $U(\mathfrak{g})^K$

Proposition 6. *If $u \in Z(\mathfrak{g})$ then $P(u) \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$.*

Proof. Let $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n} = \sum_{\lambda>0} \mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}} = \sum_{\lambda>0} \mathfrak{g}_{-\lambda}$. We enumerate $\Delta(\mathfrak{g},\mathfrak{a})^+$ as $\{\lambda_1, \ldots, \lambda_p\}$. Let $X_{j,1}, \ldots, X_{j,m(j)}$ (resp. $Y_{j,1}, \ldots, Y_{j,m(j)}$) be a basis of $\mathfrak{g}_{\lambda_j}$ (resp. $\mathfrak{g}_{-\lambda_j}$). Then set $X_j^K = (X_{j,1})^{k_1} \cdots (X_{j,m(j)})^{k_{m(j)}}$ and $Y_j^I = (Y_{j,1})^{i_1} \ldots (Y_{j,m(j)})^{i_{m(j)}}$, where $K = (k_1, \ldots k_{m(j)})$ and $I = (i_1, \ldots i_{m(j)})$. Then the Poincaré-Birkhoff-Witt Theorem implies that $u \in U(\mathfrak{g})$ can be written in a unique way as

$$(7) \qquad u = \sum_{\tilde{I},\tilde{K}} (Y_1)^{I_1} \cdots (Y_p)^{I_p} u_{\tilde{I},\tilde{K}} (X_1)^{K_1} \cdots (X_p)^{K_p}, \quad u_{\tilde{I},\tilde{K}} \in U(\mathfrak{m} \oplus \mathfrak{a}),$$

where $\tilde{I} = (I_1, \ldots, I_p)$ and $\tilde{K} = (K_1, \ldots, K_p)$. If $u \in Z(\mathfrak{g})$ then $Hu - uH = 0$ for all $H \in \mathfrak{a}$, therefore the sum (51) is restricted to all pairs $\tilde{I}, \tilde{K}$ such that $\sum |I_j| \lambda_j = \sum |K_j| \lambda_j$, which clearly implies that $P(u) = u_{\tilde{0},\tilde{0}} \in U(\mathfrak{m} \oplus \mathfrak{a})$ or more precisely that $P(u) \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$. The proposition is proved.

Since $\mathfrak{m} = \mathfrak{m}^- \oplus \mathfrak{t} \oplus \mathfrak{m}^+$ we have

$$U(\mathfrak{m}) = U(\mathfrak{t}) \oplus \left(\mathfrak{m}^- U(\mathfrak{m}) \oplus U(\mathfrak{m})\mathfrak{m}^+\right).$$

Let q denote the projection of $U(\mathfrak{m})$ onto $U(\mathfrak{t})$ corresponding to this direct sum decomposition and set $Q = q \otimes id : U(\mathfrak{m}) \otimes U(\mathfrak{a}) \to U(\mathfrak{t}) \otimes U(\mathfrak{a})$. Since $\mathfrak{t} \oplus \mathfrak{a}$ is abelian, we shall use $U(\mathfrak{t}) \otimes U(\mathfrak{a})$ and $S(\mathfrak{t}) \otimes S(\mathfrak{a}) = S(\mathfrak{t} \oplus \mathfrak{a})$ interchangeably.

Recall the following notation: if $\alpha \in P^+$ is a simple root such that $Y_\alpha \neq 0$ ($H_\alpha = Y_\alpha + Z_\alpha$, $Y_\alpha \in \mathfrak{t}$, $Z_\alpha \in \mathfrak{a}$) set $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$ where $X_{-\alpha} \neq 0$ in $\mathfrak{g}_{-\alpha}$. Also we put

$$B_\alpha = \{b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a}) : E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1) E_\alpha^n, n \in \mathbf{N}\}.$$

Let $\tilde{\nu}, \sigma \in (\mathfrak{t} \oplus \mathfrak{a})^*$ be defined by $\tilde{\nu}|_\mathfrak{t} = \alpha|_\mathfrak{t}$, $\tilde{\nu}(Z_\alpha) = -\alpha(Y_\alpha)$ and $\sigma|_\mathfrak{t} = 0$, $\sigma(Z_\alpha) = 1$.

Lemma 7. *An element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to B_α if and only if*

$$(8) \qquad Q(b)(t\sigma + \tilde{\mu} + t\tilde{\nu} - \sigma) = Q(b)(-t\sigma + \tilde{\mu} - \sigma)$$

for all $\tilde{\mu} \in (\mathfrak{t} \oplus \mathfrak{a})^*$ such that $\tilde{\mu}(Z_\alpha) = -\tilde{\mu}(Y_\alpha)$ and all $t \in \mathbf{N}$.

Proof. We enumerate $\Delta(\mathfrak{m},\mathfrak{t})^+ = \{\beta_1, \ldots, \beta_q\}$ and choose a basis $X_1, \ldots, X_q$ of $\mathfrak{m}^+$ with $X_j \in \mathfrak{m}_{\beta_j}$. Also let $Y_1, \ldots, Y_q$ be a basis of $\mathfrak{m}^-$ with $Y_j \in \mathfrak{m}_{-\beta_j}$. Moreover let $H_1, \ldots, H_l$ be a basis of $\mathfrak{t}$. If $I, K \in \mathbf{N}_0^q$ then set $X^K = (X_1)^{k_1} \cdots (X_q)^{k_q}$, $Y^I = (Y_1)^{i_1} \cdots (Y_q)^{i_q}$. If $J \in \mathbf{N}_0^l$ then put $H^J = (H_1)^{j_1} \cdots (H_l)^{j_l}$. Then the Poincaré-Birkhoff-Witt Theorem implies that the elements $Y^I H^J X^K \otimes Z_\alpha^s$ form a basis of $U(\mathfrak{m}) \otimes U(\mathfrak{a})$.

Now if $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, $b = \sum a_{I,J,K,s} Y^I H^J X^K \otimes Z_\alpha^s$ then $a_{I,J,K,s} \neq 0$ and $I \neq 0$ imply $K \neq 0$. Therefore $b \in B_\alpha$ if and only if for all $t \in \mathbf{N}$

$$\sum a_{I,J,K,s} E_\alpha^t Y^I H^J X^K (t - Y_\alpha - 1)^s \equiv \sum a_{I,J,K,s} Y^I H^J X^K (-t - Y_\alpha - 1)^s E_\alpha^t$$

which is equivalent to

$$(9) \qquad \sum a_{0,J,0,s} E_\alpha^t H^J (t - Y_\alpha - 1)^s \equiv \sum a_{0,J,0,s} H^J (-t - Y_\alpha - 1)^s E_\alpha^t,$$

because $[\mathfrak{m}^+, E_\alpha] = 0$. Using Lemma 18 (vi) of Tirao [11] repeately (9) can be written as

(10)

$$E_\alpha^t \sum a_{0,J,0,s} H^J (t - Y_\alpha - 1)^s \equiv E_\alpha^t \sum a_{0,J,0,s} \times \big(H_1 - t\alpha(H_1)\big)^{j_1} \cdots \big(H_l - t\alpha(H_l)\big)^{j_l} \big(- t - Y_\alpha + t\alpha(Y_\alpha) - 1\big)^s.$$

By Lemma 20 of Tirao [11] E_α^t can be cancelled in both sides of (10) and then clearly the equivalence sign can be replaced by an equal sign. Thus

(11)

$$\sum a_{0,J,0,s} H^J (t - Y_\alpha - 1)^s = \sum a_{0,J,0,s} \big(H_1 - t\alpha(H_1)\big)^{j_1} \cdots \big(H_l - t\alpha(H_l)\big)^{j_l} \times \big(- t - Y_\alpha + t\alpha(Y_\alpha) - 1\big)^s.$$

If we evaluate both sides of (11) at $\mu \in \mathfrak{t}^*$ we get

$$\sum a_{0,J,0,s} H^J(\mu) \big(t - \mu(Y_\alpha) - 1\big)^s = \sum a_{0,J,0,s} H^J(\mu - t\alpha) \times \big(- t - \mu(Y_\alpha) + t\alpha(Y_\alpha) - 1\big)^s. \tag{12}$$

Let $\tilde{\mu} \in (\mathfrak{t} \oplus \mathfrak{a})^*$ be defined by $\tilde{\mu}|_{\mathfrak{t}} = \mu$ and $\tilde{\mu}(Z_\alpha) = -\mu(Y_\alpha)$. Then $t - \mu(Y_\alpha) - 1 = (t\sigma + \tilde{\mu} - \sigma)(Z_\alpha)$ and $-t - \mu(Y_\alpha) + t\alpha(Y_\alpha) - 1 = (-t\sigma + \tilde{\mu} - t\tilde{\nu} - \sigma)(Z_\alpha)$. Therefore (12) is equivalent to

$$\sum a_{0,J,0,s} (H^J \otimes Z_\alpha^s)(t\sigma + \tilde{\mu} - \sigma) = \sum a_{0,J,0,s} (H^J \otimes Z_\alpha^s)(-t\sigma + \tilde{\mu} - t\tilde{\nu} - \sigma).$$

If we change $\tilde{\mu}$ by $\tilde{\mu} + t\tilde{\nu}$ and since $Q(b) = \sum a_{0,J,0,s} H^J \otimes Z_\alpha^s$ we get that $b \in B_\alpha$ if and only if (8) holds for all $\tilde{\mu} \in (\mathfrak{t} \oplus \mathfrak{a})^*$ such that $\tilde{\mu}(Z_\alpha) = -\tilde{\mu}(Y_\alpha)$. This completes the proof of the lemma.

To make things more transparent we recall some basic facts about the structure of $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. Let $\mathbf{F}$ denote either the reals $\mathbf{R}$ or the complexes $\mathbf{C}$ and let $x \mapsto \bar{x}$ be the standard involution. For $x \in \mathbf{F}$ set $|x|^2 = x\bar{x}$.

Consider on $\mathbf{F}^{n+1}$ the quadratic form $q(x_1, \ldots, x_{n+1}) = |x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$. Then G_o is the connected component of the identity in the group of all $\mathbf{F}$-linear transformations g of $\mathbf{F}^{n+1}$ preserving q and such that $\det(g) = 1$. Then $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. If we set

$$Q = \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix},$$

where I denotes the $n \times n$ identity matrix, we have

$$G_o = \{A \in \mathrm{GL}(n+1, \mathbf{F}) : {}^t\bar{A} Q A = Q, \det(A) = 1\}_o.$$

Here the subindex "o" in the right hand side denotes the connected componet of the identity. We also have

$$\mathfrak{g}_o = \{X \in \mathfrak{gl}(n+1, \mathbf{F}) : {}^t\bar{X} Q + Q X = 0, \mathrm{Tr}(X) = 0\}.$$

The Lie algebra $\mathfrak{g}_\circ$ has a Cartan decomposition $\mathfrak{g}_\circ = \mathfrak{k}_\circ \oplus \mathfrak{p}_\circ$ where

$$\mathfrak{k}_\circ = \left\{ \begin{pmatrix} X & 0 \\ 0 & w \end{pmatrix} : {}^t\bar{X} + X = 0, w + \mathrm{Tr}(X) = 0 \right\}$$

and

$$\mathfrak{p}_\circ = \left\{ \begin{pmatrix} 0 & u \\ {}^t\bar{u} & 0 \end{pmatrix} : u \in \mathbf{F}^n \right\}.$$

In each case the Cartan involution θ is given by $\theta(X) = -{}^t\bar{X}$.

Let $E_{i,j} \in \mathfrak{gl}(n+1, \mathbf{F})$ denote the matrix with a one in the (i,j) entry and zero otherwise. Set $H_\circ = E_{1,n+1} + E_{n+1,1}$ and let $\mathfrak{a}_\circ = \{tH_\circ : t \in \mathbf{R}\}$ in both cases. As we know $\mathfrak{a}_\circ$ is a maximal abelian subspace of $\mathfrak{p}_\circ$. Let λ be the complex linear functional on $\mathfrak{a}$ defined by $\lambda(H_\circ) = 1$. Then, we have $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda\}$ if $\mathbf{F} = \mathbf{R}$ and $\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda, \pm 2\lambda\}$ if $\mathbf{F} = \mathbf{C}$. In both cases we choose $\Pi = \{\lambda\}$ as a set of simple roots. Now consider the following Cartan subalgebra of $\mathfrak{m}$:
if $\mathbf{F} = \mathbf{R}$

$$\mathfrak{t} = \{T = \sum_{j=1}^{p-1} i t_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) : t_j \in \mathbf{C}\}, \tag{13}$$

if $\mathbf{F} = \mathbf{C}$

$$\mathfrak{t} = \{T = t_1(E_{1,1} + E_{n+1,n+1}) + \sum_{j=2}^{n} t_j E_{j,j} : \mathrm{Tr}(T) = 0, t_j \in \mathbf{C}\}, \tag{14}$$

where $p - 1 = [(n-1)/2]$. Then as we know $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Now according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$ we define linear functionals λ_j on $\mathfrak{h}$ as follows,

$$\lambda_j(H) = \begin{cases} t, & j = 1 \\ t_j, & j = 2, \ldots, p \end{cases} \quad \text{and } \lambda_j(H) = \begin{cases} t_1 + t, & j = 1 \\ t_j, & j = 2, \ldots, n \\ t_1 - t, & j = n+1, \end{cases} \tag{15}$$

respectively. Here $H = T + tH_\circ$ where T is as in (13) and (14). Now a positive root system of $\mathfrak{m}$ with respect to $\mathfrak{t}$ can be discribed as follows:
if $\mathbf{F} = \mathbf{R}$

$$\Delta(\mathfrak{m}, \mathfrak{t})^+ = \begin{cases} \{\lambda_i \pm \lambda_j : 2 \le i < j \le p\} \cup \{\lambda_i : 2 \le i \le p\}, & n = 2p \\ \{\lambda_i \pm \lambda_j : 2 \le i < j \le p\}, & n = 2p - 1, \end{cases} \tag{16}$$

if $\mathbf{F} = \mathbf{C}$

$$\Delta(\mathfrak{m}, \mathfrak{t})^+ = \{\lambda_i - \lambda_j : 2 \le i < j \le n\}.$$

If $\Delta(\mathfrak{g}, \mathfrak{h})$ denotes the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, we define a positive root system $\Delta(\mathfrak{g}, \mathfrak{h})^+$ compatible with $\Delta(\mathfrak{g}, \mathfrak{a})^+$ and $\Delta(\mathfrak{m}, \mathfrak{t})^+$, as follows: we say that $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is positive if, whenever $\alpha|_\mathfrak{a} \neq 0$ then $\alpha|_\mathfrak{a} \in \Delta(\mathfrak{g}, \mathfrak{a})^+$ and if α is such that $\alpha|_\mathfrak{a} = 0$ then $\alpha|_\mathfrak{t} \in \Delta(\mathfrak{m}, \mathfrak{t})^+$. A straightforward computation shows that:

if $\mathbf{F} = \mathbf{R}$

$$\Delta(\mathfrak{g},\mathfrak{h})^+ = \begin{cases} \{\lambda_i \pm \lambda_j : 1 \le i < j \le p\} \cup \{\lambda_i : 1 \le i \le p\}, & n = 2p \\ \{\lambda_i \pm \lambda_j : 1 \le i < j \le p\}, & n = 2p-1, \end{cases}$$

if $\mathbf{F} = \mathbf{C}$

$$\Delta(\mathfrak{g},\mathfrak{h})^+ = \{\lambda_i - \lambda_j : 1 \le i < j \le n+1\}.$$

The corresponding sets of simple roots are:
if $\mathbf{F} = \mathbf{R}$

$$\Pi(\mathfrak{g},\mathfrak{h}) = \begin{cases} \{\alpha_i\}, & \alpha_i = \lambda_i - \lambda_{i+1}(1 \le i \le p-1), \alpha_p = \lambda_p, n = 2p \\ \{\alpha_i\}, & \alpha_i = \lambda_i - \lambda_{i+1}(1 \le i \le p-1), \alpha_p = \lambda_{p-1} + \lambda_p, n = 2p-1, \end{cases}$$

$$\Pi(\mathfrak{m},\mathfrak{t}) = \begin{cases} \{\alpha_1,\dots,\alpha_p\}, & n = 2p, p \ge 2 \\ \{\alpha_1,\dots,\alpha_p\}, & n = 2p-1, p \ge 3 \\ \emptyset, & n = 3; \end{cases}$$

if $\mathbf{F} = \mathbf{C}$

$$\Pi(\mathfrak{g},\mathfrak{h}) = \{\alpha_1,\dots,\alpha_n\}, \quad \alpha_i = \lambda_i - \lambda_{i+1} (i = 1,\dots,n),$$

$$\Pi(\mathfrak{m},\mathfrak{t}) = \begin{cases} \{\alpha_2,\dots,\alpha_{n-1}\}, & n \ge 3 \\ \emptyset, & n = 2. \end{cases}$$

In what follows we shall consider Q as a linear map from $U(\mathfrak{m}) \otimes U(\mathfrak{a})$ onto $S(\mathfrak{h})$. Also if $w \in W(\mathfrak{g},\mathfrak{h})$ we set

$$S(\mathfrak{h})^{\tilde{w}} = \{p \in S(\mathfrak{h}) : p(w(\mu) - \rho) = p(\mu - \rho), \text{ for all } \mu \in \mathfrak{h}^*\}.$$

Proposition 8. *Let $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. If $\alpha \in P^+$ is a simple root then an element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to B_α if and only if $Q(b) \in S(\mathfrak{h})^{\tilde{s}_\alpha}$.*

Proof. We shall consider three cases according to: (i) $G_o = \mathrm{SO}(2p-1,1)$, $p \ge 2$, (ii) $G_o = \mathrm{SO}(2p,1)$, $p \ge 2$ and (iii) $G_o = \mathrm{SU}(n,1)$ $n \ge 2$.

(i) If $p \ge 3$ then $\alpha_1 = \lambda_1 - \lambda_2$ is the unique simple root in P^+. When $p = 2$, $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_2 = \lambda_1 + \lambda_2$ are both in P^+. We shall only consider the case $\alpha = \alpha_1$, leaving the other to the reader. A simple computation gives: $H_{\alpha_1} = H_o - i(E_{2,3} - E_{3,2})$; hence $Y_{\alpha_1} = -i(E_{2,3} - E_{3,2})$ and $Z_{\alpha_1} = H_o$. Now $\tilde{\mu} \in \mathfrak{h}^*$ satisfies $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ if and only if $\tilde{\mu} = x(\lambda_1 + \lambda_2) + x_3\lambda_3 + \cdots + x_p\lambda_p$, $x, x_3, \dots, x_p \in \mathbf{C}$. We have $\tilde{\nu} = -\lambda_1 - \lambda_2$ and $\sigma = \lambda_1$ (see the definitions given right before Lemma 7). Also $\rho = (p-1)\lambda_1 + (p-2)\lambda_2 + \cdots + \lambda_{p-1}$.

We shall identify $p \in S(\mathfrak{h})$ with the polynomial function on $\mathbf{C}^p$ defined by $p(x_1,\dots,x_p) = p(x_1\lambda_1 + \cdots + x_p\lambda_p)$. Then (see (8)) the following equation

$$p(t\sigma + \tilde{\mu} + t\tilde{\nu} - \sigma) = p(-t\sigma + \tilde{\mu} - \sigma) \tag{17}$$

for all $\tilde{\mu} \in \mathfrak{h}^*$ such that $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ and all $t \in \mathbf{N}$, can be rewritten as

$$p(x-1, x-t, x_3, \dots, x_p) = p(x-t-1, x, x_3, \dots, x_p) \tag{18}$$

for all $x, x_3, \ldots, x_p \in \mathbf{C}$ and all $t \in \mathbf{N}$. For $p \in S(\mathfrak{h})$ let $\tilde{p} \in S(\mathfrak{h})$ be defined by $\tilde{p}(\mu) = p(\mu - \rho)$, $\mu \in \mathfrak{h}^*$. Then it can be easily seen that (18) is equivalent to

$$(19) \qquad \tilde{p}(x, x+t, x_3, \ldots, x_p) = \tilde{p}(x+t, x, x_3, \ldots, x_p)$$

for all $x, x_3, \ldots, x_p \in \mathbf{C}$ and all $t \in \mathbf{Z}$. Let $s\colon \mathbf{C}^p \to \mathbf{C}^p$ be the symmetry given by $s(x_1, x_2, x_3, \ldots, x_p) = (x_2, x_1, x_3, \ldots, x_p)$. If $\tilde{p}$ satisfies (19) then the zero set of $\tilde{p} \circ s - \tilde{p}$ contains an infinite number of parallel hyperplanes. Hence p satisfies (17) if and only if $\tilde{p} \circ s = \tilde{p}$. But $s_{\alpha_1}(\lambda_1) = \lambda_2$ and $s_{\alpha_1}(\lambda_j) = \lambda_j$ for $3 \leq j \leq p$. Therefore s corresponds precisely to s_{α_1} under the identification of $\mathfrak{h}^*$ with $\mathbf{C}^p$ defined above. Thus if $p = Q(b)$, $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, then $b \in B_{\alpha_1}$ if and only if (Lemma 7) $\tilde{p} \in S(\mathfrak{h})^{s_{\alpha_1}}$ as we wanted to prove.

(ii) The cases $G_o = \mathrm{SO}(2p, 1)$ $p \geq 2$, are completely similar to those considered in (i) and are left to the reader.

(iii) Now we take $G_o = \mathrm{SU}(n, 1)$ $n \geq 2$. In this case there are two simple roots $\alpha_1 = \lambda_1 - \lambda_2$ and $\alpha_n = \lambda_n - \lambda_{n+1}$ in P^+: $H_{\alpha_1} = \frac{1}{2}(E_{1,1} + E_{n+1,n+1}) - E_{2,2} + \frac{1}{2}H_o$ and $H_{\alpha_n} = -\frac{1}{2}(E_{1,1} + E_{n+1,n+1}) + E_{n,n} + \frac{1}{2}H_o$; hence $Y_{\alpha_1} = \frac{1}{2}(E_{1,1} + E_{n+1,n+1}) - E_{2,2}$, $Y_{\alpha_n} = -\frac{1}{2}(E_{1,1} + E_{n+1,n+1}) + E_{n,n}$, $Z_{\alpha_1} = Z_{\alpha_2} = \frac{1}{2}H_o$, $\rho = \frac{1}{2}\sum_{j=1}^{n+1}(n - 2j + 2)\lambda_j$.

Any $\mu \in \mathfrak{h}^*$ can be written in a unique way as $\mu = x_1\lambda_1 + \cdots + x_{n+1}\lambda_{n+1}$ with $x_j \in \mathbf{C}$ and $\sum x_j = 0$. We shall identify μ with $(x_1, \ldots, x_{n+1}) \in \mathbf{C}^{n+1}$ and $\mathfrak{h}^*$ with the corresponding subspace of $\mathbf{C}^{n+1}$.

Let us consider the case $\alpha = \alpha_1$. Then $\tilde{\mu} \in \mathfrak{h}^*$ satisfies $\tilde{\mu}(Z_{\alpha_1}) = -\tilde{\mu}(Y_{\alpha_1})$ if and only if $\tilde{\mu} = x(\lambda_1 + \lambda_2) + x_3\lambda_3 + \cdots + x_{n+1}\lambda_{n+1}$. We have $\tilde{\nu} = -\lambda_1 - \lambda_2 + 2\lambda_{n+1}$ and $\sigma = \lambda_1 - \lambda_{n+1}$. We shall identify the restriction to $\mathfrak{h}^*$ of an element $p \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ with the corresponding $p \in S(\mathfrak{h})$ by setting $p(x_1, \ldots, x_{n+1}) = p(x_1\lambda_1 + \cdots + x_{n+1}\lambda_{n+1})$, $x_j \in \mathbf{C}$ and $\sum x_j = 0$. Then the equation (17) can be written as

$$(20) \quad p(x-1, x-t, x_3, \ldots, x_n, x_{n+1}+t+1) = p(x-t-1, x, x_3, \ldots, x_n, x_{n+1}+t+1)$$

for all $x, x_3, \ldots, x_{n+1} \in \mathbf{C}$ such that $2x + \sum_{j=3}^{n+1} x_j = 0$ and all $t \in \mathbf{N}$. For $p \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ let $\tilde{p} \in \mathbf{C}[x_1, \ldots, x_{n+1}]$ be defined by $\tilde{p}(x_1, \ldots, x_{n+1}) = p(x_1 - n/2, x_2 - (n-2)/2, \ldots, x_{n+1} - (-n)/2)$; in this way $\tilde{p}(\mu) = p(\mu - \rho)$ for all $\mu \in \mathfrak{h}^*$. Then it can be easily seen that (20) is equivalent to

$$(21) \qquad \tilde{p}(x, x+t, x_3, \ldots, x_{n+1}) = \tilde{p}(x+t, x, x_3, \ldots, x_{n+1})$$

for all $x, x_3, \ldots, x_{n+1} \in \mathbf{C}$, $t \in \mathbf{Z}$ such that $2x + t + x_3 + \cdots + x_{n+1} = 0$. As before this implies that

$$\tilde{p}(x_1, x_2, x_3, \ldots, x_{n+1}) = \tilde{p}(x_2, x_1, x_3, \ldots, x_{n+1})$$

for all $x_1, \ldots, x_{n+1} \in \mathbf{C}$ such that $\sum x_j = 0$. But the symmetry $(x_1, x_2, \ldots, x_{n+1}) \mapsto (x_2, x_1, \ldots, x_{n+1})$ of $\mathbf{C}^{n+1}$ when restricted to $\mathfrak{h}^*$ coincide with s_{α_1}. Therefore if $p = Q(b)$, $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$, then $b \in B_{\alpha_1}$ if and only if (Lemma 7) $\tilde{p} \in S(\mathfrak{h})^{s_{\alpha_1}}$.

When $\alpha = \alpha_n$ the proof is exactly the same. The proof of the proposition is now complete.

The following choice of a representative in M_0' of the non-trivial element in $W = W(\mathfrak{g}, \mathfrak{a})$ will be convenient. Let

$$w = \begin{cases} \mathrm{Diag}(-1, 1, \ldots, 1, -1, 1), & \text{for } G_0 = \mathrm{SO}(2p-1, 1), p \geq 2 \\ \mathrm{Diag}(-1, \ldots, -1, 1), & \text{for } G_0 = \mathrm{SO}(2p, 1), p \geq 2 \\ \mathrm{Diag}(\xi, \ldots, \xi, -\xi), & \text{for } G_0 = \mathrm{SU}(n, 1), n \geq 2, \xi^{n+1} = -1. \end{cases}$$

Then $w \in M_0'$ and $Ad(w)H_0 = -H_0$. Moreover in the first case we have

$$\begin{aligned} Ad(w) \sum_{j=1}^{p-1} it_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) = & \sum_{j=1}^{p-2} it_{j+1}(E_{2j,2j+1} - E_{2j+1,2j}) \\ & - it_p(E_{2p-2,2p-1} - E_{2p-1,2p-2}). \end{aligned}$$

Therefore (see (13)) the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$ is $Ad(w)$- stable, $w(\lambda_j) = \lambda_j$ $(j = 2, \ldots, p-1)$ and $w(\lambda_p) = -\lambda_p$ (see (15)). Hence $\Delta(\mathfrak{m}, \mathfrak{t})^+$ is also stable under the action of w (see (16)). In the other two cases it is clear that $Ad(w)$ restricts to the identity on $\mathfrak{t}$. Thus in all cases $Ad(w)|_{\mathfrak{h}}$ defines an element in $W(\mathfrak{g}, \mathfrak{h})$, which we shall also denote by w.

Proposition 9. *Let $G_0 = \mathrm{SO}(n, 1)_e$ or $\mathrm{SU}(n, 1)$. An element $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ belongs to $\left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}}$ if and only if $Q(b) \in S(\mathfrak{h})^{\tilde{w}}$.*

Proof. When $G_0 = \mathrm{SO}(n, 1)_e$ or $\mathrm{SU}(n, 1)$ we have

$$\left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}} = \{b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a}) : Ad(w)(b(\lambda - \rho)) = b(-\lambda - \rho), \lambda \in \mathfrak{a}^*\}.$$

(See (3), also Kostant, Tirao [15, Corollary 3.3].) If $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ let $b^w \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ be defined by $b^w(\lambda - \rho) = Ad(w^{-1})(b(-\lambda - \rho))$ for all $\lambda \in \mathfrak{a}^*$. Then $b \in \left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}}$ if and only if $b^w = b$. The projection $q \colon U(\mathfrak{m}) \to U(\mathfrak{t})$ commutes with $Ad(w)$ because $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are $Ad(w)$-stable. Therefore if $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$

$$Q(b^w)(\nu, \lambda - \rho) = Q(b)(w(\nu), \lambda - \rho) \tag{22}$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*$. If we replace in (22) ν by $\nu - \rho_{\mathfrak{m}}$ and take into account that $w(\rho_{\mathfrak{m}}) = \rho_{\mathfrak{m}}$ we see that

$$Q(b^w)(\nu - \rho_{\mathfrak{m}}, \lambda - \rho) = Q(b)(w(\nu) - \rho_{\mathfrak{m}}, \lambda - \rho) \tag{23}$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*$. Now from the explicit description of $\Delta(\mathfrak{g}, \mathfrak{h})^+$ and of $\Delta(\mathfrak{m}, \mathfrak{t})^+$ it follows that $\rho|_{\mathfrak{t}} = \rho_{\mathfrak{m}}$. Then (23) is equivalent to

$$Q(b^w)(\mu - \rho) = Q(b)(w(\mu) - \rho) \tag{24}$$

for all $\mu \in \mathfrak{h}^*$. Therefore $Q(b) \in S(\mathfrak{h})^{\tilde{w}}$ if and only if $Q(b) = Q(b^w)$. Since $Q \colon U(\mathfrak{m})^M \otimes U(\mathfrak{a}) \to S(\mathfrak{h})$ is one-to-one (cf. Wallach [22, Theorem 3.2.3]) we finally have: $b \in \left(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\right)^{\widetilde{W}} \iff b = b^w \iff Q(b) = Q(b^w) \iff Q(b) \in S(\mathfrak{h})^{\tilde{w}}$, for all $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$.

Proposition 10. *If* $G_\circ = \mathrm{SO}(n,1)_e$ *or* $\mathrm{SU}(n,1)$. *Then* $\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B^{\widetilde{W}} = \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$ *and* $Q\big(\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B\big) = S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$.

Proof. If $c \in U(\mathfrak{m})^M$ it is well known (cf. Wallach [22, Theorem 3.2.3]) that $q(c)(\nu - \rho_{\mathfrak{m}}) = q(c)(\omega(\nu) - \rho_{\mathfrak{m}})$ for all $\nu \in \mathfrak{t}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. By extending each $\omega \in W(\mathfrak{m},\mathfrak{t})$ to $\mathfrak{h}$ by making it trivial on $\mathfrak{a}$ we can consider $W(\mathfrak{m},\mathfrak{t})$ as a subgroup of $W(\mathfrak{g},\mathfrak{h})$. Then for all $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ we have

$$Q(b)(\nu - \rho_{\mathfrak{m}}, \lambda - \rho) = Q(b)(\omega(\nu) - \rho_{\mathfrak{m}}, \lambda - \rho) = Q(b)(\omega(\nu) - \rho_{\mathfrak{m}}, \omega(\lambda) - \rho)$$

for all $\nu \in \mathfrak{t}^*, \lambda \in \mathfrak{a}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. Equivalently

$$Q(b)(\mu - \rho) = Q(b)(\omega(\mu) - \rho)$$

for all $\mu \in \mathfrak{h}^*, \omega \in W(\mathfrak{m},\mathfrak{t})$. Hence $Q\big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \subset S(\mathfrak{h})^{\widetilde{W(\mathfrak{m},\mathfrak{t})}}$.

From the explicit description of the corresponding sets of simple roots given before we see that:

$$W(\mathfrak{g},\mathfrak{h}) = \begin{cases} \langle s_1, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p \\ \langle s_1, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p-1 \\ \langle s_1, \dots, s_n \rangle, & \text{for } \mathbf{F} = \mathbf{C}; \end{cases}$$

$$W(\mathfrak{m},\mathfrak{t}) = \begin{cases} \langle s_2, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p, p \geq 2 \\ \langle s_2, \dots, s_p \rangle, & \text{for } \mathbf{F} = \mathbf{R}, n = 2p-1, p \geq 3 \\ \langle e \rangle & \text{for } \mathbf{F} = \mathbf{R}, n = 3 \\ \langle s_2, \dots, s_{n-1} \rangle, & \text{for } \mathbf{F} = \mathbf{C}, n \geq 3 \\ \langle e \rangle & \text{for } \mathbf{F} = \mathbf{C}, n = 2, \end{cases}$$

where $s_i = s_{\alpha_i}$ in all cases.

If $G_\circ = \mathrm{SO}(2p,1)_e, p \geq 2$ or $G_\circ = \mathrm{SO}(2p-1,1)_e, p \geq 3$, then α_1 is the unique simple root in P^+. If $G_\circ = \mathrm{SU}(n,1), n \geq 2$, then α_1 and α_n are the unique simple roots in P^+. While if $G_\circ = \mathrm{SO}(3,1)_e$ then α_1 and α_2 are in P^+. In any case we see that $W(\mathfrak{g},\mathfrak{h})$ is generated by $W(\mathfrak{m},\mathfrak{t})$ and $\{s_\alpha : \alpha \in P^+ \text{ is a simple root}\}$. Thus from Proposition 8 and from what was observed above it follows that $Q(b) \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ for all $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$.

Conversely if $p \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ there exists a unique $b \in U(\mathfrak{m})^M \otimes U(\mathfrak{a})$ such that $Q(b) = p$ (see Wallach [22, Theorem 3.2.3]). Now Propositions 8 and 9 imply that $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B^{\widetilde{W}}$. This completes the proof of our proposition.

Theorem 11. *If* $G_\circ = \mathrm{SO}(n,1)_e$ *or* $\mathrm{SU}(n,1)$ *then* $P(Z(\mathfrak{g})) = \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$.

Proof. From Theorem 37 of Tirao [11] and Proposition 6 it follows that $P(Z(\mathfrak{g})) \subset \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$. If $b \in \big(U(\mathfrak{m})^M \otimes U(\mathfrak{a})\big) \cap B$ then $Q(b) \in S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ (Proposition 10). Now $Q \circ P \colon Z(\mathfrak{g}) \colon \to S(\mathfrak{h})^{\widetilde{W(\mathfrak{g},\mathfrak{h})}}$ is the Harish-Chandra isomorphism (see Wallach [22, Theorem 3.2.3]). Hence there exists $u \in Z(\mathfrak{g})$ such that

$Q(P(u)) = Q(b)$. Since $Q\colon U(\mathfrak{m})^M \otimes U(\mathfrak{a}) \to S(\mathfrak{h})$ is injective we get $P(u) = b$, proving what we wanted.

To prove that when $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ we have $U(\mathfrak{g})^K \simeq Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$ it will be convenient to begin discussing the following concept.

Let $\Delta(\mathfrak{k}, \mathfrak{j})^+$ be a choice of a positive root system of $\mathfrak{k}$ and let Λ be the corresponding set of all dominant integral linear functions on $\mathfrak{j}$. Also let Ω be the set of all dominant integral linear functions on $\mathfrak{t}$, with respect to $\Delta(\mathfrak{m}, \mathfrak{t})^+$. A subset $X \subset \mathfrak{j}^*$ ($X \subset \mathfrak{t}^*$) is a *separating set* of $S(\mathfrak{j})_l$ ($S(\mathfrak{t})_l$) if for any $f \in S(\mathfrak{j})_l$ ($f \in S(\mathfrak{t})_l$) $f|_X = 0$ implies $f = 0$. ($S(\mathfrak{h})_l$ denotes the subspace of $S(\mathfrak{h})$ of all elements of degree $\leq l$.) For $\lambda \in \Lambda$ ($\omega \in \Omega$) let V_λ (W_ω) be a finite dimensional irreducible $\mathfrak{k}$-module ($\mathfrak{m}$-module) with highest weight λ (ω). If $\omega \in \Omega$ set

$$\Lambda(\omega) = \{\lambda \in \Lambda : \mathrm{Hom}_{\mathfrak{m}}(W_\omega, V_\lambda) \neq 0\}.$$

When $G_o = \mathrm{SO}(n,1)_e$ ($\mathrm{SU}(n,1)$) the algebra $\mathfrak{k} \simeq \mathfrak{so}(n, \mathbf{C})$ ($\mathfrak{gl}(n, \mathbf{C})$) and $\mathfrak{m} \simeq \mathfrak{so}(n-1, \mathbf{C})$ ($\mathfrak{gl}(n-1, \mathbf{C})$) corresponds to the subalgebra of all matrices in $\mathfrak{so}(n, \mathbf{C})$ ($\mathfrak{gl}(n, \mathbf{C})$) with all zeros in the first row and in the first column. Let Λ' (Ω') be the set of all $\lambda \in \Lambda$ ($\omega \in \Omega$) such that there exists a representation of $\mathrm{SO}(n, \mathbf{C})$ or $\mathrm{GL}(n, \mathbf{C})$ ($\mathrm{SO}(n-1, \mathbf{C})$ or $\mathrm{GL}(n-1, \mathbf{C})$) of highest weight λ (ω), according to $G_o = \mathrm{SO}(n,1)_e$ or $G_o = \mathrm{SU}(n,1)$.

For the proof of the following proposition we need to recall how a representation V_λ of $\mathrm{SO}(n, \mathbf{C})$ or $\mathrm{GL}(n, \mathbf{C})$ decomposes as a representation of $\mathrm{SO}(n-1, \mathbf{C})$ or $\mathrm{GL}(n-1, \mathbf{C})$, respectively. We need to distinguish three cases: $\mathrm{SO}(2\nu+1, \mathbf{C})$, $\mathrm{SO}(2\nu, \mathbf{C})$ or $\mathrm{GL}(\nu, \mathbf{C})$. In any of these cases a basis $\lambda_1, \ldots, \lambda_\nu$ of $\mathfrak{j}$ can be chosen in such a way that any $\lambda \in \Lambda'$ can be writen as $\lambda = m_1\lambda_1 + \cdots + m_\nu\lambda_\nu$ where

$$\begin{cases} m_1 \geq \cdots \geq m_\nu \geq 0, \ m_i \text{ all integers or half-integers,} & \text{for } \mathrm{SO}(2\nu+1, \mathbf{C}) \\ m_1 \geq \cdots \geq m_{\nu-1} \geq |m_\nu|, \ m_i \text{ all integers or half-integers,} & \text{for } \mathrm{SO}(2\nu, \mathbf{C}) \\ m_1 \geq \cdots \geq m_\nu \geq 0, \ m_i \text{ all integers,} & \text{for } \mathrm{GL}(\nu, \mathbf{C}). \end{cases}$$

Now the following branching formulas hold (see Foulton, Harris [4,§25.3]).

The restriction from $\mathrm{SO}(2\nu+1, \mathbf{C})$ to $\mathrm{SO}(2\nu, \mathbf{C})$ is determined by the following spectral formula

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_\nu)} \tag{25}$$

the sum over all $(p_1, \ldots, p_\nu)$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq m_\nu \geq |p_\nu|$, with the p_i and m_i simultaneously all integers or all half-integers.

When we restrict from $\mathrm{SO}(2\nu, \mathbf{C})$ to $\mathrm{SO}(2\nu-1, \mathbf{C})$ we have

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_{\nu-1})}$$

the sum over all $(p_1, \ldots, p_{\nu-1})$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq |m_\nu|$, with the p_i and m_i simultaneously all integers or all half-integers.

For $\mathrm{GL}(\nu-1, \mathbf{C}) \subset \mathrm{GL}(\nu, \mathbf{C})$ the restriction of V_λ $\lambda = (m_1, \ldots, m_\nu)$ from $\mathrm{GL}(\nu, \mathbf{C})$ to $\mathrm{GL}(\nu-1, \mathbf{C})$ is given by

$$V_{(m_1,\ldots,m_\nu)} = \sum W_{(p_1,\ldots,p_{\nu-1})}$$

the sum over all $(p_1, \ldots, p_{\nu-1})$ with $m_1 \geq p_1 \geq m_2 \geq p_2 \geq \cdots \geq p_{\nu-1} \geq m_\nu \geq 0$, with the p_i and m_i all integers.

Proposition 12. *Let $G_\circ = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$. The set Y_l of all $\omega \in \Omega$ such that $\Lambda(\omega)$ is a separating set of $S(\mathfrak{j})_l$ is a separating set of $S(\mathfrak{t})_n$ for all $n \in \mathbf{N}$.*

Proof. If $\omega \in \Omega'$ let $\Lambda'(\omega) = \{\lambda \in \Lambda' : \mathrm{Hom}_{\mathfrak{m}}(W_\omega, V_\lambda) \neq 0\}$ and $Y_l' = \{\omega \in \Omega' : \Lambda'(\omega)$ is a separating set of $S(\mathfrak{j})_l\}$. Then clearly $\Lambda'(\omega) \subset \Lambda(\omega)$ and $Y_l' \subset Y_l$ for all $\omega \in \Omega', l \in \mathbf{N}$. Thus it will be enough to prove that Y_l' is a separating set of $S(\mathfrak{t})$.

If $G_\circ = \mathrm{SO}(2\nu+1,1)_e$ and $\omega = (p_1,\dots,p_\nu), p_1 \geq p_2 \geq \dots \geq p_{\nu-1} \geq |p_\nu|$, p_i simultaneously all integers or all half-integers, then from (25) it follows that $\Lambda'(p_1,\dots,p_\nu) = \{\lambda = (m_1,\dots,m_\nu) : m_1 \geq p_1 \geq m_2 \geq p_2 \geq \dots \geq p_{\nu-1} \geq m_\nu \geq |p_\nu|, p_i$ and m_i all integers or all half-integers$\}$. Now we claim that $\Lambda'(p_1,\dots,p_\nu)$ is a separating set of $S(\mathfrak{j})_l$ if and only if $l(p_1,\dots,p_\nu) = \min\{p_1-p_2, p_2-p_3,\dots,p_{\nu-1}-|p_\nu|\} \geq l$. In fact, if $x_1,\dots,x_\nu$ is the dual basis of $\lambda_1,\dots,\lambda_\nu$ then any element of $S(\mathfrak{j})$ can be viewed as a polynomial in $x_1,\dots,x_\nu$. Thus if $l(p_1,\dots,p_\nu) \geq l$, $f = f(x_1,\dots,x_\nu) \in S(\mathfrak{j})_l$ and $f(m_1,\dots,m_\nu) = 0$ for all $(m_1,\dots,m_\nu) \in \Lambda'(p_1,\dots,p_\nu)$ then clearly $f = 0$, i.e. $\Lambda'(p_1,\dots,p_\nu)$ is a separating set of $S(\mathfrak{j})_l$. Conversely, if $p_{i-1} - |p_i| < l$ for some $i = 2,\dots,\nu$ then $f(x_1,\dots,x_\nu) = \prod(x_i - m_i)$ (the product over all m_i such that $p_{i-1} \geq m_i \geq |p_i|$ p_i and m_i both integers or both half-integers) is a nonzero element in $S(\mathfrak{j})_l$ which vanishes on $\Lambda'(p_1,\dots,p_\nu)$. Therefore $Y_l' = \{\omega = (p_1,\dots,p_\nu) \in \Omega' : l(p_1,\dots,p_\nu) \geq l\}$ which obviously implies that Y_l' is a separating set of $S(\mathfrak{t})$.

In a completely similar way the proposition is proved when $G_\circ = \mathrm{SO}(2\nu,1)$ or $G_\circ = \mathrm{SU}(\nu,1)$.

Corollary 13. *Let $a_1,\dots,a_m$ be a linearly independent subset of $Z(\mathfrak{k})$ and let $p_1,\dots,p_m \in U(\mathfrak{t})$. Then $\sum_i a_i p_i \equiv 0 \mod (U(\mathfrak{k})\mathfrak{m}^+)$ implies $p_i = 0$, $i = 1,\dots,m$.*

Proof. Let $l = \max\{\deg(a_i), \deg(p_i) : i = 1,\dots,m\}$. Given $\omega \in Y_l$ and $\lambda \in \Lambda(\omega)$ let $0 \neq v \in V_\lambda$ be a highest weight vector of $\mathfrak{m}$ of weight ω. Let $\gamma\colon U(\mathfrak{k}) \to U(\mathfrak{j})$ be the Harish-Chandra projection defined by the direct sum decomposition $U(\mathfrak{k}) = U(\mathfrak{j}) \oplus (\mathfrak{k}^- U(\mathfrak{k}) \oplus U(\mathfrak{k})\mathfrak{k}^+)$. Then an element $a \in Z(\mathfrak{k})$ acts on V_λ by multiplication by $\gamma(a)(\lambda)$. Therefore

$$\left(\sum_{i=1}^m \gamma(a_i)(\lambda) p_i(\omega)\right) v = \sum_{i=1}^m a_i p_i \cdot v = 0,$$

hence $\sum_i \gamma(a_i)(\lambda) p_i(\omega) = 0$ for all $\lambda \in \Lambda(\omega), \omega \in Y_l$. Now we claim that the linear span L of $\{(\gamma(a_1)(\lambda),\dots,\gamma_m(\lambda)) : \lambda \in \Lambda(\omega)\}$ is $\mathbf{C}^m$. In fact, let $\xi = (\xi_1,\dots,\xi_m)$ be an element in the annihilator of L. Thus $\xi_1\gamma(a_1)(\lambda) + \dots + \xi_m\gamma(a_m)(\lambda) = 0$ for all $\lambda \in \Lambda(\omega)$. Since $\Lambda(\omega)$ is a separating set of $S(\mathfrak{j})_l$ it follows that $\xi_1\gamma(a_1) + \dots + \xi_m\gamma(a_m) = 0$. But $\gamma\colon Z(\mathfrak{k}) \to U(\mathfrak{j})$ is injective, therefore $\xi_1 a_1 + \dots + \xi_m a_m = 0$ which implies that $\xi = 0$, because by assumption $a_1,\dots,a_m$ are linearly independent. From this we get that $p_i(\omega) = 0$ for all $\omega \in Y_l, i = 1,\dots,m$. Since Y_l is a separating set of $S(\mathfrak{t})_l$ we finally get that $p_i = 0$, $i = 1,\dots,m$, as we wanted to prove.

Proposition 14. *Let $G_\circ = \mathrm{S0}(n,1)_e$ or $\mathrm{SU}(n,1)$. Take a linearly independent subset $a_1,\dots,a_m$ of $Z(\mathfrak{k})$ and elements $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i = 1,\dots,m$.*
(i) If $\sum_i a_i c_i \in B$ then $c_i \in B$ for $i = 1,\dots,m$.
(ii) If $\sum_i a_i c_i \in B^{\widetilde{W}}$ then $c_i \in B^{\widetilde{W}}$ for $i = 1,\dots,m$.

Proof. We enumerate $\Delta(\mathfrak{m},\mathfrak{t})^+ = \{\beta_1,\ldots,\beta_q\}$ and choose bases $Y_1,\ldots,Y_q$ of $\mathfrak{m}^-$, $X_1,\ldots,X_q$ of $\mathfrak{m}^+$ with $Y_j \in \mathfrak{m}_{-\beta_j}$, $X_j \in \mathfrak{m}_{\beta_j}$. Also let $H_1,\ldots,H_l$ be a basis of $\mathfrak{t}$. Let $E = E_\alpha, Y = Y_\alpha, Z = Z_\alpha$, where $\alpha \in P^+$ is a simple root. If $I, K \in \mathbb{N}_0^q$ set $Y^I = (Y_1)^{i_1}\cdots(Y_q)^{i_q}$, $X^K = (X_1)^{k_1}\cdots(X_q)^{k_q}$. If $J \in \mathbb{N}_0^l$ put $H^J = (H_1)^{j_1}\cdots(H_l)^{j_l}$. Then the Poincaré-Birkhoff-Witt Theorem implies that the elements $Y^I H^J X^K \otimes Z^s$ form a basis of $U(\mathfrak{m})\otimes U(\mathfrak{a})$. Let $c_i = \sum_{i,s,I,J,K} c_{i,s,I,J,K} Y^I H^J X^K \otimes Z^s$.

The element $b = \sum_i a_i c_i \in B$ if and only if (see (2))

$$E^n b(n-Y-1) \equiv b(-n-Y-1)E^n \qquad \bmod (U(\mathfrak{k})\mathfrak{m}^+).$$

Now, using Lemma 18 (vi) of Tirao [11] and the hypothesis, we obtain

$$\begin{aligned} E^n b(n-Y-1) &= E^n \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J X^K (n-Y-1)^s \\ &= E^n \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J \\ &\qquad \times \big(n-Y-1+(k_1\beta_1+\cdots+k_q\beta_q)(Y)\big)^s X^K \\ &\equiv E^n \sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (n-Y-1)^s. \end{aligned} \tag{26}$$

Similarly, and taking into account that $[\mathfrak{m}^+, E] = 0$, we get

$$\begin{aligned} b(-n-Y-1)E^n &= \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J X^K (-n-Y-1)^s E^n \\ &= \sum_{i,s,I,J,K} a_i c_{i,s,I,J,K} Y^I H^J \\ &\qquad \times \big(-n-Y-1+(k_1\beta_1+\cdots+k_q\beta_q)(Y)\big)^s E^n X^K \\ &\equiv \sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (-n-Y-1)^s E^n \\ &= E^n \sum_{i,s,J} a_i c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s. \end{aligned} \tag{27}$$

Hence if $b \in B$, from (26) and (27) and using Lemma 20 of Tirao [11], we get

$$\sum_{i,s,J} a_i c_{i,s,0,J,0} H^J (n-Y-1)^s \equiv \sum_{i,s,J} a_i c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s.$$

If we set $p_i = \sum_{s,J} c_{i,s,0,J,0}\big[H^J(n-Y-1)^s - (H-n\alpha(H))^J(-n-Y-1+n\alpha(Y))^s\big] \in U(\mathfrak{t})$ and apply Corollary 13 to $\sum_i a_i p_i \equiv 0$ we get that $p_i = 0$ for $i = 1,\ldots,m$. Therefore

$$\sum_{s,J} c_{i,s,0,J,0} H^J (n-Y-1)^s = \sum_{s,J} c_{i,s,0,J,0} \big(H-n\alpha(H)\big)^J \big(-n-Y-1+n\alpha(Y)\big)^s \tag{28}$$

for $i=1,\ldots,m$. If we multiply (28) on the left by E^n and follow the steps leading to (26) and (27) backwards, we see that

$$E^n c_i(n-Y-1) \equiv c_i(-n-Y-1)E^n,$$

i.e. $c_i \in B$ for $i=1,\ldots,m$, proving (i).

To prove (ii) we just need to observe that for $w \in M_o' - M_o$, $\lambda \in \mathfrak{a}^*$ (see (3)) $Ad(w)(b(\lambda-\rho)) = b(-\lambda-\rho)$ is equivalent to $\sum_i a_i Ad(w)(c_i(\lambda-\rho)) = \sum_i a_i c_i(\lambda-\rho)$ which implies that $Ad(w)(c_i(\lambda-\rho))$ for all $i=1,\ldots,m$, because $Z(\mathfrak{k})Z(\mathfrak{m}) \simeq Z(\mathfrak{k}) \otimes Z(\mathfrak{m})$. This finishes the proof of our proposition.

Theorem 15. *If $G_o = \mathrm{SO}(n,1)_e$ or $\mathrm{SU}(n,1)$ then $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is a surjective isomorphism.*

Proof. Let us first consider the case $G_o = \mathrm{SO}(n,1)_e$. The proof will be by induction on $n \geq 2$. For $n=2$ an s-triple $\{H,X,Y\}$ can be chosen in $\mathfrak{g}$ with $H \in \mathfrak{k}$. Set $\zeta = H^2 - 2H + 4XY$. Then $Z(\mathfrak{g}) = \mathbf{C}[\zeta]$, $Z(\mathfrak{k}) = \mathbf{C}[H]$ and $\{X^iY^iH^j\}$ is a basis of $U(\mathfrak{g})^K$. From this it is clear that $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is a surjective isomorphism. For $n \geq 2$ let $K_n = \mathrm{SO}(n) \times \mathrm{SO}(1) \simeq \mathrm{SO}(n)$, $M_n = \mathrm{SO}(1) \times \mathrm{SO}(n-1) \times \mathrm{SO}(1) \simeq \mathrm{SO}(1) \times \mathrm{SO}(n-1)$ and let $\mathfrak{g}_n$, $\mathfrak{k}_n$, $\mathfrak{m}_n$ denote respectively the complexifications of the Lie algebras of $\mathrm{SO}(n,1)_e$, K_n and M_n. Also let η be the automorphism of $\mathfrak{gl}(n,\mathbf{C})$ which interchanges the first and the last row and the first and the last column of a matrix. Since η is given by conjugation by an orthogonal matrix it clearly restricts to an automorphism of $\mathfrak{k}_n$.

Now assume the theorem has been already proved for $G_o = \mathrm{SO}(n-1,1)_e$, $n \geq 3$. Then

$$(29) \qquad U(\mathfrak{k}_n)^{M_n} = \eta\big(U(\mathfrak{g}_{n-1})^{K_{n-1}}\big) = \eta\big(Z(\mathfrak{g}_{n-1})\big)\eta\big(Z(\mathfrak{k}_{n-1})\big) = Z(\mathfrak{k}_n)Z(\mathfrak{m}_n).$$

Let us return to our old notation for $G_o = \mathrm{SO}(n,1)_e$. Given $u \in U(\mathfrak{g})^K$ set $b = P(u) \in B^{\widetilde{W}} \subset U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Then we can write (see (29)) $b = \sum_{i=1}^m a_i c_i$ where $a_1,\ldots,a_m$ are linearly independent in $Z(\mathfrak{k})$ and $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i=1,\ldots,m$. From Proposition 14 we know that $c_i \in B^{\widetilde{W}}$. Now by Theorem 13 there exist $u_i \in Z(\mathfrak{g})$ such that $c_i = P(u_i)$. Then $\sum_i a_i u_i \in U(\mathfrak{g})^K$ and $P(\sum_i a_i u_i) = P(u)$, hence $u = \sum_i a_i u_i \in Z(\mathfrak{k})Z(\mathfrak{g})$. This proves that $\mu\colon Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \to U(\mathfrak{g})^K$ is surjective. As we pointed out in the introduction this establishes the theorem for $G_o = \mathrm{SO}(n,1)_e$.

The proof for $\mathrm{SU}(n,1)$ will be also by induction on $n \geq 2$. For $n=2$ we have $U(\mathfrak{k})^M = Z(\mathfrak{k})Z(\mathfrak{m})$ (Lemma 1). Given $u \in U(\mathfrak{g})^K$ set $b = P(u) \in B^{\widetilde{W}} \subset U(\mathfrak{k})^M \otimes U(\mathfrak{a})$. Then $b = \sum_{i=1}^m a_i c_i$ where $a_1,\ldots,a_m$ are linearly independent in $Z(\mathfrak{k})$ and $c_i \in Z(\mathfrak{m}) \otimes U(\mathfrak{a})$ for $i=1,\ldots,m$. As before from Proposition 14 and Theorem 11 it follows that $u \in Z(\mathfrak{k})Z(\mathfrak{g})$, proving the theorem for $\mathrm{SU}(2,1)$. For $n \geq 2$ let $K_n = \mathrm{S}\big(\mathrm{U}(n) \times \mathrm{U}(1)\big)$ and

$$M_n = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} : a \in U(1), A \in U(n-1), a^2 \det A = 1 \right\}.$$

Also set $\mathfrak{g}_n$, $\mathfrak{k}_n$, $\mathfrak{m}_n$ denote respectively the complexifications of the Lie algebras of $\mathrm{SU}(n,1)$, K_n and M_n. Now take $n \geq 3$ and assume the theorem has been proved for $G_o = \mathrm{SU}(n-1,1)$. Then $\mathfrak{k}_n \simeq \mathfrak{gl}(n,\mathbf{C}) = \mathfrak{z}(\mathfrak{gl}(n,\mathbf{C})) \oplus \mathfrak{sl}(n,\mathbf{C}) = \mathfrak{z}(\mathfrak{gl}(n,\mathbf{C})) \oplus \mathfrak{g}_{n-1}$. Let

$$\bar{M}_n = \left\{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} : a \in U(1), A \in U(n-1), a^2 \det A = 1 \right\}$$

$$\bar{K}_{n-1} = \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} : a \in U(1), A \in U(n-1), a \det A = 1 \right\}$$

and observe that

$$\begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \in \bar{M}_n \text{ if and only if } a^{1/n} \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in \bar{K}_{n-1}.$$

Thus $U(\mathfrak{g}_{n-1})^{\bar{M}_n} = \eta\big(U(\mathfrak{g}_{n-1})^{\bar{K}_{n-1}}\big)$. Therefore

$$\begin{aligned} U(\mathfrak{k}_n)^{M_n} &\simeq U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big) U(\mathfrak{g}_{n-1})^{\bar{M}_n} \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big) \eta\big(U(\mathfrak{g}_{n-1})^{\bar{K}_{n-1}}\big) \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big) \eta\big(Z(\mathfrak{g}_{n-1})\big) \eta\big(Z(\mathfrak{k}_{n-1})\big) \\ &= U\big(\mathfrak{z}(\mathfrak{gl}(n,\mathbf{C}))\big) Z(\mathfrak{g}_{n-1}) Z(\mathfrak{m}_n) \\ &\simeq Z(\mathfrak{k}_n) Z(\mathfrak{m}_n). \end{aligned}$$

From this the proof is completed in the same way as in the case of $G_o = \mathrm{SO}(n,1)_e$.

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ON THE SUFFICIENT CONDITIONS OF MONOGENEITY FOR FONCTIONS OF COMPLEX-TYPE VARIABLE

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ABSTRACT. The theories of functions of hyperbolic and dual complex variable were deeply investigated between 1935 and 1941 as parallel theories with the classical complex analysis (see e.g. [2-6], [13-20]).

In some recent papers [7-8], [10-11], these theories present interest by some applications in the interpretation of physical phenomenoms.

In this spirit of ideas, the purpose of this paper is firstly to prove by counter-examples that the sufficient conditions of monogeneity in [5, p.148] and in [14, Theorem V, p.258] are false and secondly, to consider new correct conditions of monogeneity which moreover have the advantage of an unitary presentation.

1. INTRODUCTION

It is well known that a two-component number system forming an algebraic ring can be written in the form $z=a+qb,\ a,b \in R$, where q satisfies the equation $q^2=\alpha q+\beta$ with fixed $\alpha,\beta \in R$. An important result states that all the systems $C_q=\{z=a+qb;a,b\in R\}$ are ring isomorphic with one of the following three types (see e.g. [9]):

(i) C_q with $q^2=-1$, called the system of usual complex number, if $\alpha^2/4+\beta<0$;

(ii) C_q with $q^2=0$, called the system of dual complex numbers, if $\alpha^2/4+\beta=0$;

(iii) C_q with $q^2=+1$, if $\alpha^2/4+\beta>0$. In this case, a number in C_q is called binary [9], or double [21], or perplex [7-8], or anormal complex [1], or hyperbolic complex [4-6], [13].

While the theory of functions of usual complex variable is well known and does not represents the aim of the present note, the teory of functions of hyperbolic complex and dual complex variable was deeply investigated between 1935 and 1941 in e.g. [2-6], [13-20] (see also the more recent monograph [12] for generalisations to functions of hypercomplex variables).

In some recent papers (see e.g. [7-8], [10-11]), these theories were been taken in attention by some applications in the interpretation of physical phenomenoms.

In this spirit of ideas we firstly prove by counter-examples that the sufficient conditions of monogeneity in [5, p.148] and that the Theorem V in [14, p.258] are false and secondly, we consider new correct conditions of monogeneity which present the advantage that all the three cases $q^2=-1$, $q^2=0$ and $q^2=+1$ can be more unitaryly treated.

Throughout in this paper we will consider $q^2=+1$, or $q^2=0$, or $q^2=-1$ and a number $z=a+qb$ will be called q - complex number.

2. CONDITIONS OF MONOGENEITY

Keeping the notations in Introduction we can consider the following

DEFINITION 2.1 ([13], [14]). *If* $z=a+bq\in C_q$ *then* $|z|=\sqrt{a^2+b^2}$ *represents the modulus of the q - complex number z, in all the three cases* $q^2=+1$, $q^2=0$ *and* $q^2=-1$. *Also,* $N_q(z)=a^2-q^2b^2$ *represents the q - norm of the q - complex number z.*

THEOREM 2.2 ([13], [14]). *If* $q^2=0$ or $q^2=+1$ *then the set of all divisors of 0 in* C_q *is given by* $Z_q=\{z=a+qb; N_q(z)=0\}$. *Also, if* $z\in C_q\setminus Z_q$ *then z is invertible.*

REMARK. If $q^2=-1$ then $Z_q=\{0\}$ and C_q is even a field.

Let $D\subset C_q$ be and $f:D\to C_q$. Then we can write: $f(z)=u(x,y)+qv(x,y)$, for all $z=x+qy\in D$, where u and v are real functions of two real variables.

DEFINITION 2.3 ([5], [14]). *f is called q-monogenic in* $z_0\in D$ *if there exists the limit*

$$\lim_{\substack{z\to z_0\\ z-z_0\notin Z_q}}[f(z)-f(z_0)]/(z-z_0)=f'(z_0)$$

Concerning this concept, the following results are known.

THEOREM 2.4 ([5, p. 147]). *Let* $q^2=+1$. *If f is q-monogenic in* $z_0=x_0+qy_0\in D$, *then u and v have partial derivatives of order one in* (x_0,y_0) *and the equalities*

(1) $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$, $[\partial u/\partial y](x_0,y_0)=[\partial v/\partial x](x_0,y_0)$

hold.

THEOREM 2.5 ([5, p. 148]). *Let* $q^2=+1$. *If u and v have continuous partial derivatives of order one in* (x_0,y_0) *which satisfy* (1), *then is q-monogenic in* $z_0=x_0+qy_0$.

THEOREM 2.6 ([14, Theorem V, p.258]). *Let* $q^2=0$. *The function f is q-monogenic in* $z_0=x_0+qy_0\in D$ *if and only if u and v are differentiable in* (x_0,y_0) *and satisfy*

(2) $[\partial u/\partial y](x_0,y_0)=0$, $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$.

Firstly, we will prove by counter - examples that the Theorems 2.5 and 2.6 are false.

Indeed, let us define $u(x,y)=x^2+y^2$, $v(x,y)=0$ and $f(z)=u(x,y)+qv(x,y)=u(x,y)$, for all $z=x+qy$.

Obviously u and v have continuous partial derivative of order one in (0,0), which implies that u is differentiable in (0,0). Also, we immediately get

$$[\partial u/\partial x](0,0)=[\partial v/\partial y](0,0)=0,\ [\partial u/\partial y](0,0)=[\partial v/\partial x](0,0)=0.$$

Let $q^2=+1$. We have

$$\lim_{\substack{z\to z_0\\ z\notin Z_q}} [f(z)-f(0)]/z = \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} u(x,y)/(x+qy) = \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} (x^2+y^2)(x-qy)/(x^2-y^2) =$$

$$\lim_{\substack{x,y\to 0\\ |x|\neq|y|}} x(x^2+y^2)/(x^2-y^2) - q \lim_{\substack{x,y\to 0\\ |x|\neq|y|}} y(x^2+y^2)/(x^2-y^2).$$

But if we choose, for example, $x_n = 1/\sqrt{n}$, $y_n = 1/\sqrt{n+1}$ we get $x_n \to 0$, $y_n \to 0$, $|x_n| \neq |y_n|$ and

$$x_n\left(x_n^2+y_n^2\right)/\left(x_n^2-y_n^2\right) = \left(1/\sqrt{n}\right)\cdot\left(1/n+1/(n+1)\right)/\left[1/n-1/(n+1)\right] =$$

$$n(n+1)\cdot(2n+1)/\left[n(n+1)\sqrt{n}\right] = (2n+1)/\sqrt{n} \to +\infty, \text{ for } n\to+\infty.$$

Analogously,

$$y_n(x_n{}^2+y_n{}^2)/(x_n{}^2-y_n{}^2) = (2n+1)\sqrt{n+1} \to +\infty, \text{ for } n\to+\infty.$$

As conclusion, f is not monogenic in $z=0$ although u and v satisfy the conditions in Theorem 2.5. This means that Theorem 2.5 is false.

Now, let $q^2=0$. We get

$$\lim_{\substack{z\to 0\\ z\notin Z_q}} \left[f(z)-f(0)\right]/z = \lim_{\substack{x,y\to 0\\ x\neq 0}} u(x,y)/(x+qy) =$$

$$\lim_{\substack{x,y\to 0\\ x\neq 0}} \left(x^2+y^2\right)\left(x-qy\right)/x^2 = \lim_{\substack{x,y\to 0\\ x\neq 0}} (x^2+y^2)/x - q\cdot \lim_{\substack{x,y\to 0\\ x\neq 0}} \left(x^2+y^2\right)/x^2$$

But choosing $x=y^3$, $y\neq 0$, we obtain

$(x^2+y^2)/x = y^3+y^2/y^3 = y^3+1/y \to +\infty$, for $y\to 0$

and

$y(x^2+y^2)/x^2 = y+y^3/y^6 = y+1/y^3 \to +\infty$, for $y\to 0$.

As conclusion, f is not monogenic in $z=0$, although u and v are differentiable in (0,0) and satisfy the relations (2) in Theorem 2.6. This means that the sufficient conditions in Theorem 2.6. are false.

Now, let $f(z) = u(x,y)+qv(x,y)$, $z = \mathbf{x}+qy$, $q^2=0$, where

$$u(x,y) = \begin{cases} x, x\neq 0, y\in R \\ |y|, x=0, y\in R \end{cases}, \quad v(x,y) = \begin{cases} y, x\neq 0, y\in R \\ 0, x=0, y\in R \end{cases}$$

We have $u(0,0) = v(0,0) = f(0) = 0$ and

$$\lim_{\substack{z\to 0\\ z\notin Z_q}} [f(z)-f(0)]/z = \lim_{\substack{x,y\to 0\\ x\neq 0}} [u(x,y)+qv(x,y)]/(x+qy) =$$

$$\lim_{\substack{x,y\to 0\\ x\neq 0}} (x+qy)/(x+qy) = 1 = f'(0)$$

i.e. f is monogenic in z=0.

On the other hand, $(\partial u/\partial y)(0,0)$ does not exists because

$$\lim_{\substack{y\to 0\\ y\neq 0}}[u(0,y)-u(0,0)]/y = \lim_{\substack{y\to 0\\ y\neq 0}}|y|/y$$

As conclusion the necessary conditions in Theorem 2.6. also are false.

In the sequel we will give correct versions for the above Theorems 2.5. and 2.6.

Firstly, we will introduce the following.

DEFINITION 2.7. *Let* $u: M \to R, M \subset R^2$ *be and* $(x_0, y_0) \in M$. *We say that u is q-differentiable in* (x_0,y_0) *if there exist* $A, B \in R$ *and* $\omega = \omega(x,y)$ *with*

$$\lim_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega(x,y) = \omega(x_0,y_0) = 0 \text{ where } z = x+qy,\ z_0 = x_0+qy_0 \text{ such that}$$

$$u(x,y)-u(x_0,y_0) = A(x-x_0)+B(y-y_0)+\omega(x,y)\cdot N_q(z-z_0)/|z-z_0|, \text{for all } (x,y)\in M$$

with $z-z_0 \notin Z_q$.

REMARKS. 1). Obviously we have

$$N_q(z-z_0)/|z-z_0| = \left[(x-x_0)^2 - q^2(y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

2). If $q^2 = -1$ then the Definition 2.7 becomes the usual definition of differentiability in (x_0,y_0). Concerning the q - differentiability we can prove the following.

LEMMA 2.8. (i) *Let* $q^2 = +1$. *If u is q - differentiable in* (x_0,y_0) *then there exists* $[\partial u/\partial x](x_0,y_0) = A$ *and* $[\partial u/\partial y](x_0,y_0) = B$.

(ii) *Let* $q^2 = 0$. *If u is q - differentiable in* (x_0,y_0) *then there exists* $[\partial u/\partial x](x_0,y_0) = A$. *If moreover there exist* $\delta > 0$ *such that* $F(x) = u(x,y)$ *is continuous as function of x* **in** $x_0, \forall |y-y_0| < \delta$, *then there exists* $[\partial u/\partial y](x_0,y_0) = B$.

PROOF. (i) Taking in Definition 2.7 $x = x_0$ and $y \neq y_0$ (which implies $z-z_0 \notin Z_q$), we obtain

$$u(x_0,y)-u(x_0,y_0) = B(y-y_0)+\omega(x_0,y)\cdot\left[-(y-y_0)^2\right]/|y-y_0|.$$

Dividing by $y-y_0 \neq 0$ and then passing to limit with $y \to y_0$ we get

$$\lim_{\substack{y\to y_0\\ y\neq y_0}} [u(x_0,y)-u(x_0,y_0)]/(y-y_0) = B - \lim_{\substack{y\to y_0\\ y\neq y_0}} \omega(x_0,y)(y-y_0)/|y-y_0| = B$$

since

$$\lim_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega(x,y) = \lim_{\substack{y\to y_0\\ y\neq y_0}} \omega(x_0,y)=0$$

Analogously, taking in Definition 2.7 $y=y_0$ and $x \neq x_0$ and reasoning as above, we get that there exists $[\partial u/\partial y](x_0,y_0)=B$.

(ii) Taking in Definition 2.7 $y=y_0$ and $x\neq x_0$ (which implies $z-z_0\notin Z_q$), we obtain

$u(x,y_0)-u(x_0,y_0) = A(x-x_0)+\omega(x,y_0)\cdot|x-x_0|, \forall x\neq x_0$.

Dividing by $x-x_0\neq 0$ and passing to limit with $x\to x_0$ we immediately get $[\partial u/\partial x](x_0,y_0)=A\cdot$

Now, let $|y-y_0|<\delta$, $y\neq y_0$ be fixed. Passing to limit with $x\to x_0$, $x\neq x_0$ in Definition 2.7, we obtain

$$u(x_0,y)-u(x_0,y_0)=B(y-y_0)+\lim_{\substack{x\to x_0\\ x\neq x_0}} \omega(x,y)(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}$$

for all $|y-y_0|<\delta$, $y\neq y_0$.

But by $\lim_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} \omega(x,y)=0$ follows that for $\varepsilon>0$, there exists $\delta_1>0$ such that $|\omega(x,y)|<\varepsilon$, for all $|x-x_0|<\delta_1$, $x\neq x_0$ and all $|y-y_0|<\delta_1$.

Denote $\delta_0=\min\{\delta,\delta_1\}$ and let $|y-y_0|<\delta_0$, $y\neq y_0$.

We get $\lim_{\substack{x\to x_0\\ x\neq x_0}}|\omega(x,y)|\leq\varepsilon$, for all $|y-y_0|<\delta_0$, $y\neq y_0$. Since

$(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}=|x-x_0|\cdot|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq|x-x_0|$, we obtain

$$\lim_{\substack{x\to x_0\\ x\neq x_0}}|\omega(x,y)|\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq\varepsilon\cdot\lim_{\substack{x\to x_0\\ x\neq x_0}}|x-x_0|=0 \text{ for all } |y-y_0|<\delta_0,\ y\neq y_0$$

As conclusion,

$u(x_0,y)-u(x_0,y_0)=B(y-y_0),\ \forall\ y\neq y_0,\ |y-y_0|<\delta_0\cdot$

Therefore, dividing with $y-y_0\neq 0$ and then passing to limit with $y\to y_0$ we obtain $[\partial u/\partial y](x_0,y_0)=B$, which proves the lemma.

A correct version of Theorem 2.5 is the

THEOREM 2.9 *Let* $q^2=+1$ *be and* $f:D\to C_q; D\subset C_q$, $f(z)=u(x,y)+qv(x,y)$,

$z = x + qy \in D$, $z_0 = x_0 + qy_0 \in D$.

If u and v are q - differentiable in (x_0, y_0) and satisfy the relations (1) *in Theorem 2.5, then f is q-monogenic in z_0.*

PROOF. By hypothesis and by Lemma 2.8, (i) we get

$$u(x,y) - u(x_0,y_0) = a(x-x_0) + b(y-y_0) + \omega_1(x,y)\cdot\left[(x-x_0)^2 - (y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2},$$

$$v(x,y) - v(x_0,y_0) = b(x-x_0) + a(y-y_0) + \omega_2(x,y)\cdot\left[(x-x_0)^2 - (y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2},$$

for all $x - x_0 + q(y-y_0) = z - z_0 \notin Z_q$, where $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ z-z_0\notin Z_q}} \omega_j(x,y) = 0$, j=1,2.

By simple calculus we obtain

$$f(z) - f(z_0) = (a+bq)(z-z_0) + [\omega_1(x,y) + q\omega_2(x,y)]\cdot\left[(x-x_0)^2 - (y-y_0)^2\right]/\sqrt{(x-x_0)^2+(y-y_0)^2}.$$

Dividind by $z - z_0 \notin Z_q$ and then multiplying by $1=[(x-x_0)-q(y-y_0)]/[(x-x_0)-q(y-y_0)]$ on the right hand, the above equality becomes

$$[f(z)-f(z_0)]/(z-z_0) = a+bq+[(x-x_0)-q(y-y_0)]\cdot[\omega_1(x,y)+q\omega_2(x,y)]/\sqrt{(x-x_0)^2+(x-x_0)^2}$$

$$= a+bq+(x-x_0)\cdot\omega_1(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} - (y-y_0)\cdot\omega_2(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} +$$

$$q\left[(x-x_0)\cdot\omega_2(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2} - (y-y_0)\cdot\omega_1(x,y)/\sqrt{(x-x_0)^2+(y-y_0)^2}\right]$$

By $|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2} \le 1$ and $|y-y_0|/\sqrt{(x-x_0)^2+(y-y_0)^2} \le 1$, passing to limit with $z \to z_0$, $z - z_0 \notin Z_q$ (which is equivalent with $x \to x_0, y \to y_0$, $|x-x_0| \ne |y-y_0|$), we immediately get that there exists $\lim\limits_{\substack{z\to z_0\\ z-z_0\notin Z_q}} [f(z)-f(z_0)]/(z-z_0) = a+qb$ which proves the theorem.

Now, a correct version of Theorem 2.6 is the

THEOREM 2.10. *Let $q^2=0$ and $f: D \to C_q$, $D \subset C_q$, $f(z) = u(x,y) + qv(x,y)$, $z = x+qy \in D$, $z_0 = x_0 + qy_0 \in D$, such that $F(x) = u(x,y)$ and $G(x) = v(x,y)$ are continuous as functions of x in x_0, for all y belonging to a neighbourhood of y_0, denoted by $V(y_0)$.*

If f is q-monogenic in z_0, then u and v satisfy the relations (2) *in Theorem 2.6.*

Conversely, if u and v are q-differentiable in (x_0, y_0) and satisfy the relations (2) *in Theorem 2.6, then f is q-monogenic in z_0.*

PROOF. Let suppose that f is q-monogenic in z_0.

Let us denote $h(z) = [f(z) - f(z_0)]/(z - z_0) - f'(z_0) =$

$$=[f(z)-f(z_0)]/(z-z_0)-(a+bq)=h_1(x,y)+qh_2(x,y),\ \ z-z_0\notin Z_q(i.e.\ x\neq x_0).$$

By hypothesis we get $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} h_i(x,y)=0,\ i=\overline{1,2}$ and

$$h_1(x,y)+qh_2(x,y)=[u(x,y)-u(x_0,y_0)+q(v(x,y)-v(x_0,y_0))]/[(x-x_0)+q(y-y_0)]-(a+qb), x\neq x_0.$$

By simple calculus, for all $x\neq x_0$ and all y with $z,z_0,z-z_0\in D$, we obtain

(3) $u(x,y)-u(x_0,y_0)=a(x-x_0)+h_1(x,y)(x-x_0)$,

(4) $v(x,y)-v(x_0,y_0)=b(x-x_0)+a(y-y_0)+h_2(x,y)(x-x_0)+h_1(x,y)(y-y_0)$

Taking $y=y_0$ in (3), dividing with $x-x_0\neq 0$ and then passing to limit with $x\to x_0, x\neq x_0$, it follows that $[\partial u/\partial x](x_0,y_0)=a$, since $\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}} h_1(x,y)=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y_0)=0$.

Then, passing with $x\to x_0$ in (3) and taking into account that $F(x)=u(x,y)$ is continuous in x_0, we get

(5) $u(x_0,y)-u(x_0,y_0)=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot 0, \forall y\in V(y_0).$

But reasoning exactly as in the proof of Lemma 2.8, (ii), (for $\omega(x,y)\equiv h_1(x,y)$), there exists a neighbourhood $V_1(y_0)$ such that $|\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)|=\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} |h_1(x,y)|\leq\varepsilon,\ for\ all\ y\in V_1(y_0)$

Combining with (5) we obtain

$$u(x_0,y)-u(x_0,y_0)=0, \forall y\in V(y_0)\bigcap V_1(y_0).$$

This obviously implies $(\partial u/\partial y)(x_0,y_0)=0$.

Analogously, taking $y=y_0$ in (4) as above we have $[\partial v/\partial x](x_0,y_0)=b$.

Then passing to limit with $x\to x_0$ in (4) and taking into account that $G(x)=v(x,y)$ is continuous in x_0, it follows

$$v(x_0,y)-v(x_0,y_0)=a(y-y_0)+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_2(x,y)\cdot 0+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot(y-y_0),\ \text{for all}$$

$y\in V(y_0)$.

Reasoning as above, there exists $V_i(y_0)$ such that

$$v(x_0,y)-v(x_0,y_0)=a(y-y_0)+\lim\limits_{\substack{x\to x_0\\ x\neq x_0}} h_1(x,y)\cdot(y-y_0), \forall\ y\in V(y_0)\bigcap V_1(y_0).$$

Dividing by $y-y_0\neq 0$ and then passing to limit with $y\to y_0$ we get

$$[\partial v/\partial y](x_0,y_0)=a+\lim\limits_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0, y\neq y_0}} h_1(x,y)=a+0=a$$

As conclusion, $[\partial u/\partial y](x_0,y_0)=0$ and $[\partial u/\partial x](x_0,y_0)=[\partial v/\partial y](x_0,y_0)$.

Now let suposse that u and v are q-differentiable in (x_0,y_0) and satisfy the relations (2) in Theorem2.6.

By Lemma 2.8, (ii) and by hypothesis we get

$u(x,y)-u(x_0,y_0)=a(x-x_0)+\omega_1(x,y)\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}$,

$v(x,y)-v(x_0,y_0)=A(x-x_0)+a(y-y_0)+\omega_2(x,y)\cdot(x-x_0)^2/\sqrt{(x-x_0)^2+(y-y_0)^2}$, for all $x\neq x_0$, y, such that $z,z_0,z-z_0\in D$, where

$\lim_{\substack{x\to x_0\\ y\to y_0\\ x\neq x_0}}\omega_i(x,y)=0$, $i=1,2$ and $a=[\partial u/\partial x](x_0,y_0), A=[\partial v,\partial x](x_0,y_0)$

By simple calculus we get

$f(z)-f(z_0)=(a+qA)\cdot(z-z_0)+(x-x_0)^2\cdot[\omega_1(x,y)+q\omega_2(x,y)]/\sqrt{(x-x_0)^2+(y-y_0)^2}$

Dividing by $z-z_0\notin Z_q$ and then multiplying with $1=[(x-x_0)-q(y-y_0)]/[(x-x_0)-q(y-y_0)]$ on the right hand, we arrive at

$[f(z)-f(z_0)]/(z-z_0)=a+qA+\omega_1(x,y)\cdot(x-x_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}+$
$q\cdot[\omega_2(x,y)\cdot(x-x_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}-\omega_1(x,y)\cdot(y-y_0)/\sqrt{(x-x_0)^2+(y-y_0)^2}]$

Passing to limit with $z\to z_0, z-z_0\notin Z_q$ (which is equivalent with $x\to x_0, y\to y_0, x\neq x_0$) by

$|x-x_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq 1, |y-y_0|/\sqrt{(x-x_0)^2+(y-y_0)^2}\leq 1$, and by the hypothesis on $\omega_i(x,y)$ we immediately get

$\lim_{\substack{z\to z_0\\ z-z_0\notin Z_q}}[f(z)-f(z_0)]/(z-z_0)=a+qA$, which proves the theorem.

REMARKS. 1). If $q^2=-1$ it is known that the q-differentiability of u and v in (x_0,y_0) together with the Cauchy-Riemann conditions in (x_0,y_0) is even equivalent with the monogeneity of f in $z_0=x_0+qy_0$.

2). In the cases when $q^2=+1$ or $q^2=0$, there exist functions $f=u+qv$ with u and v q-differentiable in (x_0,y_0) and satisfying (1) or (2), respectively.

Indeed, for $q^2=+1$ let us define

$u(x,y)=\begin{cases}0, |x|=|y|\\ |x^2-y^2|, |x|\neq|y|\end{cases}$, $v(x,y)\equiv 0, f(z)=u(x,y), z=x+qy$. We have

$$[\partial u/\partial x](0,0)=\lim_{\substack{x\to 0\\ x\neq 0}}[u(x,0)-u(0,0)]/x=\lim_{\substack{x\to 0\\ x\neq 0}}x^2/x=0,$$

$$[\partial u/\partial y](0,0)=\lim_{\substack{y\to 0\\ y\neq 0}}[u(0,y)-u(0,0)]/y=\lim_{\substack{y\to 0\\ y\neq 0}}y^2/y=0,$$

$[\partial v/\partial x](0,0)=[\partial v/\partial y](0,0)=0.$

Also, $u(x,y)-u(0,0)=0\cdot x+0\cdot y+\omega(x,y)\cdot|x^2-y^2|/\sqrt{x^2+y^2}$ for all $|x|\neq|y|$, where $\omega(x,y)=\sqrt{x^2+y^2}$ satisfies $\lim\limits_{\substack{x\to 0\\ y\to 0\\ |x|\neq|y|}}\omega(x,y)=0$, i.e. u is q-differentiable in (0,0).

Analougously, for $q^2=0$ we define

$u(x,y)=\begin{cases}x^2, x\neq 0, y\in R\\ 0, x=0, y\in R\end{cases}$, $v(x,y)\equiv 0, f(z)=u(x,y), z=x+qy$. It is easy to check that

$[\partial u/\partial x](0,0)=[\partial u/\partial y](0,0)=0$ and u is q-differentiable in (0,0) with $\omega(x,y)=\sqrt{x^2+y^2}$.

3). Let $q^2=+1$. The sufficient conditions of q-monogenity in Theorem 2.9 however are not necessary. Indeed, let us define $f(z)=u(x,y)+q(x,y), z=x+qy, z_0=0,$

$u(x,y)=\begin{cases}x(x^2+y^2), |x|\neq|y|\\ 0, |x|=|y|\end{cases}$, $v(x,y)=\begin{cases}y(x^2+y^2), |x|\neq|y|\\ 0, |x|=|y|\end{cases}$

We have

$$f'(0)=\lim_{\substack{z\to 0\\ |x|\neq|y|}}[f(z)-f(0)]/z=\lim_{\substack{x,y\to 0\\ |x|\neq|y|}}[u(x,y)+qv(x,y)]/(x+qy)=$$

$$\lim_{\substack{x,y\to 0\\ |x|\neq|y|}}(x^2+y^2)\cdot(x+qy)/(x+qy)=\lim_{\substack{x,y\to 0\\ |x|\neq|y|}}(x^2+y^2)=0,$$

wich means that f is monogenic in $z_0=0$.

On the other hand u is not q-differentiable in (0,0). Indeed, let suppose that u is q-differentiable. We easily get $[\partial u/\partial x](0,0)=[\partial u/\partial y](0,0)=0$ and therefore by Lemma 2.8, (i) we get

$u(x,y)=\omega(x,y)\cdot[x^2-y^2]/\sqrt{x^2+y^2}$, for all $|x|\neq|y|$, with $\lim\limits_{\substack{x,y\to 0\\ |x|\neq|y|}}\omega(x,y)=0.$

It follows $x(x^2+y^2)=\omega(x,y)\cdot[x^2-y^2]/\sqrt{x^2+y^2}$, which implies

$\omega(x,y)=x(x^2+y^2)^{3/2}/[x^2-y^2]$, for all $|x|\neq|y|$.

Now, choosing for example $x_n=1/\sqrt{n}, y_n=1/\sqrt{n+1}\to 0, |x_n|\neq|y_n|$, by simple calculus we obtain

$\omega(x_n,y_n)=(2n+1)^{3/2}/[n\sqrt{n+1}]\xrightarrow{n\to+\infty}2$, contradiction.

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Characterization of the Moment Space of a Sequence of Exponentials

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Abstract

We consider the moment problem for the sequence $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ in $L^2(0,T)$ $(0 < T \leq \infty)$, being $\{\lambda_i\}_{i\in N}$ a sequence of positive real numbers such that $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$. We prove properties of the moment space M of that sequence. In [**K**] it is shown that M is a moment space. Our main result is that M is a Hilbert space and moreover, that is the image of ℓ^2 by the operator $G^{1/2}$, the square root of the Gram matrix G of the sequence. The operator $G^{1/2}$ is proved to be the limit in $B(\ell^2)$ of a sequence of simple operators of finite rank. We also obtain an upper bound for the norm of the operator G. We find different expressions for the solution of minimum norm of the stated moment problem, extending some results of [**Z**].

1 Introduction

We consider the moment problem of the sequence:

$$\left\{e^{-\lambda_i t}\right\}_{i\in N} \tag{1}$$

in $L^2(0,T)$ $(0 < T \leq \infty)$, being $\{\lambda_i\}_{i\in N}$ a sequence of positive real numbers such that:

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$$

Remark: This condition implies that the sequence **(1)** is not dense in $L^2(0,T)$.

Our main goal is to characterize the moment space M of that sequence. In the first section we introduce the moment problem and recall some well known results about it. In the second section we prove the following properties of M:

*) M is a dense and proper subspace in ℓ^2.

*) M does not depend on T.

*) M is a Hilbert space, and there exists a continous inmersion in ℓ^2.

In the third section we obtain the operator G. It is defined by the Gram matrix of the sequence (1) as the limit in $B(\ell^2)$ of a sequence of simple operators of finite rank. This allows us to show that $G^{1/2}$ is a compact operator.

In section four we prove that M is the image of ℓ^2 by the operator $G^{1/2}$. In the last two sections we find different expressions for the solution of minimal norm of the moment problem of our interest.

2 The moment problem.

Let H be a real Hilbert space, provided by an inner product $(\cdot,\cdot)$. Let $\{f_k\}_{k\in N}$ a sequence of elements of H such that any finite subfamily of this sequence is linearly independent. We note by $\{c_k\}_{k\in N}$ an arbitrary real sequence. So, the inner product (f, f_k) , $k \in N$ is called *the nth. moment of* f, and the sequence $\{(f, f_k)\}_{k\in N}$ is *the moment sequence of* f. Then in the theory of moments the following problem arises:

Does there exist an element $f \in H$ *such that* : $(f, f_k) = c_k, k = 1, 2, ...$?

The *moment space* M of $\{f_k\}$ is then the collection of all the moment sequences $M = \{(f, f_k) : f \in H\}$. Thus a numerical sequence $\{c_k\}_{k\in N}$ belongs to M if and only if there exist $f \in H$ such that $c_k = (f, f_k)$, $k = 1, 2, ...$.

M is a Banach space with the norm defined by:

$$\|c\|_M^2 = \sup_{n\in N} \sum_{k,l=1}^{n} \sigma_{l,k}^{(n)} c_k c_l = \lim_{n\to\infty} \sum_{k,l=1}^{n} \sigma_{l,k}^{(n)} c_k c_l$$

where $\sigma_{l,k}^{(n)}$ is the (l,k) element of the inverse of the Gram matrix G_n of $\{f_1, f_2, ..., f_n\}$. The last equality is valid because:

$$\sum_{k,l=1}^{n} \sigma_{k,l}^{(n)} c_k c_l$$

does not decrease as n increases **[K]**. It is easily proved that M is also a Hilbert space (cf. Lema 2).

Remark: To avoid confussion we use a subscript denoting the space we are refering to; for example $(\cdot,\cdot)_H$ or $\|\cdot\|_H$.

3 The moment space of a sequence of exponentials. Some properties.

Let $H = H(T) = L^2(0,T)$, $0 < T \leq \infty$ and let $f_k(t) = e^{-\lambda_k t}$, k = 1,2,..., being $\{\lambda_k\}_{k\in N}$ a sequence of positive real numbers such that $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ and $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$. In what follows, we will call $M(T)$ the moment space of (1) if $0 < T < \infty$, and M if $T = \infty$. We will study properties of M and $M(T)$.

If $T < \infty$,let

$$G_n(T) = \left(\frac{1 - e^{-(\lambda_i + \lambda_j)T}}{\lambda_i + \lambda_j} \right)_{1 \leq i,j \leq n}$$

be the Gram matrix of $\left\{ e^{-\lambda_k t} \right\}_{1 \leq k \leq n}$, $n \in N$, and

$$G(T) = \left(\frac{1 - e^{-(\lambda_i + \lambda_j)T}}{\lambda_i + \lambda_j} \right)_{i,j \in N}$$

be the Gram matrix of $\left\{ e^{-\lambda_k t} \right\}_{k \in N}$.

If $T = \infty$,then

$$G_n = \left[\frac{1}{\lambda_i + \lambda_j} \right]_{1 \leq i,j \leq n} \quad n \in N \qquad G(T) = \left[\frac{1}{\lambda_i + \lambda_j} \right]_{i,j \in N}$$

PROPOSITION:

a) $M(T) \subset \ell^2$, $M(T) \neq \ell^2, \forall T > 0$

b) $M(T) = M$, $\forall T > 0$

c) M is dense in ℓ^2 ,and the inmersion i: $M \rightarrow \ell^2$ is continuous.

Proof:

a) Let $\gamma_1^{(n)}(T)$ be the greatest eingenvalue of $G_n(T)$, and $\gamma_n^{(n)}(T)$ be the smallest one. Then

$$\gamma_1^{(n)}(T) = \max_{x \in R, x \neq 0} \frac{(x, G_n(T)x)}{\|x\|^2} \qquad , x = (x_i)_{1 \leq i \leq n}$$

and

$$(x, G_n(T)x) = \sum_{i,j=1}^{n} \frac{1 - e^{-(\lambda_i + \lambda_j)T}}{\lambda_i + \lambda_j} x_i x_j \leq \sum_{i,j=1}^{n} \frac{1}{\lambda_i + \lambda_j} |x_i| \, |x_j| =$$

$$\sum_{i,j=1}^{n} \frac{\left(\lambda_i \lambda_j\right)^{1/2}}{\lambda_i + \lambda_j} \frac{|x_i|}{(\lambda_i)^{1/2}} \frac{|x_j|}{(\lambda_j)^{1/2}} \leq \tfrac{1}{2} \left(\sum_{i=1}^{n} \frac{x_i}{(\lambda_i)^{1/2}} \right)^2 \leq Tr\, G_n \|x\|^2$$

where $Tr\ G_n$ is the trace of G_n. Then $\gamma_1^{(n)}(T) \leq Tr\ G_n$, $\forall n \in N$, (1) is a Bessel sequence **[Y]**, and $M(T) \subset \ell^2$.

Since

$$\gamma_n^{(n)}(T) \leq \frac{1 - e^{-2\lambda_n T}}{2\lambda_n}$$

then $\gamma_n^{(n)}(T) \to 0$ if $n \to \infty$, and (1) is not a Riesz-Fischer sequence. Then $M(T) \neq \ell^2$.

b) $(G_n - G_n(T))_{i,j} = \int\limits_T^\infty e^{-\lambda_i t} e^{-\lambda_j t} dt$ then $G_n - G_n(T)$ is the Gram matrix of $\left\{e^{-\lambda_i t}\right\}_{1 \leq i \leq n}$ in $L^2(T, \infty)$. So $G_n - G_n(T)$ is positive definite. It follows that $G_n \geq G_n(T)$.

In addition to this, the following result is valid

LEMMA 1: $G_n^{-1}(T) \geq G_n^{-1}$.
Proof:

Let L be a linear transformation such that **[CH]** $L^T G_n(T) L = Id$ and $L^T G_n L = D$ where $D = (d_{i,j})_{1 \leq i,j \leq n}$ is the diagonal matrix of order n such that

$$d_{i,j} = \begin{cases} \rho_i & i = j \\ 0 & i \neq j \end{cases}$$

Then $G_n - G_n(T) \geq 0$ implies that $\rho_i \geq 1$, $1 \leq i \leq n$. Also $L^{-1} G_n^{-1}(T)(L^T)^{-1} = Id$ and $L^{-1} G_n^{-1}(T)(L^T)^{-1} = \widetilde{D}$, where $\widetilde{D} = (\widetilde{d}_{i,j})_{1 \leq i,j \leq n}$ is the diagonal matrix of order n such that

$$\widetilde{d}_{i,j} = \begin{cases} 1/\rho_i & i = j \\ 0 & i \neq j \end{cases}$$

Then $Id - \widetilde{D} \geq 0$ and $G_n^{-1}(T) \geq G_n^{-1}$.

As a consequence of Lemma 1, $M(T) \subseteq M$. Also, there exists a constant $K = K(T)$ such that:

$$\frac{1}{K(T)} G_n \leq G_n(T).$$

In fact, let $c = (c_j)_{j \in N} \in \omega$, and $c(n) = (c_1, c_2, ..., c_n) \in R^n$

$$(c(n), G_n c(n)) = \int\limits_0^\infty \left(\sum_{i=1}^n c_i e^{-\lambda_i t}\right) \left(\sum_{j=1}^n c_j e^{-\lambda_j t}\right) dt = \|P(t)\|^2_{L^2(0,\infty)}$$

where $P(t) := \sum\limits_{i=1}^n c_i e^{-\lambda_i t}$. In an analogous way,

$$(c(n), G_n(T) c(n)) = \|P(t)\|^2_{L^2(0,T)} .$$

Acording to a result proved by Scwartz **[S]** there exists a constant $K = K(T)$ such that

$$\|P(t)\|_{L^2(0,\infty)} \leq K(T)\,\|P(t)\|_{L^2(0,T)}\,.$$

Hence $\frac{1}{K(T)}G_n \leq G_n(T)$ and $G_n^{-1}(T) \leq K(T)G_n^{-1}$. Therefore $M \subseteq M(T)$.

c) Let $x \in \ell^2$ be such that $(x,c)_{\ell^2} = 0$, $\forall c \in M$. Since $c \in M$ there exists $\Psi\,(t) \in L^2(0,T)$ such that:

$$\int_0^T \Psi\,(t)\,e^{-\lambda_j t}dt = c_j, \forall j \in N.$$

Then $\sum_{i=1}^{\infty} x_i c_i = \sum_{i=1}^{\infty} x_i \int_0^T \Psi\,(t)\,e^{-\lambda_i t}dt = 0$, $\forall \Psi\,(t) \in L^2(0,T)$. By the continuity of the inner product

$$\lim_{N\to\infty} \int_0^T \left(\sum_{i=1}^{N} x_i e^{-\lambda_i t}\right) \Psi\,(t)\,dt = 0.$$

Since $\sum_{i=1}^{\infty} x_i e^{-\lambda_i t} \in L^2(0,T)$, it follows that $\int_o^T \left(\sum_{i=1}^{\infty} x_i e^{-\lambda_i t}\right) \Psi\,(t)\,dt = 0.$

The sequence (1) is minimal in the sense that each element of the sequence lies outside the closed linear span of the others. Then there exists a biorthogonal sequence [Y] $\{g_i(t)\}_{i\in N}$ such that taking $\Psi(t) = g_i(t)$ will give $x_i = 0$, $\forall i \in N$. Then $x \equiv 0$.

To show that the inmersion $i : M \longrightarrow \ell^2$ is continuous, we shall show that:

$$\|c\|^2_{\ell^2} \leq Tr\,G\,\|c\|^2_M\,.$$

This is inmediate since

$$\left(c(n), G_n^{-1}c(n)\right) = \|c\,(n)\|^2\,\frac{(c(n), G_n^{-1}c(n))}{\|c\,(n)\|^2} \geq \|c\,(n)\|^2 \left(\gamma_1^{(n)}\right)^{-1} \geq$$

$$\|c\,(n)\|^2\,(Tr\,G_n)^{-1} \qquad \bullet$$

LEMMA 2: $(M; \|\cdot\|_M)$ is a Hilbert space.

4 An approximation to the Gram matrix.

The Gram matrix:

$$G = \left(\frac{1}{\lambda_i + \lambda_j}\right)_{1\leq i,j<\infty}$$

generates a bounded operator on ℓ^2 because $\|G\| \leq Tr\, G$. This result is a particular case of the following one:

LEMMA 3: If $G = (g_{i,j})_{1 \leq i,j < \infty}$ *is the Gram matrix of a system* $\{f_i\}_{i \in N}$ *such that* $\sum\limits_{i=1}^{\infty} g_{i,i} < \infty$ *then* $|g_{i,j}| = |(f_i, f_j)| \leq \|f_i\| \, \|f_j\| \leq (g_{i,i})^{1/2} (g_{j,j})^{1/2}$, $1 \leq i,j < \infty$ and $\left| \sum\limits_{i,j=1}^{\infty} g_{i,j} x_i x_j \right| \leq \left(\sum\limits_{i=1}^{\infty} g_{i,i} \right) \left(\sum\limits_{i=1}^{\infty} |x_i|^2 \right)$. Hence $\|G\| \leq \sum\limits_{i=1}^{\infty} g_{i,i} = Tr\, G$.

LEMMA 4: $\|G\| < TrG$.

Let G_n be the nth. section of G, $G_n = (g_{i,j})_{1 \leq i,j \leq n}$.

Then the infinite matrix $\widetilde{G_n} = (\tilde{g}_{i,j})_{1 \leq i,j < \infty} = \begin{cases} g_{i,j} \, , 1 \leq i,j \leq n \\ 0 \, , i > n \text{ or } j > n \end{cases}$ defines a bounded operator $\widetilde{G_n} : \ell^2 \to \ell^2, \forall n \in N$

LEMMA 5: $\widetilde{G_n} \to G$ *on* $B(\ell^2)$ *if* $n \to \infty$.

Proof:

Let $R_n := G - \widetilde{G_n}$ and let $x \in \ell^2$, $y = R_n x$. Then

$$y_i = \sum_{j=n+1}^{\infty} g_{i,j} x_j \quad i = 1, 2, ..., n \qquad y_{n+i} = \sum_{j=1}^{\infty} g_{n+i,j} x_j \quad i = 1, 2, ...$$

thus, if $1 \leq i \leq n$,

$$y_i^2 \leq \left(\sum_{j=n+1}^{\infty} g_{i,j}^2 \right) \cdot \left(\sum_{j=n+1}^{\infty} x_j^2 \right) \text{ and } y_{n+i}^2 \leq \left(\sum_{j=1}^{\infty} g_{n+i,j}^2 \right) \cdot \left(\sum_{j=1}^{\infty} x_j^2 \right).$$

Hence

$\sum\limits_{i=1}^{n} y_i^2 \leq \left(\sum\limits_{i=1}^{n} \sum\limits_{j=n+1}^{\infty} g_{i,j}^2 \right) \cdot \left(\sum\limits_{j=n+1}^{\infty} x_j^2 \right) \leq \frac{1}{2} \left(\frac{1}{2} \sum\limits_{i=1}^{\infty} \frac{1}{\lambda_i} \right) \left(\sum\limits_{j=n+1}^{\infty} \frac{1}{\lambda_j} \right) \left(\sum\limits_{j=1}^{\infty} x_j^2 \right) =$

$= \frac{1}{2} \tau \left(\sum\limits_{j=n+1}^{\infty} \frac{1}{\lambda_j} \right) \|x\|_{\ell^2}^2$ where $\tau := \frac{1}{2} \sum\limits_{i=1}^{\infty} \frac{1}{\lambda_i}$. In an analogous way results

$$\sum_{i=n+1}^{\infty} y_i^2 \leq \frac{1}{2} \tau \left(\sum_{j=n+1}^{\infty} \frac{1}{\lambda_j} \right) \|x\|_{\ell^2}^2 .$$

Hence $\left\| G - \widetilde{G_n} \right\|^2 \leq \tau \left(\sum\limits_{j=n+1}^{\infty} \frac{1}{\lambda_j} \right)$ and $\widetilde{G_n} \to G$ on $B(\ell^2)$ if $n \to \infty$. •

Remark: It can be proved in a similar way that Lemma 5 is valid if $G = (g_{i,i})_{1 \leq i,j < \infty}$ is a Gram matrix such that $\sum\limits_{i=1}^{\infty} g_{i,i} < \infty$

The operators $\widetilde{G_n}$ are of finite rank and positive (recall that a bounded linear operator T on a Hilbert space H is said to be *positive* if $(Tf, f) \geq 0$, $\forall f \in H$). Therefore G is a compact and positive operator. Since

$$\left(x, \widetilde{G_n}x\right) = \sum_{i,j=1}^{\infty} g_{i,j}x_ix_j \leq \left(\sum_{i=1}^{\infty} g_{i,i}\right) \cdot \left(\sum_{i=1}^{\infty} x_i^2\right) \leq \tau \|x\|_{\ell^2}^2$$

it follows that $0 \leq \widetilde{G_n} \leq \tau \ Id$, $\forall n \in N$, and $0 \leq G \leq \tau \ Id$. Hence for every natural number n there exists a unique operator T_n such that $T_n^2 = \widetilde{G_n}$ and a unique operator T such that $T^2 = G$. We will denote them by $\widetilde{G_n}^{1/2}$ and $G^{1/2}$ respectively. Now, because of the uniqueness, it follows that

$$\widetilde{G_n}^{1/2} = \begin{pmatrix} Q_n & \dots & 0 & \dots \\ \vdots & & \vdots & \\ 0 & \dots & 0 & \dots \\ \vdots & & \vdots & \end{pmatrix}$$

where Q_n is the only matrix such that $Q_n \geq 0$ and $Q_n^2 = G_n$.

LEMMA 6: $\widetilde{G_n}^{1/2} \to G^{1/2}$ *on* $B(\ell^2)$ *if* $n \to \infty$.
Proof:

Let $\{P_k(\lambda)\}_{k \in N}$ be a sequence of polinomials with real coefficients that converges uniformly to the function $\rho(\lambda) = \lambda^{1/2}$, $\lambda \in [0, \tau]$. Let T be a selfadjoint operator such that $0 \leq T \leq \tau.Id$. Then

$$\|P_m(T) - P_n(T)\| \leq \max_{\lambda \in [0,\tau]} \|P_m(\lambda) - P_n(\lambda)\|.$$

Therefore $\{P_k(T)\}_{k \in N}$ is a Cauchy sequence in $B(\ell^2)$. Accordingly, there exists an operator $\widetilde{T} \in B(\ell^2)$ satisfying:

i) $P_m(T) \to \widetilde{T}$, if $m \to \infty$
ii) $\widetilde{T}^2 = T$
iii) $\widetilde{T} \geq 0$
iv) $\widetilde{T}$ is the only operator with the properties *i)-iii)*.

We note $T^{1/2} = \widetilde{T}$.We choose an arbitrary positive small ϵ and find an index k such that

$$\sup_{\lambda \in [0,\tau]} \left|P_k(\lambda) - \lambda^{1/2}\right| < \frac{\epsilon}{3}.$$

For that k we have: $\left\|P_k(G) - G^{1/2}\right\| < \frac{\epsilon}{3}$ and $\left\|P_k(\widetilde{G_n}) - \widetilde{G_n}^{1/2}.\right\| < \frac{\epsilon}{3}$. Let $n_0 = n_0(\epsilon)$ be such that $\left\|P_k(\widetilde{G_n}) - P_k(G)\right\| < \frac{\epsilon}{3}$, $\forall n > n_0$. Hence

$$\left\|\widetilde{G_n}^{1/2} - G^{1/2}\right\| < \epsilon, \forall n > n_0. \qquad \bullet$$

5 A characterization of M.

THEOREM 1: $M = G^{1/2}(\ell^2)$.
Proof:

Let $c \in M$. Then $(c(n), G_n^{-1}c(n)) \le K$, $\forall n \in N$. We denote

$$x\,(n) = \left(G_n^{-1}\right)^{1/2} c(n).$$

Hence $\|x(n)\| \le K$, $\forall n \in N$, and $c(n) = G_n^{1/2}x(n)$. We define the elements

$$\widetilde{x}_{n,i} := \begin{cases} x_i\,(n) & if\ 1 \le i \le n \\ 0 & if\ i > n \end{cases}$$

and we denote $\widetilde{x}_n = (\widetilde{x}_{n,i})_{i \in N}$. As $\|\widetilde{x}_n\|_{\ell^2} = \|x(n)\|_{R^n} \le K$, $\forall n \in N$,we can suppose that $\{\widetilde{x}_n\}_{n \in N}$ is weak convergent in ℓ^2 (if it is not the case, it is sufficient to consider a subsequence with this property). Then

$$(\widetilde{x}_n, y) \to (x, y)\ \ if\ n \to \infty, \forall y \in \ell^2.$$

Since $G^{1/2}$ is a compact operator $G^{1/2}\widetilde{x}_n \to G^{1/2}x$ if $n \to \infty$ and $G^{1/2}\widetilde{x}_n \to c$ if $n \to \infty$, then $c = G^{1/2}x$.

To show that $G^{1/2}(\ell^2) \subseteq M$, let c be an element of $G^{1/2}(\ell^2)$. Then there exists $x \in \ell^2$ such that $G^{1/2}x = c$. We now introduce the elements

$$u^{(s)} := \widetilde{G}_s^{\,1/2} x$$

We assume for an instant that $u^{(s)} \in M$, $\forall n \in N$.. Then we have

$$\left\|\left(G_n^{-1}\right)^{1/2} c(n)\right\| \le \left\|\left(G_n^{-1}\right)^{1/2}\left(c(n) - u_{(n)}^{(s)}\right)\right\| + \sup_{n \in N}\left\|\left(G_n^{-1}\right)^{1/2} u_{(n)}^{(s)}\right\| \le$$

$\le \left\|(G_n^{-1})^{1/2}\left(c(n) - u_{(n)}^{(s)}\right)\right\| + K$,being K a constant. So

$$\left\|\left(G_n^{-1}\right)^{1/2} c(n)\right\| \le \left\|\left(G_n^{-1}\right)^{1/2}\left(c(n) - \lim_{s \to \infty} u_{(n)}^{(s)}\right)\right\| + K = K$$

because $u^{(s)} \to c$ in ℓ^2 if $s \to \infty$. Thus $c \in M$.

To show that $u^{(s)} \in M$, $\forall n \in N$ let's introduce the set

$$\widetilde{R}_s = \left\{\alpha = (\alpha_i)_{i \in N} \in \ell^2 : \alpha_i = 0\ \forall i > s\right\}$$

and consider $\{g_i\}_{i \in N}$ a biorthogonal sequence to the sequence (1). Next we define $g = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_s g_s$, $g \in L^2(0, \infty)$. Then

$$(g, e^{-\lambda_j t}) = \begin{cases} \alpha_i & i \le s \\ 0 & i > s \end{cases}$$

and hence $\widetilde{R}_s \subseteq M$, $\forall s \in N$. $\bullet$

Remark: Now the part *a)* of the proposition is obvious.

6 Solution of the moment problem.

If $\varphi_n(t)$ is the solution with minimun norm of the truncated moment problem

$$\left(\varphi_n(t), e^{-\lambda_j t}\right) = c_j \quad j = 1, 2, \ldots, n$$

then **[K]**

$$\varphi_n(t) = \sum_{i=1}^{n} \gamma_i e^{-\lambda_i t}$$

where $\gamma_i = \sum_{i=1}^{n} \sigma_{j,i}(n) c_j$ and $\sigma_{i,j}(n)$ is the (i,j)-element of G_n^{-1}. It can be proved that

$$\sigma_{i,j}(n) = \frac{4\lambda_i\lambda_j}{\lambda_i + \lambda_j} \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\lambda_k + \lambda_i}{\lambda_k - \lambda_i} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{\lambda_k + \lambda_j}{\lambda_k - \lambda_j} .$$

If we call $\alpha_i(n) = 2\lambda_i \prod_{\substack{k=1 \\ k \neq i}}^{n} \frac{\lambda_k + \lambda_i}{\lambda_k - \lambda_i}$ we can write $\sigma_{i,j}(n) = \frac{1}{\lambda_i + \lambda_j} \alpha_i(n) \alpha_j(n)$. The moment problem has a solution if and only if there exist a constant $K > 0$ such that **[K]** $\|\varphi_n(t)\| \leq K$, $\forall n \in N$. Let $D_n = (d_{i,j})_{1 \leq i,j \leq n}$ be a diagonal matrix of order n such that

$$d_{i,j} = \begin{cases} \alpha_i(n) & i = j \\ 0 & i \neq j \end{cases}$$

then $G_n^{-1} = D_n G_n D_n$ and

$$\varphi_n(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\alpha_i(n)}{\lambda_i + \lambda_j} c_i \right) \alpha_j(n) e^{-\lambda_j t} = \sum_{j=1}^{n} d_j(n) \alpha_j(n) e^{-\lambda_j t}$$

where $d(n) = \begin{pmatrix} d_1(n) \\ d_2(n) \\ \vdots \\ d_n(n) \end{pmatrix} = G_n D_n c(n).$

The condition $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$ implies convergence of the infinite products $\lim_{n \to \infty} \alpha_i(n) = \alpha_i$, $\forall i \in N$ [C] . For every $i \in N$ the sequence $\{d_i(n)\}_{n \in N}$ has also a finite limit when $n \to \infty$. Then we write $d_i = \lim_{n \to \infty} d_i(n)$.

In fact, let $P_n(t) = \sum_{i=1}^{n} c_i(n) \alpha_i(n) e^{-\lambda_i t}$; then $\|P_n(t)\| = \|\varphi_n(t)\| \leq K$, $\forall n \in N$ and $\left(P_n, e^{-\lambda_i t}\right) = d_i(n)$. This shows that $\{P_n(t)\}_{n \in N}$ is a sequence of elements in $L^2(0, \infty)$ such that the norms form a nondecreasing sequence of real numbers with K as an upper bound. Then there exists $P \in L^2(0, \infty)$ such that $P_n \to P$ if $n \to \infty$.

The following theorem is valid

THEOREM 2: If there exist a constant $\beta > 0$ such that $\lambda_{n+1} - \lambda_n \geq \beta$, $\forall n \in N$, *and* $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$ *then*

$$\varphi(t) = \sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t}$$

is the solution with minimun norm of the moment problem

$$\int_0^{\infty} \varphi(t) e^{-\lambda_i t} = c_i \ , \ i \in N.$$

Proof:

First,

$$\sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t} \in L^2(0, \infty),$$

is a consequence of a theorem of Schwartz [S]. In fact, as $\varphi_n(t) = \sum_{i=1}^{n} d_i(n) \alpha_i(n) e^{-\lambda_i t}$ is the solution of minimun norm of the problem of order n:

$$\int_0^{\infty} \varphi(t) e^{-\lambda_i t} = c_i \ , \ 1 \leq i \leq n,$$

there exists $\varphi(t) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i(n) d_i(n) e^{-\lambda_i t} \in L^2(0, \infty)$, being $\varphi(t)$ the solution of minimum norm of the moment problem [K]. Then $\varphi(t)$ belongs to the clausure of the subspace of $L^2(0, \infty)$ generated by $\left\{e^{-\lambda_i t}\right\}_{i \in N}$ and can be written as a Dirichlet series [S]

$$\varphi(t) = \sum_{i=1}^{\infty} k_i e^{-\lambda_i t}$$

As $\left\{e^{-\lambda_i t}\right\}_{i \in N}$ is a minimal system [S] it follows that $k_i = \alpha_i d_i$, $\forall i \in N$, i.e.:

$$\varphi(t) = \sum_{i=1}^{\infty} \alpha_i d_i e^{-\lambda_i t} \in L^2(0, \infty)$$

It remains to prove that $\sum_{j=1}^{\infty} d_j \alpha_j e^{-\lambda_j t}$ is a solution. As

$$\left(\sum_{i=1}^{\infty} d_i \alpha_i e^{-\lambda_i t} , e^{-\lambda_k t} \right) = \sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k}$$

then we must prove that:

$$\sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k} = c_k\,,\ \forall k \in N.$$

If $k \leq n$, $\sum_{i=1}^{n} d_i(n)\alpha_i(n)\frac{1}{\lambda_i+\lambda_k} = (G_n D_n G_n D_n c(n))_k = c_k$ then

$$\lim_{n\to\infty} \sum_{i=1}^{n} d_i(n)\alpha_i(n)\frac{1}{\lambda_i + \lambda_k} = c_k\,,\ \forall k \in N.$$

But $\sum_{i=1}^{\infty} \alpha_i d_i e^{-\lambda_i t} \in L^2(0,\infty)$ then

$$c_k = \sum_{i=1}^{\infty} d_i \alpha_i \frac{1}{\lambda_i + \lambda_k} \qquad \bullet$$

7 Another expression for the solution

The solution of minimun norm of the problem of order n $\varphi_n(t) = \sum_{j=1}^{\infty} d_j(n)\alpha_j(n)e^{-\lambda_j t}$ can be written as $\varphi_n(t) = \sum_{j=1}^{\infty} \gamma_j(n)e^{-\lambda_j t}$ with $\gamma(n) = (\gamma_i(n))_{1\leq i\leq n} = D_n G_n D_n c(n)$. But $D_n G_n D_n = G_n^{-1}$, then

$$\gamma(n) = (\gamma_i(n))_{1\leq i\leq n} = G_n^{-1}c(n).$$

The goal of this section is to find an analogue expression for the solution $\varphi(t)$. In section 5 we proved that there exists $P(t) \in L^2(0,\infty)$ such that

$$P(t) = \lim_{n\to\infty} P_n(t) = \lim_{n\to\infty} \sum_{i=1}^{n} c_i(n)\alpha_i(n)e^{-\lambda_i t}$$

Then $P(t)$ belongs to the clausure of the subespace of $L^2(0,\infty)$ generated by the system $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ and $P(t)$ can be developed in a Dirichlet series

$$P(t) = \sum_{i=1}^{\infty} h_i e^{-\lambda_i t}.$$

But $\left\{e^{-\lambda_i t}\right\}_{i\in N}$ is a minimal system, then $h_i = \alpha_i c_i\,,\ \forall i \in N$,

$$P(t) = \sum_{i=1}^{\infty} c_i \alpha_i e^{-\lambda_i t} \in L^2(0,\infty).$$

Then $\left(P(t)\,,\, e^{-\lambda_j t}\right) = \sum_{i=1}^{\infty} \frac{c_i \alpha_i}{\lambda_i + \lambda_j}$ converges and

$$d_i = \lim_{n\to\infty} d_i(n) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{c_j \alpha_j(n)}{\lambda_i + \lambda_j} = \sum_{j=1}^{\infty} \frac{c_j \alpha_j}{\lambda_i + \lambda_j}.$$

Then $\varphi(t) = \sum_{i=1}^{\infty} d_i \alpha_i e^{-\lambda_i t} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j} c_j e^{-\lambda_i t}$.

If we define the operator DGD as the one generated by the infinite matrix $\left(\frac{\alpha_i \alpha_j}{\lambda_i + \lambda_j}\right)_{i,j}$ and the operator GD as the one generated by the infinite matrix $\left(\frac{\alpha_i}{\lambda_i + \lambda_j}\right)_{i,j}$ it follows that $\varphi(t) = \sum_{i=1}^{\infty} (DGDc)_i e^{-\lambda_i t}$.

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CROWNS.
A UNIFIED APPROACH TO STARSHAPEDNESS

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ABSTRACT: *It is observed that many papers concerning starshaped sets have similar structure and objectives. Those papers usually deal with construction of the convex kernel, dimension of the kernel and Krasnoselsky-type theorems. Furthermore, the logical connections among these different topics are almost the same in the different papers. The aims of the present note are to exhibit these logical connections and to sketch a unified theory of starshapedness. A third implicit aim is the development of a brief survey of some aspects of this part of Convexity Theory. The main tool to obtain these objectives is the notion of crown of a starshaped set.*

1.- INTRODUCTION.

More than thirty years ago, F. A. Valentine, in his classical book [15] on Convexity, posed several problems regarding starshaped sets. The first and more important two problems were :

(P_0) *Characterize the starshapedness of* S *in terms of the maximal convex subsets of* S. (Problem 9.3 of [5]).

(P_1) *Determine neccessary and sufficient conditions that the convex kernel of* S *have dimension* α *, where* $0 \leq \alpha \leq d$ *, and* d *is the space dimension.* (Problem 1.1 of [5]).

Problem (P_0) was completely solved in [11] , but its solution provoked a similar and more general type of problem :

(P_2) *Describe the convex kernel of a starshaped set* S *as the intersection of a certain family of subsets of* S.

In 1946 Krasnoselsky [8] proved that a compact set $S \subset \mathbf{R}^n$ is starshaped if and only if for each subset of n+1 points of S there exists a point of S than can see via S all these points. This theorem, perhaps the most important result in the theory of starshapedness, suggested a new angle of research about starshaped sets and visibility. The results of this new approach are usually labelled as *Krasnoselsky-type theorems* , and provide answers to the following problem :

(P_3) *Describe properties (related to visibility and starshapedness) of the set* S *by means of conditions upon each subset of* k *points of* S *, where* k *is an integer related to the space dimension.*

The literature on starshapedness and related matters includes scores of particular solutions of problems (P_1) , (P_2) and (P_3) . We will mention some of those solutions in Paragraph 3. The main purpose of this note is to exhibit the logical connections among these problems. We intend to show that a solution to any of these problems can produce solutions to the remaining ones.

2.- BASIC DEFINITIONS.

Unless otherwise stated, all the points and sets considered here are included in a real locally convex linear topological space E. The interior, closure, boundary, convex hull and affine hull of a set S are denoted by int S , cl S , bdry S, conv S and aff S , respectively. The open segment joining x and y is denoted (x y) . The substitution of one or both parentheses by square ones indicates the adjunction of the correspondig extremes. The ray issuing from x and going through y is denoted $R(x \to y)$, while $R(y\ x \to)$ is the ray issuing from x and going in the opposite direction. All rays are considered closed. We say that x *sees* y *via* S if $[x\ y] \subset S$. The *star* of x in S is the set st(x,S) of all the points of S that see x via S. A *star-center* of S is a point $x \in S$ such that st(x,S) = S . The *kernel* (*convex kernel*, *mirador*) of S is the set ker S of all the star-centers of S . Finally, S is *starshaped* if ker S is not empty.

A *crown* of the starshaped set S is a collection $\mathfrak{R}$ of subsets of S whose intersection is ker S. If S is a starshaped set and $\mathfrak{R}$ is a crown of S, a *subcrown* is a subfamily $\mathfrak{I} \subset \mathfrak{R}$ such that $\mathfrak{I}$ itself be a crown of S. A *minimal crown* of S is a crown that admits no proper subcrown. A *covering crown* of S is a crown whose union is S. A *finite crown* is one with a finite number of members. Any other qualification of the word *"crown"* (e.g.: *convex crown, closed crown,* etc.) indicates that the same adjective applies to each of the members of the crown. That is, $\mathfrak{R}$ is a convex crown if and only if it is a crown and each of its members is convex. We are naturally inclined to try to prove, by means of a nonconstructive approach (i.e. Zorn's Lemma, well ordering principle, or the like), a theorem that assures that every crown admits a minimal subcrown. Unfortunately, such a theorem would be false, as a counterexample given below shows.

3.- EXAMPLES OF CROWNS.

In this paragraph we consider seven examples of crowns already in the literature. We shall restrict our exposition to the basic definition in each case, and the statement that identifies the crown considered.

THEOREM 3.1 *If* S *is a starshaped set, the family* $\mathfrak{R} = \{st(x,S) \mid x \in S\}$ *is a crown of* S .

No proof is needed here. This is just a different way to state the definition of the convex kernel of S. An interesting type of problem is to describe, in different environments and settings, a minimal subcrown of the crown just defined. Theorem 3.3 and Theorem 3.6, stated below, present two different approaches in this direction. A *convex component* of S is a maximal convex subset of S.

THEOREM 3.2 (Toranzos, [11]) *If* S *is a starshaped subset, a covering family of convex components of* S *is a covering and convex crown of* S.

The original statement of this result refers to the family of <u>all</u> convex components of S, but the proof applies to the present statement. It is important to remark that both previous theorems omit any topological and/or dimensional requirement, either on the space or on the starshaped set S.

The *relative interior* of a set M, denoted 'relint M', is the interior of M in the relative topology of aff M . A k-*simplex* is the convex hull of k+1 affinely independent points. A point $x \in S$ is a k-*extreme point* if no k-simplex $\Delta \subset S$ exists such that $x \in \text{relint } S$. Of course, in these two definitions k is not larger than the space dimension. The set of all the k-extreme points of S is denoted by $\text{ext}_k S$.

THEOREM 3.3 ([6], [10]) *Let* S *be a compact starshaped subset of* $\mathbf{R}^d$. *The family* $\mathfrak{R} = \{\text{st}(x,S) \mid x \in \text{ext}_{d-1} S\}$ *is a crown of* S .

This statement was proved simultaneously and independently by Tidmore [10] and by Kenelly et al. [6] . It is easy to construct, even in $\mathbf{R}^3$, counterexamples to show that the set of regular extreme points of S , that is $\text{ext}_1 S$ in the previous definition, is not enough to describe the convex kernel as intersection of its stars.

The point y *sees clearly* x *via* S if there exists a neighborhood $\mathcal{U}_x$ of x such that $\mathcal{U}_x \subset \text{st}(y,S)$. The *nova* (or *clear star*) of x in S is the set nova(x,S) of all points of S that see clearly x via S . A point $x \in S$ is a point of *local convexity* of S if there exists a neighborhood $\mathcal{U}_x$ of x such that $\mathcal{U}_x \cap S$ be convex. Otherwise, x is a *point of local nonconvexity* of S. The set of all points of local nonconvexity [local convexity] of S is denoted by lnc S [lc S].

THEOREM 3.4 (Stavrakas, [9]) *Let* S *be a compact connected subset of* $\mathbf{R}^d$. *Then, the family of novae of points of local nonconvexity of S is a crown of* S.

This theorem has recently been generalized in Theorem 2.2 of [14] where the requirement of finite dimension is dropped, and the condition of compactness of S is substituted by that of lnc S. As we remark here, these improvements yield easily better results about the dimension of the kernel and new Krasnoselsky-type theorems.

Let p and q be points of S. The point p *has higher visibility via* S *than* q if st(q,S) $\subset$ st(p,S). The *visibility cell of* p *in* S is the set vis(p,S) of all the points of S having higher visibility via S than p. Of course, p $\in$ vis(p,S) always.

THEOREM 3.5 (Toranzos, [12]) *Let* S *be a closed connected set such that* lnc S *be compact. The family of visibility cells of all points of local nonconvexity of* S *is a convex crown of* S.

A *simple smooth Jordan domain* is a compact set $S \subset \mathbf{R}^2$ whose boundary is a simple closed smooth Jordan curve having a finite number of inflection points.

THEOREM 3.6 (Forte Cunto, [2]) *Let* S *be a simple smooth Jordan domain. The family of stars of the inflection points of* bdry S *is a finite crown of* S.

Let y $\in$ bdry S and x $\in$ st(y,S). We say that R(x $\rightarrow$ y) is an *inward ray through* y if there exists t $\in$ R(x y $\rightarrow$) such that (y t) $\subset$ int S. Otherwise, we say that R(x $\rightarrow$ y) is an *outward ray through* y. The *inner stem of* y *in* S is the set ins(y,S) formed by y and all the points of st(y,S) that issue outward rays through y . A *regular domain* is a set S having connected interior and such that S = cl int S .

THEOREM 3.7 (Toranzos, [13]) *Let* S *be a nonconvex regular domain. Then the family* $\Im = \{\text{ins}(x,S) \mid x \in \text{Inc } S\}$ *is a crown of* S .

EXAMPLE 3.8 *Example of a crown without minimal subcrowns.*

Let S be a planar set consisting of three quarters of a circular disk, that is, using polar cordinates :

$$S = \left\{(r,w) \in \mathbb{R}^2 \,\middle|\, 0 \le r \le 1;\, \tfrac{p}{2} \le w \le 2p\right\}.$$

Let O be the origin, $p = \left(1, \frac{p}{2}\right)$ and $q = (1,p)$. The convex components of S are the closed semidisks obtained by intersection of S with a halfplane limited by a line through O . Each of these convex components is characterized by the point of the arc $[\overline{p \quad q}]$ where its limiting line intersects this arc. If x is a point of this arc, let K_x be the corresponding convex component. It is easy to verify that if L is a subset of the mentioned circular arc such that the points p and q are accumulation points of L, then the family $\Re_L = \{K_x \mid x \in L\}$ is a crown of S . Consider now the family $\mathscr{P}$ of all the convex components of S, with the exception of K_p and K_q . Then $\mathscr{P}$ is a crown of S that has no minimal subcrown.❑

4.- REPRESENTATION AND DIMENSION OF THE CONVEX KERNEL.

The natural way to begin a study on starshapedness is to prove a theorem of representation (or construction) of the convex kernel of a starshaped set. The format of such a theorem is :

THEOREM 4.1 *Let* S *be a starshaped set with property* $\mathcal{P}$ *included in the space* E *with structure* $\mathcal{Q}$ *. Then the family* $\mathfrak{R}$ *of subsets of* S *is a crown of* S.

Unless we determine explicitly the property (or properties) $\mathcal{P}$, the structure $\mathcal{Q}$ and the family $\mathfrak{R}$, this statement is not a real theorem but a *theorem-format* i.e. a logical template that can be filled with real mathematical contents. All of the theorems quoted in the previous paragraph fit into this format. The proof of a theorem having this format is a particular solution of the Problem (P_2) stated in the first paragraph. Once solved the *Representation Problem* of the convex kernel, the *Dimension Problem,* stated above as Problem (P_1) , can be approached in the same way by means of another *theorem-format* .

THEOREM 4.2 *Let* S *be a set with property* $\mathcal{P}$ *included in the space* E *that has structure* $\mathcal{Q}$ *, and let* $\mathfrak{R}$ *be a crown of* S *. Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F *, a point* $x \in \operatorname{relint}(F \cap S)$ *, and a neighborhood* $\mathcal{U}_x$ *of* x *such that for each* $M \in \mathfrak{R}$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset M$.

<u>Proof</u> : The *'if'* part is simple since the definition of crown implies $(\mathcal{U}_x \cap F \cap S) \subset M$ where the set between brackets has dimension α . For the converse implication it is enough to take $F = \operatorname{aff} \ker S$ and $x \in \operatorname{relint} \ker S$.❑

Let us now apply this *theorem-format* to the examples of crowns that were introduced in the previous paragraph.

THEOREM 4.3 *Let* E *be a locally convex linear topological space, and* S *a starshaped subset of* E *. Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in \operatorname{relint}(F \cap S)$, *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall\, t \in S$, $(\mathcal{U}_x \cap F \cap S) \subset \operatorname{st}(t,S)$.

<u>Proof</u> : This is just the conjunction of Theorem 3.1 and Theorem 4.2 .❑

THEOREM 4.4 *Let* E *be a locally convex linear topological space,* S *a starshaped subset of* E, *and* $\mathfrak{R}$ *a covering family of convex components of* S. *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall K \in \mathfrak{R}$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset K$.

Proof: Conjunction of Theorem 3.2 and Theorem 4.2.❑

THEOREM 4.5 *Let* $E = \mathbf{R}^d$, *and* S *be a compact starshaped subset of* E. *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall t \in ext_{d-1} S$ *holds* $(\mathcal{U}_x \cap F \cap S) \subset st(t,S)$.

Proof: Conjunction of Theorem 3.3 and Theorem 4.2 .❑

THEOREM 4.6 *Let* $E = \mathbf{R}^d$ *and* S *be a compact connected subset of* E. *Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that* $\forall t \in$ Inc S *holds* $(\mathcal{U}_x \cap F \cap S) \subset nova(t,S)$.

Proof: Conjunction of Theorem 3.4 and Theorem 4.2 . It is important to recall that precisely the present result was proved in [9], where Stavrakas introduced the notion of clear visibility. ❑

THEOREM 4.7 *Let* E *be a locally convex linear topological space and* S *a closed connected subset of* E *such that* Inc S *be compact. Then* dim (ker S) $\geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F, *a point* $x \in$ relint $(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* $(\mathcal{U}_x \cap F \cap S)$ *has higher visibility via* S *than each of the points of local nonconvexity of* S.

Proof: This is the conjunction of Theorem 3.5 and Theorem 4.2 .❑

THEOREM 4.8 *Let* $E = \mathbf{R}^2$ *and* S *be a simple smooth Jordan domain. Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in \text{relint}(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* $(\mathcal{U}_x \cap F \cap S)$ *see via* S *every inflection point of* bdry S .

Proof : Conjunction of Theorem 3.6 and Theorem 4.2 .❑

THEOREM 4.9 *Let* E *be a locally convex linear topological space and* S *a nonconvex regular domain included in* E . *Then* $\dim(\ker S) \geq \alpha \geq 0$ *if and only if there exists an* α*-dimensional flat* F , *a point* $x \in \text{relint}(F \cap S)$ *and a neighborhood* $\mathcal{U}_x$ *of* x *such that every point of* $(\mathcal{U}_x \cap F \cap S)$ *issues outward rays through each of the points of local nonconvexity of* S .

Proof : This is the conjunction of Theorem 3.7 and Theorem 4.2 .❑

We have shown, by means of these seven examples, that any solution to the Problem (P_2) of representation of the convex kernel by a crown yields almost inmediately, via the theorem-format 4.2, a solution to the problem (P_1) of the dimension of the convex kernel.

5.- KRASNOSELSKY-TYPE THEOREMS.

Every theorem that fits into the theorem-format 4.1 of representation of the convex kernel by means of a crown is essentially a result about the intersection of a certain family of sets. The literature on Convexity has, in the finite-dimensional case, a large *corpus* of theory usually labelled as **Helly-type Theorems** , that deals with the intersection of families of sets and has a strong combinatorial flavor. The conjunction of this type of result with the theorems exhibited in the previous paragraph is highly desirable, but a technical problem arises : Helly-type theorems usually refer to families of **convex** sets, while the members of a crown are not necessarily convex. The difficulty is solved by means of an auxiliary lemma whose proof is usually far from simple.

LEMMA 5.1 (K-lemma) *Let* S *be a set with property* $\mathfrak{P}$ *included in the space* E *that has structure* $\mathfrak{Q}$ *and* $\mathfrak{R}$ *be a nonconvex crown of* S . *Let* $x \in S$ *but* $x \notin \ker S$. *Then,* $\exists\, M \in \mathfrak{R}$ *such that* $x \notin \operatorname{conv} M$ $[x \notin \operatorname{cl} \operatorname{conv} M]$.

This lemma implies inmediately that $\ker S$ is the intersection of the convex hulls [the closed convex hulls] of the members of the crown $\mathfrak{R}$. The use of the alternative enclosed in square brackets depends on the topological conditions of the crown considered. It is clear that this lemma is superfluous if the crown is convex, as in examples 3.2 and 3.5 above. We quote here for later reference the three most commonly used Helly-type theorems.

THEOREM 5.2 (Helly,[4]) *Let* $E = \mathbf{R}^d$ *and* $\mathfrak{R}$ *be a finite family of convex subsets of* E *such that each subfamily of* k *members of* $\mathfrak{R}$, *with* $k \leq d+1$, *has nonempty intersection. Then, the intersection of all the members of* $\mathfrak{R}$ *is nonempty. The condition of finiteness of* $\mathfrak{R}$ *can be dropped if it is required the compactness of all its members.*

THEOREM 5.3 (Grünbaum, [3]) *Let* $E = \mathbf{R}^d$ *and* $\mathfrak{R}$ *be a finite family of convex subsets of* E . If N *denotes the set of positive integers, we define a function* $g : \mathrm{N}\times\mathrm{N} \to \mathrm{N}$ *by* $g(n,1) = 2n$, $g(n,n) = n+1$, *and* if $n > k > 1$ *then* $g(n,k) = 2n-k$. *Any other value of* $g(n,k)$ *is irrelevant. The dimension of the* intersection of all *the members of* $\mathfrak{R}$ *is greater than or equal to* α *if and only if the dimension of the intersection of every subfamily of* $\mathfrak{R}$ *that has at most* $g(d,\alpha)$ *members is at least* α .

THEOREM 5.4 (Klee, [7]) *Let* $E = \mathbf{R}^d$, $\mathfrak{R}$ *be a finite family of convex subsets of* E *and* $\delta > 0$. *The intersection of all the members of* $\mathfrak{R}$ *contains a ball of radius* δ *if and only if for every subfamily of* d+1 *members of* $\mathfrak{R}$, *its intersection contains such a ball. As in Theorem 5.2, the finiteness of* $\mathfrak{R}$ *can be dropped provided the compactness of all its menbers is required.*

The knowledge of a crown for a certain class of starshaped sets, plus the previous theorems, produce three different Krasnoselsky-type theorem-formats. As we have observed at the beginning of thiis paragraph, either the crown considered is convex or it must verify a K-Lemma that follows the format of Lemma 5.1.

THEOREM 5.6 (Krasnoselsky-type 1) *Let* $E = \mathbf{R}^d$, S *be a compact subset of* E , *and* $\mathfrak{R}$ *be a crown of* S *that either is convex or verifies Lemma 5.1. Then* S *is starshaped if and only if the intersection of every subfamily of* d+1 *members of* $\mathfrak{R}$ *is nonempty.*

Proof : Theorem 5.2 and , if needed, Lemma 5.1. The compactness of S can be substituted by the finiteness of the crown $\mathfrak{R}$.❑

THEOREM 5.6 (Krasnoselsky-type 2) *Let* $E = \mathbf{R}^d$, S *be a subset of* E , *and* $\mathfrak{R}$ *be a finite crown of* S *that either is convex or verifies Lemma 5.1. If* N *denotes the set of positive integers, define a function* $g : \mathrm{N}\times\mathrm{N} \to \mathrm{N}$ *by* g(n,1) = 2n , g(n,n) = n+1 , *and for* n > k > 1 g(n,k) = 2n-k . *Any other value of* g(n,k) *is irrelevant. Then,* S *is starshaped and* dim ker S $\geq \alpha$ *if and only if the dimension of the intersection of each subfamily of* g(d,α) *members of the crown is at least* α .

Proof : Theorem 5.3 and, if the crown is not convex, Lemma 5.1. In this case the finiteness of the crown is essential and admits no substitution by any compactness condition.❑

THEOREM 5.7 (Krasnoselsky-type 3) *Let* $E = \mathbf{R}^d$, S *be a compact subset of* E , *and* $\mathfrak{R}$ *be a crown of* S *that either is convex or verifies Lemma 5.1. Then* S *is starshaped and* ker S *contains a ball of radius* $\delta > 0$ *if and only if the intersection of each subfamily of* $\mathfrak{R}$ *having at most* d+1 *members contains a ball of radius* δ .

Proof : Theorem 5.4 and, if needed, Lemma 5.1. Once more, the compactness of S can be substituted by the finiteness of the crown.❑

These three theorem-formats combined with the seven types of crowns described in Paragraph 3 can give rise to twenty one Krasnoselsky-type theorems. Some of those results are already known. M. Breen (in [1] and other papers) has derived several Krasnoselsky-type theorems from the Stavrakas' crown described in Theorem 3.4. The theorem that can be obtained by the conjunction of Theorem 3.1 and Theorem 5.5 is the original 1946 Krasnoselsky's Theorem [9]. The nine Krasnoselsky-type theorems that can be derived from Theorems 3.5, 3.6 and 3.7 have already been proved in the papers ([12], [2] and [13]) where the respective crowns were described. As an example we state the theorems that can be derived from the crown described in Theorem 3.2.

THEOREM 5.8 *Let* $E = \mathbf{R}^d$, S *be a subset of* E , *and* $\mathfrak{R}$ *be a covering family of convex components of* S . *Then* S *is starshaped if and only if every* d+1 *members of* $\mathfrak{R}$ *have nonempty intersection.*

THEOREM 5.9 *Let* $E = \mathbf{R}^d$, S *be a subset of* E , *and* $\mathfrak{R}$ *be a finite covering family of convex components of* S . *If* N *denotes the set of positive integers, define a function* $g : N \times N \to N$ *by* $g(n,1) = 2n$, $g(n,n) = n+1$, *and for* $n > k > 1$ $g(n,k) = 2n-k$. *Any other value of* g(n,k) *is irrelevant. Then,* S *is starshaped and* $\dim \ker S \geq \alpha$ *if and only if the dimension of the intersection of each subfamily of* $g(d,\alpha)$ *members of* $\mathfrak{R}$ *is at least* α .

THEOREM 5.10 *Let* $E = \mathbf{R}^d$, S *be a subset of* E , *and* $\mathfrak{R}$ *be a covering family of convex components of* S . *Then* S *is starshaped and* ker S *contains a ball of radius* δ *if and only if the intersection of each subfamily of* $\mathfrak{R}$ *that has at most* d+1 *members contains a ball of radius* δ.

6.- CONCLUDING REMARKS.

In the previous sections we have shown that once proved a theorem about the construction of the convex kernel that fits the format of Theorem 4.1 and, in the case that it would be necessary, a *K-Lemma* like 5.1, the whole Starshapedness Theory including theorems about construction and dimension of the kernel and Krasnoselsky-type theorems follows easily. The main tool in this development has been the idea of *crown of a starshaped set.* We claim that this notion is worthy of systematic study. The study of minimal crowns generated by some of the known types of crowns seems specially promising.

In the Krasnoselsky-type theorems that fit the theorem-format 5.6 , sometimes it is possible to obtain a slight improvement if the analogous theorem of Katchalsky [5] is substituted instead of Grünbaum's Theorem 5.3. The application of different Helly-type theorems and/or adimensional theorems regarding intersections of convex sets to the present approach remains to be studied in the future.

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ESTIMATES ON THE $(L^p(w), L^q(w))$ OPERATOR NORM OF THE FRACTIONAL MAXIMAL FUNCTION.

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Abstract: In $\mathbb{R}^n$, given $\gamma \in [0,n)$ and $p \in (1, n/\gamma)$, it is well known that $w^q \in A^r$, with $1/q = 1/p - \gamma/n$ and $r = 1 + q\frac{p-1}{p}$, is a necessary and sufficient condition for the boundedness of the Maximal Fractional Operator M_γ between $L^p(w^p)$ and $L^q(w^q)$ spaces. In this work we study the dependence of the operator norm on the constant of the A_r condition. The result extends the obtained by S. Buckley for the Hardy-Littlewood Maximal Function (i.e.: $\gamma = 0$).

§1.

Let μ be a positive Borel measure in $\mathbb{R}^n$. For each γ in $(0,n)$, the fractional maximal operator M_γ with respect to μ is defined by

$$M_\gamma f(x) = \sup_{x \in Q} \frac{1}{\mu(Q)^{1-\frac{\gamma}{n}}} \int |f|\, d\mu, \tag{1.1}$$

for $f \in L^1_{loc}(d\mu)$, where the sup is taken over all cubes in $\mathbb{R}^n$ containing x. It is well known that for each p in $(1, n/\gamma)$ there exists a constant C, independent of f, such that the inequality

$$\left(\int_{\mathbb{R}^n} (|M_\gamma f|\, w)^q \, d\mu\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} (|f|\, w)^p \, d\mu\right)^{\frac{1}{p}}, \tag{1.2}$$

holds with $1/q = 1/p - \gamma/n$ for every f in $L^p(w^p d\mu)$ if and only if w is a weight in the $A(p,q)$ class with respect to μ, that is, w is a non negative function satisfying

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$$(1.3) \qquad K_{w,p,q} = \sup_Q \left(\frac{1}{\mu(Q)} \int_Q w^q \, d\mu \right)^{\frac{1}{q}} \left(\frac{1}{\mu(Q)} \int_Q w^{-p'} \, d\mu \right)^{\frac{1}{p'}} < \infty,$$

where the sup is taken over all cubes in $I\!R^n$ and $p' = p/(p-1)$. From the classical proofs of the above result, it can be obtained that the constant C in (1.2) depends on $K_{w,q,p}$, but they do not show explicitly the dependence. In 1993, S. Buckley ([B]) solved the problem for the Hardy-Littlewood maximal function (i.e.: $\gamma = 0$ in (1.1)). The purpose of this work is to extend that result to the general case of the operator in (1.1). Actually, our main result is the following theorem.

(1.4) **Theorem:** *If $0 \le \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$ and w is a nonnegative function on $I\!R^n$ such that, for every cube Q, (1.3) holds, then*

$$(1.5) \qquad \left(\int_{I\!R^n} (|M_\gamma f \,|\, w)^q \, d\mu \right)^{\frac{1}{q}} \le C K_{w,p,q}^{p'(1-\frac{\gamma}{n})} \left(\int_{I\!R^n} |f\, w|^p \, d\mu \right)^{\frac{1}{p}}$$

The power $K_{w,p,q}^{p'(1-\frac{\gamma}{n})}$ is the best possible.

As it can be seen in §2, our techniques to prove the above theorem are extensions of those used by Buckley in the case $\gamma = 0$. An important point in order to obtain these extensions is to recall the obvious relation between the $A(p,q)$ classes, defined as in (1.3), and the Muckenhoupt's classes A_r with respect to μ. In fact, since a weight w is in A_r, $1 < r < \infty$, when

$$(1.6) \qquad B_{w,r} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \, d\mu \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{r-1}} \, d\mu \right)^{r-1} < \infty,$$

where the sup is taken over all cubes in $I\!R^n$, it is clear that w belongs to $A(p,q)$ if and only if w^q belongs to $A_{1+q/p'}$, with $p' = p/(p-1)$. Moreover, we have $B_{w^q,1+q/p'} = K_{w,p,q}^q$.

§2

As in the case $\gamma = 0$, we are going to prove theorem (1.4) by using an argument of interpolation. For this reason, let us first to state the following version of the Marcinkiewicz's interpolation theorem with respect to a positive Borel measure μ.

(2.1) **Theorem:** *Suppose that a quasi-linear operator T is simultaneously of weak types (p_1, q_1) y (p_2, q_2), $1 \le p_i, q_i \le \infty$, $q_1 \ne q_2$, with norms M_1 y M_2 respectively (i.e.: $\mu(\{x : Tf(x) > \alpha\}) \le \left(\frac{M_i}{\alpha} \|f\|_{L^{p_i}(d\mu)} \right)^{q_i}$, $i = 1, 2$). Then for any (p,q) with*

$$\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_2} \quad , \quad \frac{1}{q} = \frac{t}{q_1} + \frac{(1-t)}{q_2} \, , \qquad 0 < t < 1,$$

the operator T is of strong type (p,q), and we have

$$\|Tf\|_{L^q(d\mu)} \le H\, M_1^t M_2^{1-t}\, \|f\|_{L^p(d\mu)}$$

Proof: See [Z], p. 111, vol.2.■

(2.2) **Remark:** From the proof of (2.1) in the case $p_1 \le p_2$ y $q_1 < q_2$, it follows that

$$H^q = 2^q q \left(\frac{(p_1/p)^{\frac{q_1}{p_1}}}{q-q_1} + \frac{(p_2/p)^{\frac{q_2}{p_2}}}{q_2-q} \right)$$

To apply the above theorem we need weak type inequalities for M_γ . They will be given by the next two results. The first one was proved by S. Buckley and provides an estimate concerning a known property of A_r classes. The proof of the second one is due to B. Muckenhoupt and R. Wheeden ([MW]). However, accordingly to our purpose, here we are going to examine carefully that proof in order to obtain a more precise conclusion.

(2.3) **Theorem:** *If w satisfies A_p then w satisfies $A_{p-\varepsilon}$ with $\varepsilon \sim B_{w,p}^{1-p'}$ and $B_{w,p-\varepsilon} \le C B_{w,p}$, where $C = C(n,p)$.*

Proof: See [B], p. 255, lemma 2.1.■

(2.4) **Theorem:** *If $0 \le \gamma < n$, $1 < p < n/\gamma$, $1/q = 1/p - \gamma/n$, $\alpha > 0$, E_α is the set where $M_\gamma f > \alpha$, and w is a nonnegative function on $I\!R^n$ satisfying (1.3) then there is a C , independent of f , such that*

$$\left(\int_{E_\alpha} w^q \, d\mu \right)^{\frac{1}{q}} \le C \frac{K_{w,p,q}}{\alpha} \left(\int_{I\!R^n} |fw|^p \, d\mu \right)^{\frac{1}{p}} \tag{2.5}$$

Proof: Fix $M > 0$ and let $E_{\alpha,M} = E_\alpha \cap B(0,M)$. It is clear that for each $x \in E_{\alpha,M}$ there exists a cube Q containing x such that

$$\frac{1}{\alpha \mu(Q)^{1-\frac{\gamma}{n}}} \int_Q |f| \, d\mu > 1$$

Using Besicovitch's theorem we can choose a sequence $\{Q_k\}$ of these cubes such that $E_{\alpha,M} \subset \cup Q_k$ and no point of $I\!R^n$ is on more that $C = C(n)$ of these cubes , i.e. $\sum \chi_{Q_k} \le C$. Then, since $p/q \le 1$ and w satisfies (1.3), we have

$$
\begin{aligned}
\left(\int_{E_{\alpha,M}} w^q\,d\mu\right)^{\frac{p}{q}} &\leq \sum_k \left(\int_{Q_k} w^q\,d\mu\right)^{\frac{p}{q}}\\
&\leq \sum_k \left(\int_{Q_k} w^q\,d\mu\right)^{\frac{p}{q}} \left(\frac{1}{\alpha\mu(Q_k)^{1-\frac{\gamma}{n}}}\int_{Q_k}|f|\,d\mu\right)^p\\
&\leq \sum_k \left(\int_{Q_k} w^q\,d\mu\right)^{\frac{p}{q}} \frac{1}{\alpha^p\mu(Q_k)^{(1-\frac{\gamma}{n})p}}\\
&\leq \left(\int_{Q_k}|f|^p\ w^p d\mu\right)\left(\int_{Q_k} w^{-p'}\,d\mu\right)^{\frac{p}{p'}}\\
&\leq \frac{K^p_{w,p,q}}{\alpha^p}\sum_k\left(\int_{Q_k}|f|^p\ w^p d\mu\right)\\
&\leq \frac{K^p_{w,p,q}C}{\alpha^p}\left(\int_{I\!R^n}|f|^p w^p d\mu\right)
\end{aligned}
$$

Finally, letting $M \to \infty$ we get (2.5).∎

Now, we are able to proceed with the proof of our main result.

Proof of Theorem (1.4): In the next, for the sake of simplicity, we are going to denote $K_{w,p,q}$ by K. As we said in §1, the fact that w satisfies (1.3) implies w^q belongs to A_r with $r = 1 + q/p'$ and $B_{w^q,r} = K^q$. Then, from (2.2), there exists $\varepsilon \sim K^{q(1-r')}$ such that w^q belongs to A_s with $s = r - \varepsilon > 1$ and $B_{w^q,s} \leq CK^q$, $C = C(n,p,q)$. Now, we choose numbers p_1 and q_1 such that $1 < p_1 < p$, $1/q_1 = 1/p_1 - \gamma/n$ and $s = 1 + q_1/p_1'$. So w^{q/q_1} satisfies $A(p_1,q_1)$ with $K_{w^{q/q_1},p_1,q_1} \leq CK^{q/q_1}$, $C = C(n,p,q)$. Then, by theorem (1.4),

$$
\int_{\{M_\gamma f>\alpha\}} w^q d\mu \leq C\frac{K^q}{\alpha^{q_1}}\left(\int_{I\!R^n}|f|^{p_1}\,w^{q\,p_1/q_1}d\mu\right)^{\frac{q_1}{p_1}} \quad . \tag{2.6}
$$

By defining $Tg\,(x) = M_\gamma\left(g\,v^{\frac{\gamma}{n}}\,(x)\right)$, with $v = w^q$, and taking $f = g\,v^{\frac{\gamma}{n}}\,(x)$, it is clear that (2.6) can be written in the form

$$
\int_{\{Tg(x)>\alpha\}} v\,d\mu \leq C\frac{K^q}{\alpha^{q_1}}\left(\int_{I\!R^n}|g|^{p_1}\,v d\mu\right)^{\frac{q_1}{p_1}} \tag{2.7}
$$

In the following step of the proof we shall asume $\varepsilon \leq \frac{q}{4}\frac{n}{n-\gamma}$. This hypothesis can be ensured by taking ε min $\left(1, \frac{q}{4}\frac{n}{n-\gamma}\right)$ instead of the original ε in the choice of p_1 and q_1 (note that this change preserves the relation between ε and K). Now, we can pick q_2 and p_2 such that $1/q - 1/q_2 = 1/q_1 - 1/q$ and $1/q_2 = 1/p_2 - \gamma/n$.

It is clear that $1 + q_2/p_2' > 1 + q/p'$, so $v \in A_{1+q_2/p_2'}$ with $B_{v,1+q_2/p_2'} \le CK^q$, $C = C(n,q,p)$. Then, by reasoning as before, we get

$$\int_{\{Tg(x)>\alpha\}} v\, d\mu \le C\frac{K^q}{\alpha^{q_2}} \left(\int_{\mathbb{R}^n} |g|^{p_2}\, v\, d\mu\right)^{\frac{q_2}{p_2}} \tag{2.8}$$

Since there exists $t \in (0,1)$ such that

$$\frac{1}{p} = \frac{t}{p_1} + \frac{(1-t)}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{t}{q_1} + \frac{(1-t)}{q_2},$$

theorem (2.1) allows us to obtain, from (2.7) and (2.8), the inequality

$$\|Tg\|^q_{L^q(v)} \le CH^qK^q\, \|g\|^q_{L^p(v)}, \tag{2.9}$$

where $C = C(n,p,q)$ and H is as in (2.2). From our choice of q_1, p_2 and q_2 and the assumption on ε, we have

$$q_1 = q - \frac{\varepsilon n}{n-\gamma},$$

$$q_2 = \frac{qq_1}{2q_1 - q} = q\frac{q - \frac{\varepsilon n}{n-\gamma}}{q - \frac{2\varepsilon n}{n-\gamma}} \le \frac{q^2}{q - 2q/4} = 2q,$$

$$p_2 = \frac{nq_2}{n + \gamma q_2} \le 2q.$$

Then, H can be estimated as follows

$$H^q = 2^q q\left(\frac{(p_2/p)^{\frac{q_2}{p_2}}}{q_2 q} + \frac{(p_1/p)^{\frac{q_1}{p_1}}}{qq_1}\right)\frac{qq_1}{q - q_1} \le 2^q q^3\left(\left(\frac{2q}{p}\right)^{\frac{2q}{p}} + 1\right)\frac{(n-\gamma)}{\varepsilon n}.$$

The above inequality, (2.9) and the fact that $\varepsilon \sim K^{q(1-r')}$ allow us to obtain

$$\|Tg\|^q_{L^q(v)} \le CK^{qr'}\, \|g\|^q_{L^p(v)} = C\, K^{q\left(\frac{p'+q}{q}\right)}\, \|g\|^q_{L^p(v)},$$

with $C = C(n,p,q)$. Finally, (1.5) follows from the definition of T by taking $g = f\, w^{-q\frac{\gamma}{n}}$ and $v = w^q$.

To see that the power of K in (1.5) is the best possible, we give an example in $\mathbb{R}$ (a similar one works in $\mathbb{R}^n$ for any n). Let $r = 1 + q/p'$, where p and q are as in the hypothesis of the theorem, and δ belonging to $(0,1)$. By a simple computation we can see that $w(x)=|x|^{\frac{(r-1)(1-\delta)}{q}}$ satisfies $A(p,q)$ with $K_{w,p,q}\simeq\delta^{\frac{1-r}{q}}$, when μ is the Lebesgue measure. Then, from (1.5) with this weight, we have

$$\|M_\gamma f\|^q_{L^q(w^q)}\leq C\delta^{-r}\ \|f\|^q_{L^p(w^p)}\,. \tag{2.10}$$

Now, we take $f(x)=|x|^{(\delta-1)}\,\chi_{[0,1)}(x)$. It is not difficult to prove that

$$M_\gamma f(x)\geq\frac{C}{\delta}\,|x|^\gamma\,f(x),$$

for every $x\in I\!R$, where C is independent of δ. Then, the above inequality and the fact that $\|f\|^q_{L^p,w^p}=\delta^{-q/p}$ allow us to get the estimate

$$||M_\gamma f||^q_{L^q(w^q)}\geq C\,\delta^{-1-q}=C\delta^{-r}\ \|f\|^q_{L^p(w^p)}\,,$$

where C is independent of δ. Finally, we complete the proof by combining the above inequality with (2.10).∎

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PAYOFF MATRICES IN COMPLETELY MIXED BIMATRIX GAMES WITH ZERO–VALUE

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ABSTRACT. A completely mixed bimatrix game (A, B) has a unique equilibrium strategy. The values of this game for each player, are defined by $v_1 = x^T Ay$ and $v_2 = x^T By$ where (x, y) is an equilibrium strategy. We give a formula for computing the completely mixed equilibrium strategy when the bimatrix game has zero–value.

1. INTRODUCTION

For the zero-sum two-person games Kaplansky (1945) introduced the notion of completely mixed strategies and showed that in games where both players have only completely mixed optimal strategies, the payoff matrix is square and each player has a unique optimal strategy. Raghavan (1970) extended this result to the non-zero-sum bimatrix games. Also Kaplansky (1945) gave a necessary and sufficient condition on the payoff matrix for a game of value zero to be completely mixed. He showed that if the value of a game is different from zero, then the payoff matrix is nonsingular and gave a formula for computing this value. Jansen $(1981a, b)$ showed that in completely mixed bimatrix games with $A > 0$ and $B < 0$, the matrices A and B are nonsingular. He also extended the formulas for computing equilibrium strategies and the values for completely mixed bimatrix games.

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Completely mixed bimatrix games have unique equilibrium strategies. The value of these games are defined to be the payoffs that the player receive when they play equilibrium strategies. In this paper we try to see how far the results can be extended to bimatrix games with zero value.

2. GENERAL RESULTS

A bimatrix game with m pure strategies for player 1 and n pure strategies for player 2, where $1 \leq m, n < \infty$, is specified by two real $m \times n$ matrices A and B. If player 1 chooses pure strategy i and player 2 chooses pure strategy j, the payoffs to players 1 and 2 are $a_{i,j}$ and $b_{i,j}$ respectively, for $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Let

$$P_n = \left\{ x \in \Re^n : x_i \geq 0, i = 1, \ldots, n, \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}$$

and $P_n^+ = \{x \in P_n : x_i > 0, i = 1, \ldots, n\}$. Vectors are assumed to be column vectors, and ^{T} denotes transpose. The vectors in P_n are called mixed strategies and denoted by $x \geq \mathbf{0}$ where $\mathbf{0} = (0, \ldots, 0)$. The vectors in P_n^+ are called completely mixed strategies and denoted by $x > \mathbf{0}$. A pair (x, y), where $x \in P_m$ and $y \in P_n$ is defined to be a equilibrium strategy of the game specified by (A, B) if

$$x^T A y \geq \xi^T A y \qquad \text{for all } \xi \in P_m$$

$$x^T B y \geq x^T B \eta \qquad \text{for all } \eta \in P_n$$

Nash (1950) proved that this equilibrium strategy exists. Let $\mathcal{E}$ be the set of all pairs of equilibrium strategies. We say that $\mathcal{E}$ is completely mixed if the elements of $\mathcal{E}$ are completely mixed pairs. Let $(x, y) \in \mathcal{E}$ be $v(x, y, A) = x^T A y$ and $v(x, y, B) = x^T B y$ are called equilibrium values of the bimatrix game (A, B).

Let

$$S(y) = \{x \in P_m : (x, y) \in \mathcal{E}\}$$

$$T(x) = \{y \in P_n : (x, y) \in \mathcal{E}\} .$$

We say that $S(y)$ is completely mixed if all elements of $S(y)$ are in P_m^+. A similar definition applies for $T(x)$.

Theorem 1 If the set $\mathcal{E}$ is completely mixed and $v(x, y, A) = v(x, y, B) = 0$ then

i. A and B are square matrices and $rank(A) = rank(B) = n - 1$

ii. $A_{i,j}$, $B_{i,j}$ denotes the cofactor of $a_{i,j}$ and $b_{i,j}$. Then there exists an i with $1 \leq i \leq m$ such that $A_{i,1}, \ldots, A_{i,n}$ are different from zero and have the same sign. There exists a j with $1 \leq j \leq n$ such that $B_{1,j}, \ldots, B_{n,j}$ are different from zero and have the same sign.

iii. $\sum_{i,j} A_{i,j} \neq 0$, and $\sum_{i,j} B_{i,j} \neq 0$.

Proof. The necessity of (i) is an immediate corollary to Theorem 1 and 4 from the paper of Raghavan (1970).

Let (x, y) be completely mixed equilibrium strategy. Let $A_{i,j}$ be the cofactor of $a_{i,j}$. Since $x > \mathbf{0}$ and $(x, y) \in \mathcal{E}$ implies that

$$Ay = \mathbf{0}.$$

Then

$$\frac{y_1}{A_{i,1}} = \frac{y_2}{A_{i,2}} = \ldots = \frac{y_n}{A_{i,n}}. \tag{1}$$

Since $rank(A) = n - 1$ then there exists $\bar{\imath}, \bar{\jmath}$ such that $A_{\bar{\imath},\bar{\jmath}}$ is different from zero. As y is a completely mixed strategy this implies in (1) that for $i = \bar{\imath}$, and for all j, $A_{\bar{\imath},j}$ have the same sign. A similar remark applies to B. Hence the necessity of (ii) is proven.

Since $rank(B) = n - 1$ then $rank(cof(B)) = 1$ where $cof(B)$ is the matrix in which the (i, j) elements are the cofactors for $b_{i,j}$. Without loss of generality we assume that the cofactor of $b_{n,n}$, $B_{n,n} \neq 0$, then

$$B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n-1} & \sum_{j=1}^{n-1} t_j b_{1,j} \\ \vdots & & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n-1} & \sum_{j=1}^{n-1} t_j b_{n-1,j} \\ \sum_{i=1}^{n-1} (-\lambda_i) b_{i,1} & \cdots & \sum_{i=1}^{n-1} (-\lambda_i) b_{i,n-1} & \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (-\lambda_i) t_j b_{i,j} \end{bmatrix}$$

and

$$cof(B) = \begin{bmatrix} -t_1\lambda_1 B_{n,n} & \cdots & -t_{n-1}\lambda_1 B_{n,n} & \lambda_1 B_{n,n} \\ \vdots & & \vdots & \vdots \\ -t_1\lambda_{n-1} B_{n,n} & \cdots & -t_{n-1}\lambda_{n-1} B_{n,n} & \lambda_{n-1} B_{n,n} \\ -t_1 B_{n,n} & \cdots & -t_{n-1} B_{n,n} & B_{n,n} \end{bmatrix}$$

where $\lambda_i = x_i/x_n$. If $\sum\limits_{i,j} B_{i,j} = 0$, implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} B_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i(-t_j)B_{n,n} = B_{n,n}\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n}(-t_j) = 0$$

where $t_n = -1$, then

$$\sum_{j=1}^{n}(-t_j) = 0 \qquad \text{or} \qquad \sum_{j=1}^{n-1}(-t_j) = 1.$$

Since the system

$$I : \begin{cases} w^T B & = \mathbf{0}^T \\ w & \geq \mathbf{0} \end{cases}$$

has a solution x, we 'll show that the system

$$II : \begin{cases} v^T B & = \mathbf{1}^T \\ v & \geq \mathbf{0} \end{cases}$$

(where $\mathbf{1}$ is the column–vector of length n with every element equal to 1) has a solution. In fact, if the system (II) has not a solution, by Alternative Theorems (see Mangasarian book, page 34, Table 2.4.1) then (by Theorem (6) Farkas) the system

$$II^{(1)} : \begin{cases} Bz & \leq \mathbf{0} \\ \mathbf{1}^T z & > 0 \end{cases}$$

has a solution $\bar{z}$. Therefore the system

$$II^{(2)} : \begin{cases} Bs & \geq \mathbf{0} \\ \mathbf{1}^T s & < 0 \end{cases}$$

has a solution $\bar{s} = -\bar{z}$.

It suffices to analyze three different case:

$a)$ If $\bar{s}$ fulfills that

$$B\bar{s} > \mathbf{0}$$

then (by Theorem 5 (Gordan)) the system (I) has not any solution. This is a contradiction.

b) If $\bar{s}$ fulfills that

$$B\bar{s} = \mathbf{0}$$

since $rank(B) = n-1$ and $t^T = (t_1, \ldots, t_{n-1}, -1)$ fulfills that $Bt = \mathbf{0}$ and

$$\sum_{j=1}^{n} t_j = 0$$

then $t = c\bar{s}$, but

$$0 = \sum_{j=1}^{n} t_j = c\sum_{j=1}^{n} \bar{s}_j \neq 0.$$

This is a contradiction.

c) From *a*) and *b*) there exists i_1 and i_2 such that

$$\sum_j b_{i_1,j}\bar{s}_j > 0 \qquad \text{and} \qquad \sum_j b_{i_2,j}\bar{s}_j = 0$$

Let

$$I_1 = \{i : \sum_j b_{i,j}\bar{s}_j > 0\} \qquad \text{and} \qquad I_2 = \{i : \sum_j b_{i,j}\bar{s}_j = 0\}$$

We denote by $B_{I_1}(B_{I_2})$ the submatrix of B formed by the row $i \in I_1 (i \in I_2)$. Then $\bar{s}$ is a solution of system

$$II^{(3)} : \begin{cases} B_{I_1}s & = \mathbf{0} \\ B_{I_2}s & > 0 \end{cases}$$

but (by Theorem 2 (Motzkim)) the systems

$$II^{(4)} : \begin{cases} v_{I_1}^T B_{I_1} + v_{I_2}^T B_{I_2} & = \mathbf{0} \\ \qquad\qquad v_{I_2} & \geq 0 \ (v_{I_2} \neq \mathbf{0}) \end{cases}$$

has not any solution. This is a contradiction since $\bar{v}_{I_1} = x_{I_1}, \bar{v}_{I_2} = x_{I_1}$ where $x_{I_1} = \{x_i : i \in I_1\}$ and $x_{I_2} = \{x_i : i \in I_2\}$) is a solution of this system.

From $a), b)$ and $c)$ the system (II) has a solution. Let $\bar{v}$ a solution of systems (II), and $\tilde{v} = \bar{v}/\sum_i \bar{v}_i$. It is clear that $\tilde{v} \in P_m$ and is different from x.

Finally, we are showing that $(\tilde{v}, y) \in \mathcal{E}$.

$$\tilde{v}^T Ay \geq \xi^T Ay \qquad \text{for all } \xi \in P_m$$

it holds true because $Ay = \mathbf{0}$.

$$\tilde{v}^T By \geq \tilde{v}^T B\eta \qquad \text{for all } \eta \in P_n$$

it holds true because $\bar{v}^T B = \mathbf{1}^T$.

Thus $(x, y), (\tilde{v}, y) \in \mathcal{E}$. This contradicts that a completely mixed bimatrix game has a unique equilibrium strategy. Hence $\sum_{i,j} B_{i,j} \neq 0$. □

Corollary 1 If the set $\mathcal{E}$ is completely mixed and $(x, y) \in \mathcal{E}$ then

$$v(x, y, A) = \frac{\det A}{\sum_{i,j} A_{i,j}} \qquad \textit{and } v(x, y, B) = \frac{\det B}{\sum_{i,j} B_{i,j}}$$

the denominator is always different from zero.

Proof. By Theorem 4 of Raghavan (1970) we easily see that the pair (x, y) is a unique equilibrium strategy. Let

$$v(x, y, A) = v_1 \qquad \text{and } v(x, y, B) = v_2$$

then the game (C, D) given by

$$c_{i,j} = a_{i,j} - v_1 \qquad \text{and } d_{i,j} = b_{i,j} - v_2$$

is completely mixed, (x, y) is equilibrium strategy and

$$v(x, y, C) = 0 \qquad \text{and } v(x, y, D) = 0$$

In particular by Theorem 1 $\det(C) = \det(D) = 0$.

$$\det(C) = \det(A) - v_1 \sum_{i,j} A_{i,j} \qquad \text{and } \det(D) = \det(B) - v_2 \sum_{i,j} B_{i,j}$$

By Theorem 1 $\sum_{i,j} C_{i,j} \neq 0$ and $\sum_{i,j} D_{i,j} \neq 0$. The case $v_1 = 0$ or $v_2 = 0$ obviously need not be considered. Hence we have $\det(A) \neq 0$ and $\det(B) \neq 0$. □

Proposition 1 *If for the bimatrix game (A, B) there exist v_1, v_2 such that, for any $(x, y) \in \mathcal{E}$,*

$$Ay = v_1 \mathbf{1} \qquad \textit{and } x^T B = v_2 \mathbf{1}^T$$

and if (i) and (ii) of Theorem 1 hold true, then $\mathcal{E}$ is completely mixed.

Proof. By hiphotesis A is square, $rank(A) = n - 1$ and there exists i such that $(A_{i,1}, \ldots, A_{i,n})$ have the same sign. Then the vector $\bar{y}$ ($\bar{y}_j = A_{i,j} / \sum_k A_{i,k}$) belongs to P_n^+. Similarly we choose $\bar{x} \in P_n^+$ ($\bar{x}_i = B_{i,j} / \sum_k B_{i,k}$). It is clear that $(\bar{x}, \bar{y}) \in \mathcal{E}$ and $v(\bar{x}, \bar{y}, A) = v(\bar{x}, \bar{y}, B) = 0$. Since $\bar{x} > \mathbf{0}$, it follows that if $y^* \in T(\bar{x})$ then $Ay^* = \mathbf{0}$. But $rank(A) = n - 1$ assures us that $y^* = \bar{y}$ and $y^* > \mathbf{0}$. Let $(x, y) \in \mathcal{E}$, then there exists v_1, v_2 such that

$$Ay = v_1 \mathbf{1} \text{ and } x^T B = v_2 \mathbf{1}^T.$$

Thus $(\bar{x}, y), (x, \bar{y}) \in \mathcal{E}$. By the argument above we have $x = \bar{x}, y = \bar{y}$ and $x > \mathbf{0}, y > \mathbf{0}$. □

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ON THE MEASURE OF SELF-SIMILAR SETS II

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ABSTRACT. In §1 we show a condition for $\mathcal{H}^s(K_b) > 0$ for almost all $b = (b_1, \ldots, b_\ell) \in R^{n\ell}$ where $K_b = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K_b)$ and ψ_i are similitudes, $\psi_i(x) : R^n \to R^n$ defined by $\psi_i(x) = k_i A_i x + b_i$, A_i an orthogonal matrix, $0 < k_i < 1/3$, b_i a vector of R^n.

In §2 we give a (geometrical) criterion for a set $K = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K)$ to be $\mathcal{H}^s(K) = 0$ if the Hausdorff dimension is equal to its similarity dimension.

In §3 we develop a method for calculating the measure of $K = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K)$ when K meets certain conditions, generalizing a method shown in [7]. We also calculate dimensions of sets K such that their dimensions do not coincide with their similarity dimensions.

Finally we give some examples (Sierpinski sets with overlapping).

§1

Let $\psi_i(x)$, $i = 1, \ldots, \ell$, be similitudes in R^n *i.e.* $\psi_i(x) : R^n \to R^n$, $\psi_i(x) = k_i A_i x + b_i$ with $0 < k_i < 1$, A_i an orthogonal matrix and b_i a vector. Let $b = (b_1, \ldots, b_\ell) \in R^{n\ell}$ and let K_b be the (unique) compact set such that

$$K_b = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K_b) \tag{0}$$

The following theorem is due to Falconer

Theorem 1 [1]. *If $max\ k_i < 1/3$, then the Hausdorff dimension of K_b is $inf\,(n, s)$, where $\sum_{i=1}^{\ell} k_i^s = 1$, for almost all $b \in R^{n\ell}$ in the sense of the Lebesgue measure $\mathcal{L}^{n\ell}$.*

The number s, $\sum_{i=1}^{\ell} k_i^s = 1$, is usually known as the similarity dimension of K_b. In this paragraph we **assume that** $\{max\ k_i\} < 1/3$ **and** $b \in \mathcal{A} :=\{ a \in R^{n\ell} : K_a$ has Hausdorff dimension s with $\sum_{i=1}^{\ell} k_i^s = 1\}$.

It is easy to show that $\mathcal{H}^s(K_b) < \infty$ for $b \in \mathcal{A}$ (see [3] pg.122).

One natural question is to ask whether $\mathcal{H}^s(K_b) > 0$ for $b \in \mathcal{A}$. (It should be noted that if ψ_i are affine contractions instead of similitudes then $\mathcal{H}^s(K_b)$ may be infinite, cf. [4].).

We prove

Theorem 2. *Let $g := \{min\ k_i\}$ and $G := \{max\ k_i\}$. Suppose*

$$(1) \qquad \ell^{\left(\frac{2log\ g}{log\ G}\right)} \cdot g^n < 1 \equiv \ell^2.G^n < 1$$

Then $\mathcal{H}^s(K_b) > 0$ for almost all $b \in \mathcal{R}^{n\ell}$.

Condition (1) right implies that the similarity dimension is less than n and therefore by theorem 1, $\mathcal{A}$ is almost all $\mathcal{R}^{n\ell}$. That both formulas in (1) are equivalent follows from taking logarithm to the left hand formula i.e. $log\left[\ell^{\left(\frac{2log\ g}{log\ G}\right)} g^n\right] < 0 \equiv \frac{log\ g}{log\ G} log\ \ell^2 + n\, log\ g < 0$. Multiply this last expression by $\frac{log\, G}{log\, g}$, getting the right hand formula.

We recall a theorem of McLaughlin [5], generalized by Falconer [2], which we shall use. It should be noted that theorem 3, lemma 1 and corollary 1 are true **without** assuming $\{max\ k_i\} < 1/3$ or $b \in \mathcal{A}$ (or both).

Theorem 3 [2]. *Suppose that K_b has the following property: there exist a natural number m and α , $r_o > 0$ such that for any set $N \subset K_b$ with $|N| < r_o$ there are sets N_j with $N \subset \bigcup_{j=1}^{m} N_j$ and mappings $\varphi_j :\ N_j \to K_b\,(1 \leqslant j \leqslant m)$ such that*

($d(.,.)$ is the euclidean distance)

$$\alpha\ d(x,y) \leqslant |N|\, d(\varphi_j(x), \varphi_j(y))$$

for $x, y \in N_j$. Then $\mathcal{H}^s(K_b) > 0$.

To prove Theorem 2 we need

Lemma 1 [6]. *Fix b. If $\psi_i(K_b) \cap \psi_j(K_b) = \emptyset$ for $i \neq j$ then $\mathcal{H}^s(K_b) > 0$*

Proof.

One can use theorem 3 or one can notice that K_b satisfies an open set condition. See [3] ■

Let $\mathcal{C}(K)$ denote the convex hull of a subset K of R^n. Let i_j be natural numbers such that $1 \leqslant i_j \leqslant \ell$. I stands for a finite tuple of such i_j i.e. $I = i_1 \ldots i_m$, and $|I|$ denotes the length of such a tuple. We write for short $\psi_I(\cdot) = \psi_{i_1}(\ldots(\psi_{i_m}(\cdot))\ldots)$.

Given I,J two tuples, we say that I is a curtailment of J (we write $I \leqslant J$) iff $I = i_1 \ldots i_m$; $J = i_1 \ldots i_m\, , j_{m+1} \ldots j_s,\ s \geqslant m$. It is not difficult to see that $\leqslant$ defines a partial order in the set of all finite tuples.

Corollary 1. *Suppose K_b has the following property: there exists a finite family of tuples $\mathcal{F}$ (not necesarily of equal length) such that*

(i) *For any I' , $|I'| \geqslant \max_{J \in \mathcal{F}} |J|$, there exists $I \in \mathcal{F}$ such that $I \leqslant I'$ ($\mathcal{F}$ is secure).*

(ii) *$\psi_I(K_b) \cap \psi_J(K_b) = \emptyset$ for any pair $I, J \in \mathcal{F}$ with $i_1 \neq j_1$*

Then we have $\mathcal{H}^s(K_b) > 0$.

Proof.

Suppose $p \in \psi_i(K_b)$ $1 \leqslant i \leqslant \ell$, say $p \in \psi_1(K_b)$. Choose an index $I' = 1i'_2 \dots i'_m$ such that $|I'| = \max_{J \in \mathcal{F}} |J|$ and $p \in \psi_{I'}(K_b)$. Therefore by (i) we have an index $I \in \mathcal{F}$ such that $I \leqslant I'$ and $p \in \psi_I(K_b) \supset \psi_{I'}(K_b)$ with $I = 1i'_2 \dots i'_{j(j \leqslant m)}$. A similar argument and (ii) shows that $p \notin \psi_i(K_b)$, $1 \leqslant i \neq 1 \leqslant \ell$ *i.e.* $\psi_1(K_b) \cap \psi_i(K_b) = \emptyset$ $\forall i \neq 1$. By lemma 1, $\mathcal{H}^s(K_b) > 0$.■

Proof of theorem 2.

Let $b_o \in R^{n\ell}$. There is no loss of generality if we assume $0 \in K_{b_o}$. Let Q_{b_o} be a cube in $R^{n\ell}$ centered at b_o, $\mathcal{L}^{n\ell}(Q_{b_o}) \leqslant 1$. Therefore $|K_b \cup \{0\}| < c_o$, for some constant c_o if $b \in Q_{b_o}$. This is possible since $K_b = \{b_{i_1} + k_{i_1} A_{i_1} b_{i_2} + k_{i_1} k_{i_2} A_{i_1} A_{i_2} b_{i_3} + \dots : i_j \in \{1, \dots, \ell\}\}$. We will show that $\mathcal{H}^s(K_b) > 0$ for almost all $b_o \in Q_{b_o}$. This will prove our theorem.

Let $Q^j_{b_o} = \{(\ell - 1) - tuples\ (b_1, \dots, \hat{b}_j, \dots, b_\ell) : b \in Q_{b_o}\}$. Fix $\mathcal{O}$ a large natural number and $\mathcal{F}_{\mathcal{O}}$ be the set of all tuples $I = i_1 \dots i_m$ such that $g^{\mathcal{O}} < k_{i_1} \dots k_{i_{m-1}}$ and $k_{i_1} \dots k_{i_m} \leqslant g^{\mathcal{O}}$. Then $\mathcal{F}_{\mathcal{O}}$ has property i of corollary 1 and $K_b = \bigcup_{I \in \mathcal{F}_{\mathcal{O}}} \psi_I(K_b)$ for all b. Let

$$\sum_I (b) := b_{i_1} + k_{i_1} A_{i_1} b_{i_2} + \dots + k_{i_1} \dots k_{i_{m-1}} A_{i_1} \dots A_{i_{m-1}} b_{i_m}$$

$$= \psi_I(0) \tag{2}$$

and

$$T_I(b) := \mathcal{C}(\psi_I(K_b)) = \psi_I(\mathcal{C} K_b)$$

The number of elements of $\mathcal{F}_{\mathcal{O}}$ is not greater than $c_1 \cdot \ell^{\left(\mathcal{O} \frac{\log g}{\log G}\right)}$ where $c_1 \leqslant \ell$. Let $I, J \in \mathcal{F}_{\mathcal{O}}$: $i_1 \neq j_1$. We want to measure $A_{IJ} := \{b : b \in Q_{b_o}$ and $T_I(b) \cap T_J(b) \neq \emptyset\}$. Let $b \in A_{IJ}$. Then

$$\Big|\sum_I (b) - \sum_J (b)\Big| = |\psi_I(0) - \psi_J(0)| \leqslant |\psi_I(K_b \cup \{0\}) \cup \psi_J(K_b \cup \{0\})|$$

$$\leqslant |\psi_I(\mathcal{C}(K_b \cup \{0\})) \cup \psi_J(\mathcal{C}(K_b \cup \{0\}))| \leqslant 2c_o g^{\mathcal{O}} \tag{3}$$

Therefore if $b, b' \in A_{IJ}$ and

$$b = (b_1, \dots, b_{i_1 - 1}, b_{i_1}, b_{i_1 + 1}, \dots, b_\ell)$$
$$b' = (b_1, \dots, b_{i1-1}, b'_{i_1}, b_{i_1+1}, \dots, b_\ell)$$

Then from (3) and (2) and $i_1 \neq j_1$ we get

$$\left| \left[\sum_I (b) - \sum_J (b) \right] - \left[\sum_I (b') - \sum_J (b') \right] \right| = |(b_{i_1} - b'_{i_1}) + \triangle| \leqslant 4c_o g^{\mathcal{O}}$$

with

$$|\triangle| \leqslant \frac{2G}{1-G}|b_{i_1} - b'_{i_1}|$$

Combining this last two inequalities we get $|b_{i_1} - b'_{i_1}| \leqslant c_2 \cdot g^{\mathcal{O}}$ with c_2 depending on k_i and c_o. Therefore projecting along the axis i_1 we have $\mathcal{L}^{n\ell}(A_{IJ}) \leqslant c_3 \mathcal{L}^{n(\ell-1)}(Q^{i_1}_{b_o}) \cdot g^{\mathcal{O}\cdot n} \leqslant c_3 \cdot g^{\mathcal{O}\cdot n}$ with c_3 depending on k_i, c_o and n.

If $b \notin \{\bigcup_{I,J\in\mathcal{F}_{\mathcal{O}}\,;\,i_1\neq j_1} A_{IJ}\}$ then K_b has property ii) stated in Corollary 1 and then $\mathcal{H}^s(K_b) > 0$. But the number of pairs (IJ) $I, J \in \mathcal{F}_{\mathcal{O}}$, $i_1 \neq j_1$, is not greater than $c_1^2 \cdot \ell^{2\mathcal{O}\frac{log\ g}{log\ G}}$. Therefore the set $\{\bigcup_{I,J\in\mathcal{F}_{\mathcal{O}}\,;\,i_1\neq j_1} A_{IJ}\}$ has (outer) measure at most $c_1^2 \cdot c_3 \cdot \left(g^n \cdot \ell^{\left(\frac{2log\ g}{log\ G}\right)}\right)^{\mathcal{O}}$ and this tends to zero by hypothesis if $\mathcal{O} \to \infty$. The theorem follows.■

§2

Let K be a self-similar set i.e. $K = \bigcup_{i=1}^{\ell} \psi_i(K)$ where ψ_i are similitudes of ratio $0 < k_i < 1$, K compact. Recall that $\mathcal{H}^s(K) < \infty$ if s is the similarity dimension (cf [3] pg.122). Assume that the Hausdorff dimension of K is equal to its similarity dimension. Under such hypothesis we want to give some geometrical criterion for $\mathcal{H}^s(K) = 0$. This is proposition 1 below. To prove it we need some tools. The following function $f(\delta)$ has been defined in [7] for K, s as above , with the extra condition $0 < \mathcal{H}^s(K)$: let, for $\delta > 0$,

$$f(\delta) := sup\{\mathcal{H}^s(K \cap C_\delta)/\delta^s \ :\ C_\delta\ is\ a\ convex\ compact\ set\ of\ diameter\ \delta\}$$

In [7] it was proved that $f(\delta) \leqslant 1$ for all $\delta > 0$ (see also §3 of this paper).

We follow the notation of the proof of theorem 2, $\mathcal{F}_{\mathcal{O}}$ being the set of all tuples $I = i_1 \ldots i_m$ such that $g^{\mathcal{O}} < k_{i_1} \ldots k_{i_{m-1}}$ and

$$g^{\mathcal{O}+1} < k_{i_1} \ldots k_{i_m} \leqslant g^{\mathcal{O}} \tag{4}$$

Recall $K = \bigcup_{I\in\mathcal{F}_{\mathcal{O}}} \psi_I(K)$. Write for short $T_I := \mathcal{C}(\psi_I(K))$.

Let $h(\mathcal{O})$:=the maximum number of elements $I_1 \ldots I_q \in \mathcal{F}_{\mathcal{O}}$ such that

$$d(T_{I_i}, T_{I_j}) \leqslant g^{\mathcal{O}}|K| \tag{5}$$

The function $h(\mathcal{O})$, roughly speaking, measures the overlapping of the sets of approximately equal diameter $\psi_I(K), I \in \mathcal{F}_{\mathcal{O}}$.

Proposition 1. *If the Hausdorff and similarity dimension of K are equal to s then:* $\mathcal{H}^s(K) = 0 \iff \overline{\lim_{\mathcal{O}\to\infty}}\ h(\mathcal{O}) = \infty$

Proof.

$\Rightarrow$) Suppose $\overline{\lim_{\mathcal{O}\to\infty}} h(\mathcal{O}) \leqslant \beta$. Therefore if $\mathcal{O}$ is any large natural number then by definition of $h(\mathcal{O})$ any set N such that $N \subset K$, $g^{\mathcal{O}+1} < |N| \cdot |K| \leqslant g^{\mathcal{O}}$ can be decomposed in at most β sets $N_j = \psi_{I_j}(K) \cap N$ with $I_j \in \mathcal{F}_{\mathcal{O}}$ *i.e.* $N \subset \bigcup_1^{\beta} N_j$. Apply theorem 3 with $\varphi_j = \psi_{I_j}^{-1}$.

$\Leftarrow$) Suppose $\overline{\lim_{\mathcal{O}\to\infty}}\ h(\mathcal{O}) = \infty$ and $\mathcal{H}^s(K) > 0$. Then we are in condition to define the function $f(\delta)$ as above. Also observe that $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K)) = 0$ for $1 \leqslant i \neq j \leqslant \ell$ and therefore $\mathcal{H}^s(\psi_I(K) \cap \psi_J(K)) = 0$ *if* $I \neq J \in \mathcal{F}_{\mathcal{O}}$.

By definition of $h(\mathcal{O})$ there exits $I_1 \dots I_{h(\mathcal{O})}$ such that $d(T_{I_i}, T_{I_j}) \leqslant g^{\mathcal{O}}|K|$.

Let $C_{\delta_o} = \{x : d(x, T_{I_1}) \leqslant 2g^{\mathcal{O}}|K|\}$.

Therefore (by (4)) C_{δ_o} contains $T_{I_1}, \dots, T_{I_{h(\mathcal{O})}}$ and has diameter $\delta_o \leqslant 5 \cdot g^{\mathcal{O}} \cdot |K|$. Therefore, since $\mathcal{H}^s(\psi_{I_j}(K) \cap \psi_{I_i}(K)) = 0$ we have

$$1 \geqslant f(\delta_o) \geqslant \frac{\mathcal{H}^s(K \cap C_{\delta_o})}{\delta_o^s} \geqslant \frac{h(\mathcal{O})\{\min_{i=1\dots h(\mathcal{O})} \mathcal{H}^s(\psi_{I_i}(K))\}}{5^s|K|^s g^{\mathcal{O}s}} \geqslant (by 4) \geqslant$$
$$\geqslant \frac{h(\mathcal{O})\mathcal{H}^s(K) g^s}{5^s|K|^s}$$

This is absurd taking $\mathcal{O} \to \infty$ ■

§3

In this section we assume K to be a *self* − *similar set*, i.e. $K = \bigcup_{i=1}^{\ell} \psi_i(K)$, ψ_i a similitude of ratio $0 < k_i < 1$. In [7] a method was given which permits to approximate the Hausdorff measure of $\mathcal{H}^s(K)$ assuming that:

(i) $0 < \mathcal{H}^s(K) < \infty$

(ii) K has property A (see below)

(iii) $\sum_{i=1}^{\ell} k_i^s = 1$ i.e. the Hausdorff dimension of K is equal to its similarity dimension.

In this section we want to generalize this result by dropping condition iii). Recall that for a self similar set K satisfying conditions i) and iii) one must have $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K)) = 0 \ \ \forall i \neq j \ ; \ 1 \leqslant i, j \leqslant \ell$.

Theorem 4. *Assume K to be a self similar set and* $0 < \mathcal{H}^s(K) < \infty$. *Define*

$$f(\delta) := \sup_{\substack{C_\delta \text{ convex} \\ \text{compact} \\ \text{of diameter } \delta > 0}} \mathcal{H}^s(K \cap C_\delta)/\delta^s$$

Then $f(\delta) \leqslant 1 \qquad \forall \delta > 0$.

Proof.

If the Hausdorff dimension of K is zero then K has to be a point (if K had two points at least then being K a self similar set defined by similitudes then it should have infinite points. This would contradict $\mathcal{H}^o(K) < \infty$). The theorem is true in this case.

Therefore we assume $s > 0$. Suppose $\frac{\mathcal{H}^s(K\cap C_\delta)}{\delta^s} = \frac{\mathcal{H}^s(K\cap C_\delta)}{|C_\delta|^s} \geqslant \beta > 1$ for some C_δ convex and compact. Moreover one can assume $|K \cap \partial C_\delta| = |C_\delta| = \delta$. Let $A_n = \bigcup_{|I|\geqslant n} \psi_I(K \cap C_\delta)$. Therefore $A_{n+1} \subset A_n$. Let $A = \bigcap_{i=1}^{\infty} A_i$. For A we have $\mathcal{H}^s(A) > 0$ or $\mathcal{H}^s(A) = 0$.

Assume $\mathcal{H}^s(A) > 0$. Let n_o be such that $\mathcal{H}^s(A_{n_o}/A) < \varepsilon$ and observe that $\{\psi_I(K \cap C_\delta)\}$, $|I| \geqslant n_o$ is a Vitali family for A. Since for any countable disjoint subfamily we have

$$
\begin{aligned}
\sum |\psi_I(K \cap C_\delta)|^s &\leqslant \frac{1}{\beta}\sum \mathcal{H}^s(\psi_I(K \cap C_\delta)) \\
&\leqslant \frac{\mathcal{H}^s(K)}{\beta} < \infty, \qquad (6)
\end{aligned}
$$

there exists a disjoint countable subfamily indexed by $\mathcal{F}$ such that $\mathcal{H}^s(A/\bigcup_{I\in\mathcal{F}} \psi_I(K \cap C_\delta)) = 0$ (cf.[3], pg.11). Besides, we can assume

$$
\beta\mathcal{H}^s(A) - \varepsilon \leqslant \beta\sum_{I\in\mathcal{F}} |\psi_I(K \cap C_\delta)|^s \leqslant \sum_{I\in\mathcal{F}} \mathcal{H}^s(\psi_I(K \cap C_\delta)) \leqslant \mathcal{H}^s(A_{n_o}) \leqslant \varepsilon + \mathcal{H}^s(A)
$$

This is absurd if ε is sufficiently small.

If $\mathcal{H}^s(A) = 0$, let c be a fixed positive number such that $0 < \delta < c$ and

$$
K \subset [K \cap C_\delta]_c = \{x : d(x, K \cap C_\delta) \leqslant c\} \qquad (7)
$$

For any $\varepsilon > 0$, let $n_o = n_o(\varepsilon)$ be a natural number such that $\mathcal{H}^s(A_{n_o}) \leqslant \varepsilon$. Then

$$
\mathcal{H}^s\left(\bigcup_{I\in\mathcal{F}'} \psi_I(K \cap C_\delta)\right) \leqslant \varepsilon \qquad (8)
$$

where $\mathcal{F}'$ denotes a maximal family of indices I of lenght n_o chosen in the following way: first choose I_o ($|I_o| = n_o$) such that $|\psi_{I_o}(K \cap C_\delta)| = \max_{|I|=n_o} |\psi_I(K \cap C_\delta)|$. From all the indices I ($|I| = n_o$) such that $\psi_I(K \cap C_\delta) \cap \psi_{I_o}(K \cap C_\delta) = \emptyset$ choose one such that its diameter is maximum, call this index I_1 and so on. Using (7) we see that

$$
K \subset \bigcup_{|I|=n_o} \psi_I([K \cap C_\delta]_c) \qquad (9)
$$

and by the way $\mathcal{F}'$ is chosen (recall $0 < \delta < c$)

$$K \subset \bigcup_{I \in \mathcal{F}'} \psi_I([K \cap C_\delta]_c) \tag{10}$$

Moreover

$$\frac{\mathcal{H}^s(\psi_I(K \cap C_\delta))}{|\psi_I(K \cap C_\delta)|^s} \geqslant \beta > 1 \qquad \forall I \in \mathcal{F}' \tag{11}$$

Using this last formula, (8) and (10), we get

$$\begin{aligned} \frac{\varepsilon}{\beta} \geqslant \sum_{I \in \mathcal{F}'} |\psi_I(K \cap C_\delta)|^s \geqslant \left(\frac{\delta}{3c}\right)^s \left(\sum_{I \in \mathcal{F}'} |\psi_I([K \cap C_\delta]_c)|^s\right) \geqslant \\ \left(\frac{\delta}{3c}\right)^s \mathcal{H}^s_{3c\{max\, k_i\}^{n_o}}(K) \end{aligned} \tag{12}$$

It follows that $\mathcal{H}^s(K) = 0$, an absurd.■

Corollary 2. *Assume the hypothesis of theorem 4. Then:*

(i) $f(\delta) \leqslant 1 \qquad \forall \delta > 0$

(ii) $\overline{lim}_{\delta \to 0} f(\delta) = 1$

(iii) $f(\delta)$ *is continuous from the right*

Proof.

(ii) follows from elementary density bounds (see [3], p. 24).

(iii) From Blaschke selection theorem follows that for any δ there is a compact, convex set of diameter δ, C_δ, such that $f(\delta) = \frac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s}$. Notice that $f(\delta)\delta^s$ is non decreasing. Using these last observations and the continuity of $\mathcal{H}^s(\cdot)$ we get (iii) (cf. [7] §1). ■

Property A. *We say K has property A if there exists $a > 0$ such that for any sphere $B_{r_1}(x)$ with $r_1 < a$ there exists an expanding similitude $\psi(z)$ with contraction ratio $\xi \geqslant 1$ and an index i_o, $1 \leqslant i_o \leqslant \ell$, such that*

$$\psi(B_{r_1}(x) \cap K) \subset \psi_{i_o}(K)$$

Property A is indeed quite strong.

Proposition 2. *If K is a self-similar set with property A and Hausdorff dimension s then K is an s-set i.e. $0 < \mathcal{H}^s(K) < \infty$.*

Proof.

As ψ_i are similitudes and $K = \overset{\ell}{\underset{i=1}{\cup}} \psi_i(K)$, by [2] page 550 we get $\mathcal{H}^s(K) < \infty$.

That $0 < \mathcal{H}^s(K)$ follows from the fact that property A implies the hypothesis of theorem 3. This assertion is proved as follows. Let N be any subset of K of diameter less than a of property A. Then by this property there exists an index i_o, $1 \leqslant i_o \leqslant \ell$ and ψ an expansive similitude such that $\psi(N) \subset \psi_{i_o}(K)$. Taking $\psi_{i_o}^{-1}$ in this inclusion one gets $\psi_{i_o}^{-1}\psi(N) \subset K$. If $|\psi_{i_o}^{-1}\psi(N)| < a$ then proceed as before with $\psi_{i_o}^{-1}\psi(N)$ as N. After a finite number of steps one gets

$$\psi_{i_n}^{-1}\psi' \ldots \psi_{i_o}^{-1}\psi(N) \subset K$$

$$a \leqslant |\psi_{i_n}^{-1}\psi' \ldots \psi_{i_o}^{-1}\psi(N)|$$

where $1 \leqslant i_j \leqslant \ell$ and $\psi', \ldots, \psi$ are expansive similitudes. Therefore one can define $\varphi(x) = \psi_{i_n}^{-1}\psi' \ldots \psi_{i_o}^{-1}\psi(x) : N \to K$. It is easily chequed that $a\, d(x,y) \leqslant |N|\, d(\varphi(x), \varphi(y)$ for all x, y in N.■

The following is a corollary of theorem 4.

Corollary 3. *If K has property A and $0 < \mathcal{H}^s(K) < \infty$ then $f(\delta) = 1$ for some δ_o such that $a \leqslant \delta_o \leqslant |K|$*

Proof.

Let δ be such that $0 < \delta < a$. We want to show that $f(\delta) \leqslant f(\delta\xi\{min\ k_i\}^{-1})$. For this let C_δ be a convex compact set such that $f(\delta) = \dfrac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s}$. Obviously $C_\delta \subset B_{r_1}(x)$ for some sphere with radius $r_1 < a$. From this and property A we get

$$\psi(C_\delta \cap K) \subset \psi(B_{r_1}(x) \cap K) \subset \psi_{i_o}(K)$$

and therefore $\psi_{i_o}^{-1}(\psi(C_\delta \cap K)) = \psi_{i_o}^{-1}(\psi(C_\delta)) \cap \psi_{io}^{-1}(\psi(K)) \subset K$. Intersecting this last expression with $\psi_{i_o}^{-1}(\psi(C_\delta))$ we get $\psi_{i_o}^{-1}(\psi(C_\delta \cap K)) \subset K \cap \psi_{i_o}^{-1}(\psi(C_\delta))$ and therefore

$$\mathcal{H}^s(\psi_{i_o}^{-1}(\psi(K \cap C_\delta))) = \xi^s k_{io}^{-s}\mathcal{H}^s(K \cap C_\delta) \leqslant \mathcal{H}^s(K \cap \psi_{i_o}^{-1}(\psi(C_\delta)))$$

i.e.

$$f(\delta) = \frac{\mathcal{H}^s(K \cap C_\delta)}{\delta^s} \leqslant \frac{\mathcal{H}^s(K \cap \psi_{i_o}^{-1}(\psi(C_\delta)))}{\delta^s \xi^s k_{i_o}^{-s}} \leqslant f(\delta\xi k_{i_o}^{-1})$$

(notice that $\psi_{i_o}^{-1}(\psi(C_\delta))$ is convex, compact, of diameter $\delta\xi k_{io}^{-1}$). This proves the assertion. From this, i) and ii) of corollary 2 we get that $\sup\limits_{[a,\infty)} f(\delta) = 1$ and therefore $\sup\limits_{[a,|K|]} f(\delta) = 1$ because $f(\delta) = \dfrac{\mathcal{H}^s(K)}{\delta^s}$ for $\delta > |K|$. Since $f(\delta)$ is continuous from the right and $f(\delta) \cdot \delta^s$ is non decreasing we get $f(\delta_o) = 1$ for some $a \leqslant \delta_o \leqslant |K|$.■

Definition. *K is $\varepsilon -$ discrete if $0 < \mathcal{H}^s(K) < \infty$ and there exist (non void) sets $K_1, \ldots, K_q$, $q = q(\varepsilon)$, such that:*

(i) *$K = \bigcup_{i=1}^{q} K_i$ and $\mathcal{H}^s(K_i \cap K_j) = 0$ for $i \neq j$*

(ii) *$|K_i| \leqslant \varepsilon \; \forall i$*

(iii) *the numbers $\alpha_i := \frac{\mathcal{H}^s(K_i)}{\mathcal{H}^s(K)}$, $i = 1 \ldots q$, can be calculated.*

The reader should observe that if a self similar set K is such that $0 < \mathcal{H}^s(K) < \infty$ and its Hausdorff dimension s is equal to its similarity dimension then K is ε -discrete. To see this, just take as K_i in the above definition the sets $\psi_I(K)$, $|I| = |i_1 \ldots i_{no}| = n_o$ i.e. $q = \ell^{n_o}$. Properties i), ii) are easily verified and $\dfrac{\mathcal{H}^s(K_i)}{\mathcal{H}^s(K)} = \dfrac{\mathcal{H}^s(\psi_I(K))}{\mathcal{H}^s(K)} = k_{i_1}^s \ldots k_{i_{no}}^s$.

Therefore the Sierpinski set and the Cantor set are ε-discrete. See §4 this paper for other examples.

If K is $\varepsilon -$ *discrete* with $\varepsilon < a_o$ then we can obtain approximations of $f(\delta)$ on $[a_o, |K|]$. This is theorem 5 below. Later we shall use $a_o = a$, with a of property A. For this theorem we need some definitions that **only** assume that K is $\varepsilon -$ *discrete* with $\varepsilon < a_o$.

Let $\mathcal{P}$ be the family of all non void sets $\{i_1, \ldots, i_t\}$ with $1 \leqslant i_1 < \cdots < i_t \leqslant q$. If $p \in \mathcal{P}$ **define** $G(p) = |\bigcup_{i \in p} K_i|$. It is clear that $G(\mathcal{P})$ is a finite set of non negative numbers and there exists some $d \in G(\mathcal{P})$ such that $d \leqslant \varepsilon < a_o$ (by ii) of the above definition). **Define** $\widetilde{\mathcal{U}}$ on $G(\mathcal{P})$ in the following way: $\widetilde{\mathcal{U}}(d) := \max_{\substack{p \text{ such} \\ \text{that} \\ G(p)=d}} (\sum_{i \in p} \alpha_i)$.

Next we define $\mathcal{U}(\delta)$, for $\delta \geqslant a_o$, ($\mathcal{U}(\delta) \cdot \mathcal{H}^s(K)$ will be an approximation of $f(\delta) \cdot \delta^s$). **Define** $\mathcal{U}(\delta) := \max_{\substack{d \leqslant \delta \\ d \in G(\mathcal{P})}} \widetilde{\mathcal{U}}(d)$. Easy consequences of the definition of $\mathcal{U}(\delta)$ are that it is a non decreasing function and that $\mathcal{U}(\delta)$ is constant on the intervals $[a_o, a_1), [a_1, a_2), \ldots, [a_w, \infty)$ with $a_w \leqslant |K|$ where $a_1 < \cdots < a_w$ are points of $G(\mathcal{P})$ and $\mathcal{U}(\delta) = 1$ for $\delta \in [a_w, \infty)$ by i) of the above definition. We also recall that $f(\delta) \cdot \delta^s$ is non decreasing and continuous from the right and that s does not coincide necesarily with the similarity dimension.

Theorem 5. *Assume that K is $\varepsilon -$ discrete with $\varepsilon < a_o$. Then*

(i) *$\mathcal{U}(\delta) \leqslant \frac{f(\delta)\delta^s}{\mathcal{H}^s(K)} \leqslant \mathcal{U}(\delta + 2\varepsilon)$ for $\delta \geqslant a_o$*

(ii) *Suppose that K is $\varepsilon -$ discrete for each $\varepsilon > 0$. Then*

$$\left(\sup_{\delta \in [a_o, |K|]} \frac{\mathcal{U}(\delta + 2\varepsilon)}{\delta^s} - \sup_{\delta \in [a_o, |K|]} \frac{\mathcal{U}(\delta)}{\delta^s} \right) \to 0 \; for \; \varepsilon \to 0.$$

Proof.

We assume $s > 0$. If $s = 0$ then K must be a point and the theorem is trivial.

(i) Fix δ. By Blaschke selection theorem $f(\delta)\cdot\delta^s = \mathcal{H}^s(K\cap C_\delta)$ for some C_δ convex compact. Let $d\in G(\mathcal{P}), d\leqslant\delta$. Then $\widetilde{\mathcal{U}}(d)\cdot\mathcal{H}^s(K) = \max\limits_{G(p)=d}(\sum\limits_{i\in p}\alpha_i)\mathcal{H}^s(K) = \max\limits_{G(p)=d}(\sum\limits_{i\in p}\mathcal{H}^s(K_i)) \leqslant f(\delta)\cdot\delta^s$ This last inequality because $|\bigcup\limits_{\substack{i\in p\\ G(p)=d}} K_i| = d\leqslant\delta$ and $f(\delta)\cdot\delta^s$ is non decreasing. From this the left hand inequality of i) follows. To prove the right hand inequality, suppose that C_δ is such that $f(\delta)\cdot\delta^s = \mathcal{H}^s(K\cap C_\delta)$ and let $p_o\in\mathcal{P}$ be the set of indices i such that K_i intersects C_δ. As $|K_j|\leqslant\varepsilon\ \forall j$ we get $G(p_o) = |\bigcup\limits_{i\in p_o} K_i| \leqslant \delta+2\varepsilon$ and therefore by the definition of $\mathcal{U}(\delta)$ we have

$$\begin{aligned}
f(\delta)\cdot\delta^s &= \mathcal{H}^s(K\cap C_\delta) \leqslant \mathcal{H}^s(K\cap(\bigcup_{i\in p_o} K_i)) \\
&= \left(\sum_{i\in p_o}\alpha_i\right)\mathcal{H}^s(K) \leqslant \mathcal{U}(G(p_o))\mathcal{H}^s(K) \\
&\leqslant \mathcal{U}(\delta+2\varepsilon)\mathcal{H}^s(K)
\end{aligned}$$

proving the other inequality of i).

(ii) From i) we get $\sup\limits_{\delta\in[a_o\,,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \sup\limits_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} \leqslant$
$\leqslant \sup\limits_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{(\delta+2\varepsilon)^s}\sup\limits_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s} \leqslant \sup\limits_{\delta\in[a_o+2\varepsilon,|K|+2\varepsilon]}\frac{\mathcal{U}(\delta)}{\delta^s}\sup\limits_{\delta\in[a_o,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s}$. But $\mathcal{U}(\delta)=1$ if $\delta\geqslant|K|$. Then $\sup\limits_{\delta\in[a_o+2\varepsilon,|K|+2\varepsilon]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \sup\limits_{\delta\in[a_o,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}$. From this and the above inequality we get $0 \leqslant \sup\limits_{\delta\in[a_o\,,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} - \sup\limits_{\delta\in[a_o\,,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}$
$\leqslant \sup\limits_{\delta\in[a_o\,,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s}(\sup\limits_{\delta\in[a_o\,,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s} - 1) \leqslant (by i) \leqslant$
$\leqslant \sup\limits_{\delta\in[a_o\,,|K|]}\frac{f(\delta)}{\mathcal{H}^s(K)}(\sup\limits_{\delta\in[a_o\,,|K|]}\frac{(\delta+2\varepsilon)^s}{\delta^s} - 1)$ which proves ii).■

Finally, we show how to obtain bounds for $\mathcal{H}^s(K)$. Assume

(i) K has property A

(ii) K is $\varepsilon - \textit{discrete}$ with $\varepsilon < a$, a of property A

Then, from theorem 5 and corollaries 2 i) and 3 we get

$$\sup_{\delta\in[a,|K|]}\frac{\mathcal{U}(\delta)}{\delta^s} \leqslant \frac{1}{\mathcal{H}^s(K)} \leqslant \sup_{\delta\in[a,|K|]}\frac{\mathcal{U}(\delta+2\varepsilon)}{\delta^s} \tag{13}$$

If K is $\varepsilon - \textit{discrete}$ for any $\varepsilon > 0$ we get from theorem 5 ii) that in (13) the two suprema tend to $1/\mathcal{H}^s(K)$, yielding the algorithm. For an example see §4.

We finally point out that in practice, knowing α_i in the definition of the $\varepsilon -$ *discreteness* requires the knowledge of s, the dimension of K. We prove a lemma showing how this dimension can be obtained in some cases. We **assume** that $0 < \mathcal{H}^s(K) < \infty$, which in practice will follow from theorem 3 and [2].

Lemma 2. *Let* $K = \bigcup_{i=1}^{\ell} \psi_i(K)$. *Assume* $\psi_i(K) \cap \psi_j(K)$, $i \neq j$, *is a disjoint union of sets* $K_t^{i,j}$ $t = 1, \ldots, n(i,j)$ *where* $K_t^{i,j} = \psi_t^{ij}(K)$ *and* ψ_t^{ij} *is a similitude with contraction ratio* ξ_t^{ij}. *Assume also* $\mathcal{H}^s(\psi_i(K) \cap \psi_j(K) \cap \psi_k(K)) = 0$ *for any triple* (i,j,k), $i \neq j \neq k \neq i$. *Then the Hausdorff dimension* s *verifies the following relation*

$$1 = \sum_{i=1}^{\ell} \xi_i^s - \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(i,j)} (\xi_t^{ij})^s \right)$$

(here ξ_i *is the contraction ratio of* ψ_i*)*

Proof.

From the hipothesis we know that

$$\mathcal{H}^s(\psi_i(K)) = \mathcal{H}^s \left(\psi_i(K) / (\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(i,j)} \mathcal{H}^s(K_t^{i,j}) \tag{14}$$

and therefore

$$\xi_i^s \mathcal{H}^s(K) = \mathcal{H}^s \left(\psi_i(K) / (\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(i,j)} (\xi_t^{ij})^s \mathcal{H}^s(K) \tag{15}$$

Also from the hypothesis we get

$$\mathcal{H}^s(K) = \sum_{i=1}^{\ell} \mathcal{H}^s \left(\psi_i(K) / (\bigcup_{\substack{j=1 \\ j\neq i}}^{\ell} \psi_j(K)) \right) + \sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(i,j)} \mathcal{H}^s(K_t^{i,j}) \tag{16}$$

Putting (15) in (16) we get

$$\mathcal{H}^s(K) = \left(\sum_{i=1}^{\ell} \xi_i^s \right) \cdot \mathcal{H}^s(K) - \left(\sum_{i=1}^{\ell} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \mathcal{H}^s(K) + \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \mathcal{H}^s(K)$$

As $0 < \mathcal{H}^s(K) < \infty$ and $\xi_t^{ij} = \xi_t^{ji}$,

$$1 = \sum_{i=1}^{\ell} \xi_i^s - \left(\sum_{\substack{i,j=1 \\ j<i}}^{\ell} \sum_{t=1}^{n(ij)} (\xi_t^{ij})^s \right) \blacksquare$$

§4 *Example* (*Sierpinski set with overlapping*) : Let $\triangle$ be an equilateral triangle of side 1 and let p_1, p_2, p_3 be its vertices (fig.1). Let $\triangle_1$, $\triangle_2$, $\triangle_3$, be three smaller equilateral triangles inside $\triangle$ and touching p_1, p_2, p_3, respectively. We define $\psi_i(x)$; $i = 1,2,3$, as the similitudes that transform $\triangle$ onto $\triangle_i$ (without rotation). We assume that all contracting ratios are equal to ξ , $0 < \xi < 1$. Notice that $\mathcal{C}(K) = \triangle$ ($\mathcal{C}$ = convex hull). We write for short: $\triangle_I = \psi_I(\triangle) = \psi_I(\mathcal{C}(K))$ and assume that ξ satisfies the following equation:

$$(17) \qquad 2\xi - \xi^n = 1 \; ; \; n \geqslant 3 \; , \; n \text{ an integer.}$$

We will see that we can apply the previous theory to K i.e. we will prove that K has property A and therefore by proposition 2, K is an $s-set$. We will also show that K is $\varepsilon - discrete$. To prove all these assertions one needs lemmas 3 and 4 and the following discussion.

It is easy to prove that there is a unique ξ , $0 < \xi < 1$ satisfying (17), for the polinomial $2x - x^n - 1$ has only two roots in $[1/2, 1]$, being 1 one of them.

Let $0 < \xi_3 < 1$ where ξ_3 satisfies (17) with $n = 3$. Then $\xi_3 = \frac{\sqrt{5}-1}{2}$. It is easy to prove that if $0 < \xi < 1$ and satisfies (17) then $1/2 < \xi \leqslant \xi_3$. Therefore ξ must also satisfy

$$(18) \qquad 1 \geqslant \xi + \xi^{n-1}$$

with equality only if n=3 and

$$(19) \qquad 1/2 < \xi \leqslant \xi_3 = \frac{\sqrt{5}-1}{2} < 2/3$$

From (19) it is seen that $\triangle_1 \cap \triangle_2 \cap \triangle_3 = \emptyset$ and $\triangle_i \cap \triangle_j \neq \emptyset$ for $i \neq j$. Notice that by (17), $\triangle_i \cap \triangle_j$ is an equilateral triangle of side ξ^n. From the construction we get

$$(20) \qquad \triangle_i \cap \triangle_j = \triangle_{\underbrace{ij\ldots j}_{n}} = \triangle_{\underbrace{ji\ldots i}_{n}}$$

and from this it follows that

$$(21) \qquad \psi_{\underbrace{ij\ldots j}_{n}}(x) = \psi_{\underbrace{ji\ldots i}_{n}}(x)$$

Fig. 4 a) shows K for $\xi = \xi_3$. Our aim is lemma 4 below which will be used to prove the mentioned properties of K. For the following lemma it is useful to look at figures 3, 2 a), b), c).

Lemma 3. *For any index I beginning with 1 of length $q = m(n-1)+1$, $m \geqslant 1$, such that $\triangle_I \cap \triangle_{\underbrace{12\ldots2}_{n}} \neq \emptyset$, we have a) or b) or c):*

a) if $\triangle_I \subset \triangle_{\underbrace{12\dots 2}_{n}}$ then there exists $J = \underbrace{12\dots 2}_{n} j_{n+1}\cdots j_q$ such that $\psi_I(x) = \psi_J(x)$.

b) if $\triangle_I \not\subseteq \triangle_{\underbrace{12\dots 2}_{n}}$ and $\triangle_I \cap \triangle_{\underbrace{12\dots 2}_{n}}$ is not a point then there exists $J = \underbrace{12\dots 2}_{n} j_{n+1}\cdots j_q$ such that one side of $\triangle_I$ and one side of $\triangle_J$ are on the same line and $\triangle_I \cap \triangle_J = \triangle_I \cap \triangle_{\underbrace{12\dots 2}_{n}}$ is an equilateral triangle of side of length $\xi^{(m+1)(n-1)+1}$ (Observe that there are only two possibilities, see fig.2 a, b)

c) $\triangle_I \cap \triangle_{\underbrace{12\dots 2}_{n}}$ is a point, fig. 2 c).

The same holds interchanging 1 with 2.

Only a) will be used but b) and c) are needed in the proof.

Proof.

The proposition is true if $m = 1$ because the only sets $\triangle_I$ with $|I| = n$, I beginning with 1 that can touch $\triangle_{\underbrace{12\dots 2}_{n}}$ are, by (18), $\triangle_{12\dots 22}$, $\triangle_{12\dots 21}$, $\triangle_{12\dots 23}$ and in case n=3 also $\triangle_{112}$, $\triangle_{132}$ (fig. 3). The last two satisfy c). The first one satisfies a) and the others satisfy b) with $J = \underbrace{12\dots 2}_{n}$.

Assume that the lemma is true for m. Take an index I beginning with 1 and of lenght $(m+1)(n-1)+1$ with $\triangle_I$ touching $\triangle_{\underbrace{12\dots 2}_{n}}$. Then $\triangle_I \subset \triangle_{I'}$ where $I'I'' = I$, I' of lenght m(n-1)+1 (I' is a curtailment of I). For m the lemma was assumed, therefore if $\triangle_{I'} \subset \triangle_{\underbrace{12\dots 2}_{n}}$ then there exists $J' = \underbrace{12\dots 2}_{n} j_{n+1}\cdots j_{m(n-1)+1}$ such that $\psi_{J'}(x) = \psi_{I'}(x)$ and therefore $\psi_{J'I''}(x) = \psi_{I'I''}(x) = \psi_I(x)$.

Now, assume $\triangle_{I'} \not\subseteq \triangle_{\underbrace{12\dots 2}_{n}}$. If c) is true for I' then $\triangle_I \cap \triangle_{\underbrace{12\dots 2}_{n}}$ is at most a point. If b) is true for I' then there exists J' with the mentioned properties. We assume that $\triangle_{I'}$ and $\triangle_{J'}$ are located as in fig. 2 a. As $\triangle_{I'} \cap \triangle_{J'}$ is an equilateral triangle of side $\xi^{(m+1)(n-1)+1}$ we get (by 17) that $\triangle_{I'} \cap \triangle_{J'} = \triangle_{I'} \cap \triangle_{\underbrace{12\dots 2}_{n}} = \triangle_{I'\underbrace{2\dots 2}_{n-1}} = \triangle_{J'\underbrace{3\dots 3}_{n-1}}$.

Therefore $\triangle_{I'I''} = \triangle_I$ must touch $\triangle_{I'\underbrace{2\dots 2}_{n-1}}$. But the only sets $\triangle_{I'L}$, $|I'L| = (m+1)(n-1)+1$ that could touch $\triangle_{I'\underbrace{2\dots 2}_{n-1}}$ are (by 18) $\triangle_{I'2\dots 22}$, $\triangle_{I'2\dots 23}$, $\triangle_{I'2\dots 21}$, $\triangle_{I'2\dots 212}$, $\triangle_{I'2\dots 232}$. Therefore I must be one of the above. If I is the first then as $\triangle_{I'\underbrace{2\dots 2}_{n-1}} = \triangle_{J'\underbrace{3\dots 3}_{n-1}}$ we get $\psi_{I'2\dots 2}(x) = \psi_{J'3\dots 3}(x)$ and a) follows. If I is one of the last two then c) is true and if (for example) $I'\underbrace{2\dots 23}_{n-1} = I$ then it is seen that b) is true with $J = J'\underbrace{3\dots 3}_{n-1}$. ■

Lemma 4. (a) $\triangle_1 \cap \triangle_2 \cap K = \psi_{\underbrace{12\dots 2}_{n}}(K) = \psi_{\underbrace{21\dots 1}_{n}}(K)$

(b) $\triangle_1 \cap \triangle_3 \cap K = \psi_{\underbrace{13\dots 3}_{n}}(K) = \psi_{\underbrace{31\dots 1}_{n}}(K)$

(c) $\triangle_2 \cap \triangle_3 \cap K = \psi_{\underbrace{23\dots 3}_{n}}(K) = \psi_{\underbrace{32\dots 2}_{n}}(K)$

Proof.

We prove the first proposition, the others follow from symmetry. Obviously $\psi_{\underbrace{12\dots 2}_{n}}(K) \subset \triangle_1 \cap \triangle_2 \cap K$. Let $p \in (\mathrm{int}(\triangle_1 \cap \triangle_2)) \cap K$, then there exists an index I beginning with 1 (or 2) of lenght m(n-1)+1 with m great enough such that $p \in \psi_I(K)$ and $p \in \triangle_I \subset \triangle_1 \cap \triangle_2$. Recalling (20), by Lemma 3 a) we get $p \in \psi_{\underbrace{12\dots 2}_{n}}(K)$ (or $p \in \psi_{\underbrace{21\dots 1}_{n}}(K)$ respectively). Therefore using (21) we get a).

If $p \in \partial(\triangle_1 \cap \triangle_2)$ then obvioulsy $p \in \psi_{\underbrace{12\dots 2}_{n}}(K)$. ■

Lemma 5. *K has property A.*

Proof.

Let $q_1 = \psi_{\underbrace{12\dots 2}_{n}}(p_3)$ and $q_2 = \psi_{\underbrace{12\dots 2}_{n}\underbrace{1\dots 1}_{n-1}}(p_3)$ (see fig.3). We want to show that the 'shape' of K near q_1 is the same as near q_2. Notice that from (18) the only sets $\triangle_I$, $|I| = n$ that touch q_1 are $\triangle_{12\dots 2} = \triangle_{21\dots 1}$,$\triangle_{12\dots 23}$,$\triangle_{21\dots 13}$ and eventually $\triangle_{132}$ and $\triangle_{231}$ if n=3. We assume for simplicity that $n > 3$ and let the reader fill the gaps if n=3. Therefore, if ε is small enough, from $K = \cup\{\psi_I(K) : |I| = n\}$ we get,

(22)
$$K \cap B_\varepsilon(q_1) = (\psi_{\underbrace{12\dots 2}_{n}}(K) \cup \psi_{\underbrace{12\dots 23}_{n}}(K) \cup \psi_{\underbrace{21\dots 13}_{n}}(K)) \cap B_\varepsilon(q_1)$$

By lemma 4 c) if ε is small enough we get $\psi_2(K) \cap B_{\varepsilon\xi^{1-n}}(\psi_2(p_3)) \subset \psi_{\underbrace{32\dots 2}_{n}}(K)$ and applying $\psi_{\underbrace{12\dots 2}_{n-1}}(x)$ to this last relation we get $\psi_{\underbrace{12\dots 22}_{n}}(K) \cap B_\varepsilon(q_1) \subset \psi_{\underbrace{12\dots 23}_{n}\underbrace{2\dots 2}_{n-1}}(K) \subset \psi_{\underbrace{12\dots 23}_{n}}(K)$. Applying this to (22) we get

$$K \cap B_\varepsilon(q_1) = (\psi_{\underbrace{12\dots 23}_{n}}(K) \cup \psi_{\underbrace{21\dots 13}_{n}}(K)) \cap B_\varepsilon(q_1) \tag{23}$$

Also q_2 only touches $\triangle_{12\ldots 2} = \triangle_{21\ldots 1}$ and $\triangle_{12\ldots 21}$ of sets $\triangle_I$, $|I| = n$. Therefore $K \cap B_\varepsilon(q_2) = (\psi_{\underbrace{12\ldots 22}_{n}}(K) \cup \psi_{\underbrace{12\ldots 21}_{n}}(K)) \cap B_\varepsilon(q_2)$. Using this last formula, (23) and the fact that $\psi_{\underbrace{12\ldots 23}_{n}}(x) + (q_2 - q_1) = \psi_{\underbrace{12\ldots 21}_{n}}(x)$; $\psi_{\underbrace{21\ldots 13}_{n}}(x) + (q_2 - q_1) = \psi_{\underbrace{12\ldots 22}_{n}}(x)$ we get that there exists $\varepsilon > 0$ such that

$$K \cap B_\varepsilon(q_2) = (K \cap B_\varepsilon(q_1)) + (q_2 - q_1) \text{ and } B_\varepsilon(q_2) \subset \triangle_1 \tag{24}$$

Using this last assertion one can prove that K has property A as follows.

Let $a << \varepsilon$, a will be the parameter of property A. We assume that $B_a(x)$ is a ball touching $\triangle_1 \cap \triangle_2$ (if $B_a(x)$ touches only $\triangle_1$ but neither $\triangle_2$ nor $\triangle_3$ then take $i_0 = 1$, $\psi =$ identity). Assume $B_a(x) \subset B_\varepsilon(q_1)$ then use (24) , lemma 4 a) and take $\psi(x) = x + (q_2 - q_1)$, $i_0 = 1$ in property A to get $\psi(B_a(x) \cap K) \subset \psi_1(K)$.

If $B_a(x) \not\subseteq B_\varepsilon(q_1)$ then use lemma 4 a) $\psi = identity$ and $i_0 = 1$ or 2, depending where x is placed.■

Because of proposition 2, K is an s-set, s the Hausdorff dimension of K.

Next we calculate s. Apply lemma 2 to get that s must satisfy

$$\xi^s - \xi^{ns} = 1/3 \tag{25}$$

Then, $z = \xi^s$ is a root of $1/3 = z - z^n$, $0 < z < 1$, and $s = \frac{\log z}{\log \xi}$. This last polynomial has two roots r_1 , r_2, in (0,1): $0 < r_1 < \xi_3 < r_2 < 1$. Since K contains a segment, we have $s \geqslant 1$. From (17) and (25) we get for $s = 1$, $\xi = 2/3$, which is in contradiction with (19). Then $s > 1$. From $z = \xi^s$ and (19) it follows that $z = r_1$ and $s < 2$.

K is indeed ε-discrete for any $\varepsilon > 0$. We prove this fact for $n = 3$ and a similar argument works for the other cases.

Fig.4 a) shows a decomposition of K in closed sets K_i, $i = 1, \ldots, 4$; $\mathcal{H}^s(K_i \cap K_j) = \emptyset$ for $i \neq j$; $K = K_1 \cup K_2 \cup K_3 \cup K_4$. From lemma 4 it follows that $K_1 = \psi_2(K)$, $K_3 = \psi_{11}(K)$, $K_2 = \psi_{33}(K)$ and therefore $\mathcal{H}^s(K_1) = \xi_3^s \mathcal{H}^s(K)$; $\mathcal{H}^s(K_2) = \mathcal{H}^s(K_3) = \xi_3^{2s} \mathcal{H}^s(K)$ and

$$\mathcal{H}^s(K_4) = (1 - \xi_3^s - 2\xi_3^{2s}) \mathcal{H}^s(K) \tag{26}$$

Fig. 4 b) shows a decomposition of the set K_4 in closed sets K_5, K_6, K_7; $K_4 = K_5 \cup K_6 \cup K_7$; $\mathcal{H}^s(K_i \cap K_j) = 0$, $i \neq j$ with K_5, K_6 similar to K_4 and K_7 similar to K ($K_7 = \psi_{311}(K)$). Moreover, using lemma 4 and symmetry, $\mathcal{H}^s(K_5) = \mathcal{H}^s(K_6) = \xi_3^s \mathcal{H}^s(K_4) = by$ (26) $= (\xi_3^s - \xi_3^{2s} - 2\xi_3^{3s}) \mathcal{H}^s(K)$; $\mathcal{H}^s(K_7) = \xi_3^{3s} \mathcal{H}^s(K)$. From this follows that K is ε-discrete because one can apply the decomposition of figures 4 a) and 4 b) again and again to the smaller pieces which are similar to K and K_4.

Fig. 5 shows a variant of our example where the ratios of the contractions are not equal: $\xi_1 = {\xi_2}^2, \xi_2$ a root of $x + x^2 - x^5 - 1 = 0$ (this forces $\psi_{1222}(x) = \psi_{211}(x)$) and ξ_3 such that $\triangle_3$ does not intersect $\triangle_1$ nor $\triangle_2$. In the case of the figure $\xi_3 = 1/4, \xi_2 \simeq 0,682\dots, \xi_1 \simeq 0,465\dots$.

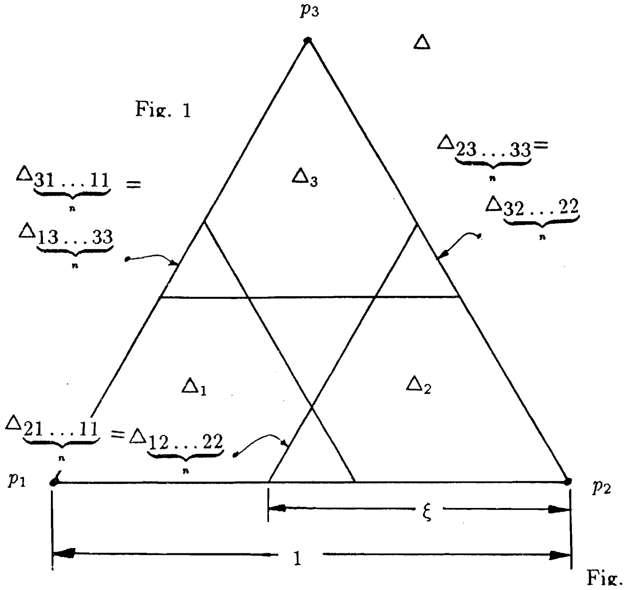


Fig. 1

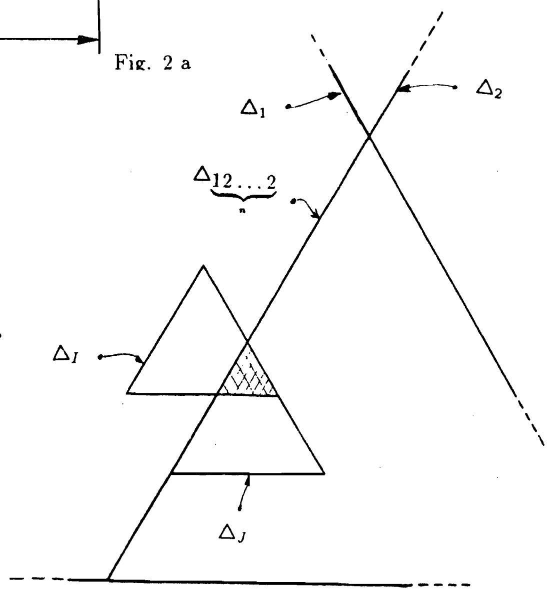


Fig. 2 a

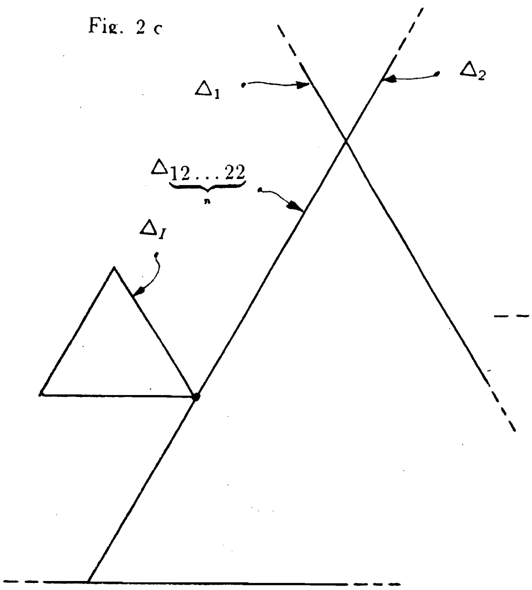


Fig. 2 c

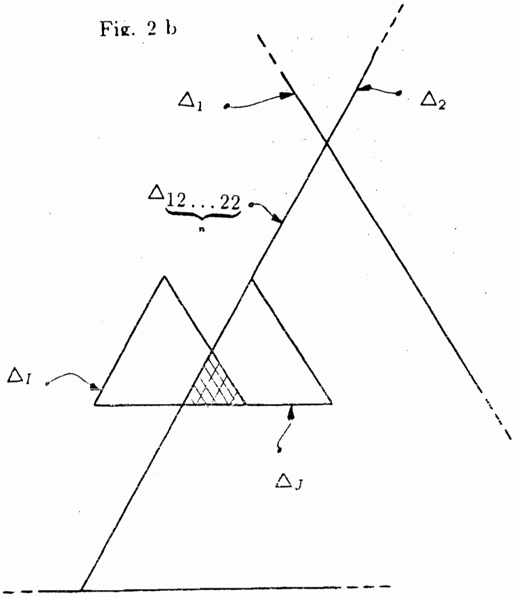


Fig. 2 b

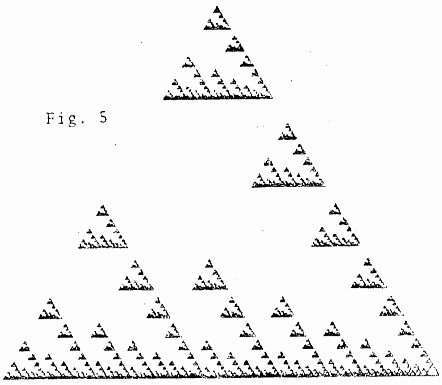


Fig. 5

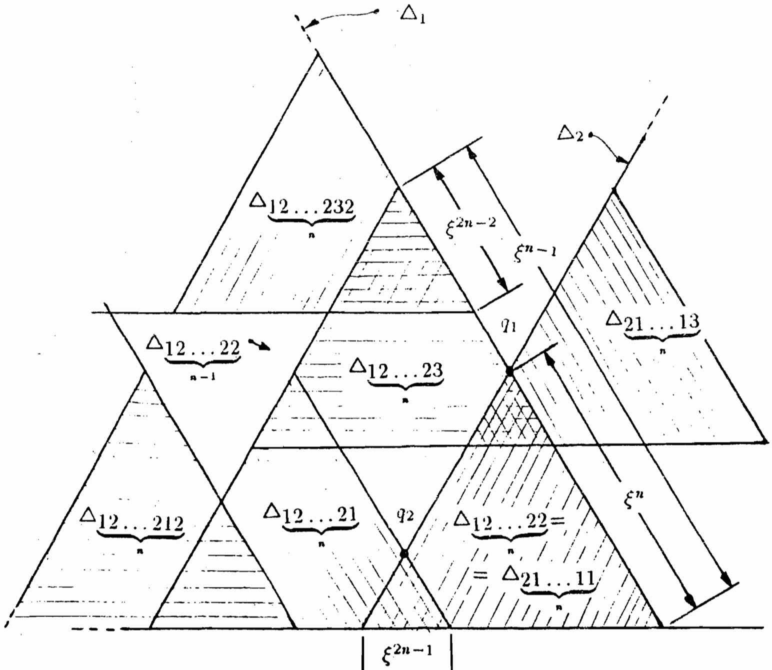
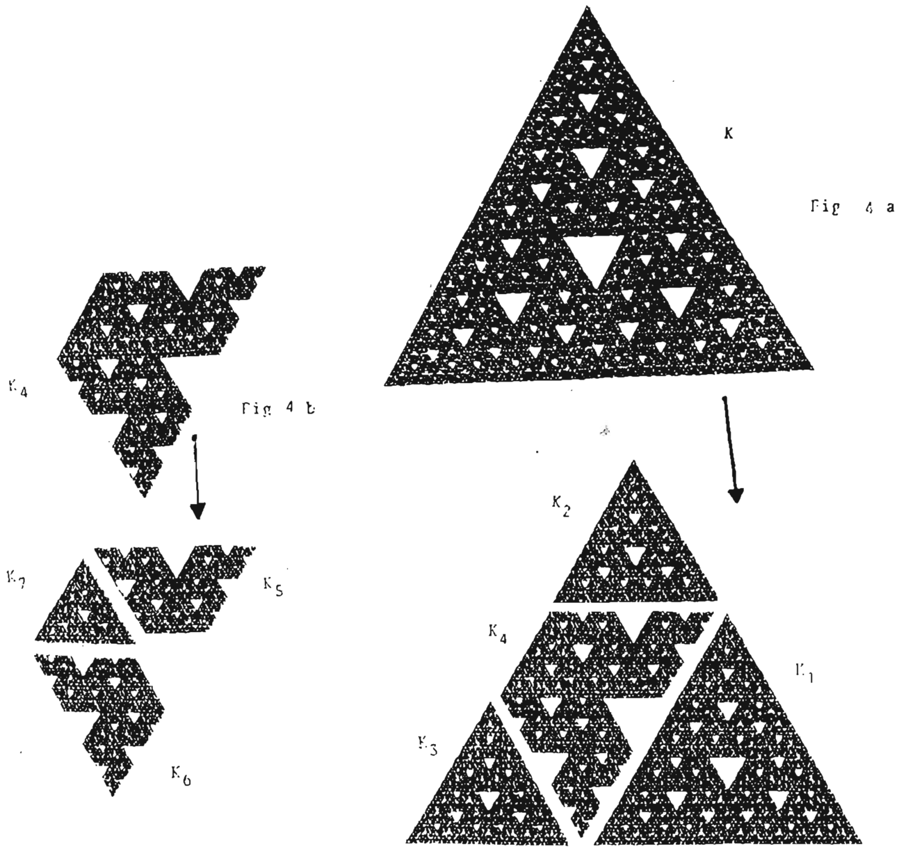


Fig. 3



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ON THE JOINT SPECTRA OF THE TWO DIMENSIONAL LIE ALGEBRA OF OPERATORS IN HILBERT SPACES

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ABSTRACT. We consider the complex solvable non-commutative two dimensional Lie algebra L, $L =< y > \oplus < x >$, with Lie bracket $[x, y] = y$, as linear bounded operators acting on a complex Hilbert space H. Under the assumption $R(y)$ closed, we reduce the computation of the joint spectra $Sp(L, E)$, $\sigma_{\delta,k}(L, E)$ and $\sigma_{\pi,k}(L, E)$, $k = 0, 1, 2$, to the computation of the spectrum, the approximate point spectrum, and the approximate compression spectrum of a single operator. Besides, we also study the case $y^2 = 0$, and we apply our results to the case H finite dimensional

1. Introduction.

In [1] we introduced a joint spectrum for complex solvable finite dimensional Lie algebras of operators acting on a Banach space E. If L is such an algebra, and $Sp(L, E)$ denotes its joint spectrum, $Sp(L, E)$ is a compact non empty subset of L^*, which also satisfies the projection property for ideals, i. e., if I is an ideal of L, and if $\Pi: L^* \to I^*$, denotes the restriction map, $Sp(I, E) = \Pi(Sp(L, E))$. In addition, when L is a commutative algebra, $Sp(L, E)$ reduces to the Taylor joint spectrum, see [5]. Moreover, in [2] we extended Slodkowski joint spectra $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$ to the case under consideration, and we proved the usual spectral properties: they are compact non empty subsets of L^*, and the projection property for ideals still holds.

In this paper we consider the complex solvable non-commutative two dimensional Lie algebra L, $L =< y > \oplus < x >$, with Lie bracket $[x, y] = y$, as bounded linear operators acting on a complex Hilbert space H, and we compute the joint spectra $Sp(L, H)$, $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, for $k = 0, 1, 2$, when $R(y)$ is a closed subspace

[1]Research supported by UBACYT

of H. Besides, by means of an homological argument, we reduce the computation of these spectra to the one dimensional case. We prove that these joint spectra are determined by the spectrum, the approximate point spectrum, and the approximate compression spectrum of x in $Ker(y)$ and $\overline{x}$ in $H/R(y)$, where $\overline{x}$ is the quotient map associated to x, ($R(y)$ and $Ker(y)$ are invariant subspaces for the operator x).

In addition, we consider the case $y^2 = 0$, (it easy to see that y is a nilpotent operator), and we obtain a relation between the spectrum of x in $R(y)$ and a subset of the spectrum of $\overline{x}$ in $H/R(y)$, which give us a more precise characterization of the joint spectrum $Sp(L, E)$. Finally, we apply our computation to the case H finite dimensional.

The paper is organized as follows. In Section 2 we review several definitions and results of [1] and [2]. In Section 3 we prove our main theorems and, in Section 4, we consider the case $y^2 = 0$ and the finite dimensional case.

2. Preliminaries.

In this section we briefly recall the definitions of the joint spectra $Sp(L, H)$, $\sigma_{\delta,k}(L, H)$ and $\sigma_\pi(L, H)$, $k = 0, 1, 2$. We restrict ourselves to the case under consideration. For a complete account of the definitions and mean properties of these joint spectra, see [1] and [2].

From now on, let L be the complex solvable two dimensional Lie algebra, $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x, y] = y$, which acts as right continuous linear operators on a Hilbert space H, i. e., L is a Lie subalgebra of $\mathcal{L}(H)^{op}$, where $\mathcal{L}(H)$ is the algebra of all bounded linear operators defined on H, and where $\mathcal{L}(H)^{op}$ means that we consider $\mathcal{L}(H)$ with its opposite product. We observe that, any complex solvable non-commutative two dimensional Lie algebra may be presented in the above form.

If f is a character of L, we consider the chain complex $(H \otimes \wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L, and $d(f)$ is the following map:

$$d_{p-1}(f): H \otimes \wedge^p L \to H \otimes \wedge^{p-1} L,$$

$$d_0(f)(a \langle y \rangle) = y(a), \qquad d_0(f)(b \langle x \rangle) = (x - f(x))(b),$$

$$d_1(f)(c \langle yx \rangle) = (-(x - 1 - f(x)))(c) \langle y \rangle + y(c) \langle x \rangle .$$

Let $H_*(H \otimes \wedge L, d(f))$ denote the homology of the complex $(H \otimes \wedge L, d(f))$, we now state our first definition.

Definition 1. *With H, L and f as above, the set $\{f \in L^*, f(L^2) = 0; H_*(\otimes \wedge L, d(f)) \neq 0\}$, is the joint spectrum of L acting on H, and it is denoted by $Sp(L, H)$.*

As a consequence of the results of [1], we have that $Sp(L, H)$ is a compact non empty subset of L^*. Besides, as a standard calculation shows that the equality $y = [x, y]^{op} = [y, x]$ implies $ny^n = [y^n, x] = [x, y^n]^{op}$, we have that y is a nilpotent operator. Thus, $Sp(< y >) = 0$, and by the projection property, if f belongs to $Sp(L, H)$, as $< y >= L^2$ is an ideal of L, $f(y) = 0$.

Now, let us consider the basis of L, A, defined by, $A = \{y, x\}$, and B, the basis of L^* dual of A. If we consider $Sp(L, H)$ in terms of the above basis, and we denote it by $Sp((y, x), H)$, i. e., $Sp((y, x), H) = \{(f(y), f(x)), f \in Sp(L, H)\}$, we have that, $Sp((y, x), H) = \{(0, f(x)), f \in Sp(L, H)\}$.

In addition, the complex $(H \otimes \wedge L, d(f))$ may be written in the following way,

$$0 \to H \xrightarrow{d_1} H \oplus H \xrightarrow{d_0} H \to 0,$$

$$d_0 = \begin{pmatrix} y & x - \lambda \end{pmatrix}, \qquad d_1 = \begin{pmatrix} -(x - 1 - \lambda) \\ y \end{pmatrix},$$

where $\lambda = f(x)$. We denote this chain complex by $(C, d(\lambda))$. Thus, as $(0, \lambda) \in Sp((y, x), H)$ if and only if $f \in Sp(L, H)$, where $\lambda = f(x)$, to compute the latter is equivalent to compute the former, and to study the exactness of the chain complex $(H \otimes \wedge L, d(f))$ is equivalent to study the exactness of $(C, d(\lambda))$.

With regard to the joint spectra $\sigma_{\delta,k}(L, H)$ and $\sigma_{\pi,k}(L, H)$, $k = 0, 1, 2$, we review, for the case under consideration, the definition of them given in [2]. If $p = 0, 1, 2$, let $\Sigma_p(L, H)$ be the set, $\Sigma_p(L, H) = \{f \in L^*, f(L^2) = 0; H_p((H \otimes \wedge L, d(f))) \neq 0\}$. We now state our second definition.

Definition 2. *With H, L and f as above,*

$$\sigma_{\delta,k}(L, H) = \bigcup_{0 \leq p \leq k} \Sigma(L, H),$$

$$\sigma_{\pi,k}(L, H) = \bigcup_{k \leq p \leq 2} \Sigma_p(L, H) \bigcup \{f \in L^*, f(L^2) = 0; R(d_k(f)) \textit{is not closed}\},$$

where $0 \leq k \leq 2$.

We observe that $Sp(L, H) = \sigma_{\delta,2}(L, H) = \sigma_{\pi,0}(L, H)$. Besides, as we have said, these joint spectra are compact non empty subsets of L^*. In addition, as in the case of the joint spectrum $Sp(L,H)$, we consider the joint spectra $\sigma_{\delta,k}(L,H)$ and $\sigma_{\pi,k}(L,H)$ in terms of the basis A and B. As these joint spectra are subsets of $Sp(L,H)$, we have that $\sigma_{\delta,k}((y,x),H) = \{(0,f(x)), f \in \sigma_{\delta,k}(L,H)\}$, and $\sigma_{\pi,k}((y,x),H) = \{(0,f(x)), f \in \sigma_{\pi,k}(L,H)\}$, where $k = 0,1,2$.

Moreover, as in the case of the joint spectrum $Sp(L,H)$, to compute $\sigma_{\delta,k}(L,H)$ and $\sigma_{\pi,k}(L,H)$, $0 \leq k \leq 2$, is equivalent to compute these joint spectra in terms of the basis A and B. Finally, to compute the latter joint spectra it is enough to study the complex $(C, d(\lambda))$, and to consider the corresponding properties involved in the definition of $\sigma_{\delta,k}(L,H)$ and $\sigma_{\pi,k}(L,H)$, $0 \leq k \leq 2$, for it.

3. The Main Result.

We begin with the characterization of $Sp(L,H)$. Indeed, we consider $Sp((y,x),H)$, and by means of an homological argument we reduce its computation to the case of a single operator.

Let us consider the chain complex $(\overline{C},\overline{d})$,

$$0 \to H \xrightarrow{\overline{d}=y} H \to 0.$$

Then an easy calculation shows that we have a short exact sequence of chain complex of the form,

$$0 \to (\overline{C},\overline{d}) \xrightarrow{i} (C,d(\lambda)) \xrightarrow{p} (\overline{C},\overline{d}) \to 0,$$

where $(i_j)_{(0\leq j\leq 2)}$ and $(p_j)_{(0\leq j\leq 2)}$ are the following maps: $i_2 = 0$, $i_1 = I_H \oplus 0$, $i_0 = I_H$, and $p_2 = I_H$, $p_1 = 0 \oplus I_H$, $p_0 = 0$.

Thus, by [4,II,4], and the fact that p is a map of degree -1, we have a long exact sequence of homology spaces of the form,

$$\to H_q(C,d(\lambda)) \xrightarrow{p_{q*}} H_{q-1}(\overline{C},\overline{d}) \xrightarrow{\partial_{q-1}} H_{q-1}(\overline{C},\overline{d}) \xrightarrow{i_{q-1*}} H_{q-1}(C,d(\lambda)) \to .$$

We observe that $H_1(\overline{C},\overline{d}) = Ker(y)$, and that $H_0(\overline{C},\overline{d}) = H/R(y)$. Moreover, as $[x,y]^{op} = y$, we have that $x(R(y)) \subseteq R(y)$, and that $x(Ker(y)) \subseteq Ker(y)$. Then, by [4,II,4], ∂_q, $q = 0,1$, are the following maps: $\partial_0([a]) = [(x-\lambda)(a)] = (\overline{x}-\lambda)[a]$, and $\partial_1(b) = -(x-\lambda-1)(b)$, where $\overline{x}: H/R(y) \to H/R(y)$ is the map obtained by passing x to the quotient space $H/R(y)$. We now give our characterization of $Sp(L,H)$.

Proposition 1. *Let L be the complex solvable non-commutative two dimensional Lie algebra $L =< y > \oplus < x >$, with Lie bracket $[x,y]=y$, which acts as right continuous linear operators on a complex Hilbert space H. If $R(y)$ is a closed subspace of H, and if we consider $Sp(L,H)$ in terms of the basis $\{y,x\}$ of L and the basis of L^* dual of the latter, we have,*

$$Sp((y,x),H) = \{0\} \times Sp(x-1, Ker(y)) \cup \{0\} \times Sp(\overline{x}, H/R(y)).$$

In addition, we have:

i) $H_0(C,d(\lambda)) = 0$ iff $\overline{x} - \lambda\colon H/R(y) \to H/R(y)$ is a surjective map,

ii) $H_2(C,d(\lambda)) = 0$ iff $x - 1 - \lambda\colon ker(y) \to Ker(y)$ is an injective map,

iii) $H_1(C,d(\lambda)) = 0$ iff $\overline{x} - 1 - \lambda$ is injective, and $x - \lambda - 1$ is surjective.

Proof.

It is a consequence of the long exact sequence of homology spaces, and the form of the maps ∂_j, $j = 0,1$.

∎

In order to characterize the joint spectra $\sigma_{\pi,k}(L,H)$, we recall the notion of approximate point spectrum of an operator T: λ is in the approximate point spectrum of T, which we denote by $\Pi(T)$, if there exists a sequence of unit vectors, $(x_n)_{n\in \mathbb{N}}$, $x_n \in H$, $\| x_n \| = 1$, such that $(T-\lambda)(x_n) \xrightarrow[n\to\infty]{} 0$. An easy calculation shows that $\lambda \notin \Pi(T)$ if and only if $Ker(T-\lambda) = 0$ and $R(T-\lambda)$ is closed in H.

We now consider the spectrum $\sigma_{\pi,2}((y,x),H)$. We observe that, as $[x,y]^{op} = y$, $(x-1)(Ker(y) \subseteq Ker(y)$. Then, we may consider $\Pi(x-1, Ker(y))$. Indeed, we shall see that $\sigma_{\pi,2}((y,x),H) = \{0\} \times \Pi(x-1, Ker(y))$.

To prove the last assertion we proceed as follows. By Definition 2, we have that $\sigma^c_{\pi,2} = \{(0,\lambda; H_2(C,d(\lambda)) = 0$, and $R(d_1(\lambda))$ is closed$\}$. However, by the definition of $d_1(\lambda)$ and $H_2(C,d(\lambda))$, $H_2(C,d(\lambda)) = Ker(x-1-\lambda) \cap Ker(y)$. Then, $H_2(C,d(\lambda)) = 0$ is equivalent to $Ker(x-1-\lambda \mid Ker(y)) = 0$. Thus, in order to conclude with our assertion, it is enough to see that the fact $R(x-1-\lambda \mid Ker(y))$ is closed, is equivalent to $R(d_1(\lambda))$ is closed.

Indeed, if $(a_n)_{n\in\mathbb{N}}$ is a sequence in $Ker(y)$ such that $(x-1-\lambda)(a_n) \xrightarrow[n\to\infty]{} b \in Ker(y)$, we have that, $d_1(\lambda)(a_n) \xrightarrow[n\to\infty]{} (-b,0)$. If $R(d_1(\lambda))$ is closed, there is a z in H such that $d_1(\lambda)(z) = (-b,0)$, i.e., $-(x-1-\lambda)(z) = -b$, and $y(z) = 0$. Thus, $z \in Ker(y)$ and $R((x-1-\lambda) \mid Ker(y))$ is closed.

On the other hand, if $R((x-1-\lambda \mid Ker(y))$ is closed, let us consider a sequence $(z_n)_{n\in\mathbb{N}}$, $z_n \in H$, such that $d_1(\lambda)(z_n) \xrightarrow[n\to\infty]{} (w_1, w_2) \in H \oplus H$. We decompose H as the orthogonal direct sum of $Ker(y)$ and $Ker(y)^\perp$, $H = Ker(Y) \oplus Ker(y)^\perp$. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $Ker(y)$ and $Ker(y)^\perp$, respectively, such that $z_n = a_n + b_n$. Then,

$$\begin{aligned} d_1(\lambda) &= d_1(\lambda)(a_n) + d_1(\lambda)(b_n) \\ &= (-(x-1-\lambda)(a_n), 0) + (-(x-1-\lambda)(b_n), \overline{y}(b_n)), \end{aligned}$$

where $\overline{y}\colon Ker(y)^\perp \to R(y)$ is the restriction of y to $Ker(y)^\perp$. We observe that, as $R(y)$ is a closed subspace of H, $\overline{y}$ is a topological homeomorphism. Besides, as $\overline{y}(b_n) \xrightarrow[n\to\infty]{} w_2$, there exists a $z_2 \in Ker(y)^\perp$ such that $b_n \xrightarrow[n\to\infty]{} z_2$, and $\overline{y}(z_2) = w_2$. Then, $-(x-1-\lambda)(b_n) \xrightarrow[n\to\infty]{} -(x-1-\lambda)(z_2)$, and $-(x-1-\lambda)(a_n) \xrightarrow[n\to\infty]{} w_1 + (x-1-\lambda)(z_2)$. As $(a_n)_{n\in\mathbb{N}}$ is a sequence in $Ker(y)$, and $R(x-1-\lambda \mid Ker(y))$ is closed, there is a $z_1 \in Ker(y)$ such that $w_1 + (x-1-\lambda)(z_2) = -(x-1-\lambda)(z_1)$. Thus, $(w_1, w_2) = d_1(\lambda)(z_1 + z_2)$, equivalently, $R(d_1(\lambda))$ is a closed subspace of $H \oplus H$.

With regard to $\sigma_{\pi,1}((y,x),H)$, we have, by Definition 2, that,

$$\sigma_{\pi,1}((y,x),H)^c = \{(0,\lambda); H_i(C,d(\lambda)) = 0, i = 1,2, \text{and } R(d_0(\lambda)) \text{ is closed}\},$$

which, by Proposition 1, is equivalent to the following conditions:

i) $x-1-\lambda\colon Ker(y) \to Ker(y)$ is an isomorphic map,

ii) $\overline{x}-\lambda\colon H/R(y) \to H/R(y)$ is an injective map,

iii) $R(d_0(\lambda))$ is closed.

We shall see that $\sigma_{\pi,1}((y,x),H) = Sp(x-1, Ker(y)) \cup \Pi(\overline{x}, H/R(y))$.

Indeed, it is clear that condition i) is equivalent to $\lambda \notin Sp(x-1, Ker(y)$. Then, it is enough to see that condition ii) and iii) are equivalent to $\lambda \notin \Pi(\overline{x}, H/R(y))$. However, by ii), it suffices to verify that the fact $R(d_0)(\lambda)$ is closed is equivalent to $R(\overline{x}-\lambda)$ is closed. Now, as the quotient map, $\Pi\colon H \to H/R(y)$, is an identification, by [3,II,6], $R = R(\overline{x}-\lambda) = \Pi(R(x-\lambda))$ is closed in $H/R(y)$ if and only if $\Pi^{-1}(R) = R(x-\lambda) + R(y) = R(d_0(\lambda))$ is closed in H.

In order to study the joint spectra $\sigma_{\delta,k}(L,H)$, $k = 0,1,2$, we recall the definition of the approximate compression Spectrum of an operator T in H: λ is in the approximate compression spectrum of T, which we denote by $\Pi C(T)$, if there exists a sequence of unit vectors in H, $(x_n)_{n\in\mathbb{N}}$, $x_n \in H$, $\| x_n \| = 1$, such that $(T-\lambda)^*(x_n) \xrightarrow[n\to\infty]{} 0$, i. e., $\Pi C(T) = \Pi(T^*)$. Besides, an easy calculation shows that λ does not belong to $\Pi(T)$ if and only if $(T-\lambda)$ is a surjective map.

We now consider the joint spectra $\sigma_{\delta,o}((y,x),H)$. However, by Definiton 2, Proposition 1, and the previous considerations about the approximate compression spectrum, it is clear that $\sigma_{\delta,k}((y,x),H) = \{0\} \times \Pi C(\overline{x}, H/R(y))$.

With regards to $\sigma_{\delta,1}((y,x),H)$, by Definition 2 and Proposition 1, we have that $(0,\lambda)$ does not belong to $\sigma_{\delta,1}((y,x),H)$, if and only if $(0,\lambda)$ satisfies the following conditions:

i) $\overline{x}-\lambda\colon H/R(y) \to H/R(y)$ is an isomorphic map,

ii) $x-1-\lambda\colon Ker(y) \to Ker(y)$ is surjective.

Then, it is obvious that, $\sigma_{\delta,1}((y,x),H) = \{0\} \times Sp(\overline{x}, H/R(y)) \cup \{0\} \times \Pi C(x-1 \mid Ker(y))$.

We now summarize our results.

Theorem 1. *Let L be the complex solvable non-commutative two dimensional Lie algebra, $L = \langle y \rangle \oplus \langle x \rangle$, with Lie bracket $[x,y]^{op} = y$, which acts as right continuous linear operators on a complex Hilbert space H. If $R(y)$ is closed, the joint spectra $Sp(L,H)$, $\sigma_{\delta,k}(L,H)$ and $\sigma_{\pi,k}(L,H)$, $k = 0,1,2$, in terms of the basis $\{y,x\}$ of L, and the basis of L^* dual of the latter, may be characterize as follows:*

i) $Sp((y,x),H) = \{0\} \times Sp(x-1, Ker(y)) \cup \{0\} \times Sp(\overline{x}, H/R(y))$,

ii) $\sigma_{\delta,0}((y,x),H) = \{0\} \times \Pi C(\overline{x}, H/R(y))$,

iii) $\sigma_{\delta,1}((y,x)) = \{0\} \times Sp(\overline{x}, H/R(y)) \cup \{0\} \times \Pi C(x-1, Ker(y))$,

iv) $\sigma_{\pi,2}((y,x),H) = \{0\} \times \Pi(x-1, Ker(y))$,

v) $\sigma_{\pi,1}((y,x),H) = \{0\} \times Sp(x-1, Ker(y)) \cup \{0\} \times \Pi(\overline{x}, H/R(y))$,

vi) $\sigma_{\delta,2}((y,x),H) = \sigma_{\pi,0}((y,x),H) = Sp((y,x),H)$.

∎

4. A Special Case.

As we have seen, y is a nilpotent operator. In this section we study the case $y^2 = 0$, and we obtain a more precise characterization of th joint spectrum $Sp(L,H)$.

We decompose H in the following way: $H = Ker(y) \oplus Ker(y)^{\perp}$. Besides, as

$R(y)$ is contained in $Ker(y)$, let us consider M, the closed subspace of H defined by, $M = Ker(y) \cap R(y)^{\perp}$. Then, we have another orthogonal direct sum decomposition of H, $H = R(y) \oplus M \oplus Ker(y)^{\perp}$. Moreover, if we recall that $x(R(y)) \subseteq R(y)$ and $x(Ker(y)) \subseteq Ker(y)$, we have that x and y have the following form,

$$y = \begin{pmatrix} 0 & 0 & \overline{y} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix},$$

where $\overline{y}$ is as in Section 3, and the maps x_{ij} ,$1 \leq i \leq j \leq 2$, are the restriction of x to the corresponding spaces. We now see that, in the case under consideration, $Sp(L, H)$ reduces essentially to the spectrum of x in $Ker(y)$.

Proposition 2. *Let L be the complex solvable non commutative two dimensional Lie algebra, $L =< y > \oplus < x >, withLiebracket[x, y]^{op} = y$, which acts as right continuous linear operators on a complex Hilbert space H. If $R(y)$ is closed and $y^2 = 0$, $Sp(L, H)$, in terms of the basis $\{y, x\}$ of L and the basis of L^* dual of the latter, may be described as follows. If x_{11} and x_{22} are the maps defined above, and if S_i, $i = 1, 2$, are the sets: $S_1 = (Sp(x_{11}, R(y)) - 1)$, and $S_2 = (Sp(x_{22}, R(y)^{\perp} \cap Ker(y))$, then, we have that,*

$$Sp((y, x), H) = \{0\} \times (S_1 \cup (S_1 + 2) \cup S_2 \cup (S_2 - 1)).$$

Proof.

An easy calculation shows that the relation $[x, y]^{op} = y$ is equivalent to $\overline{y}x_{33} - x_{11}\overline{y} = \overline{y}$. However, as $\overline{y}$ is a topological homeomorphism, $x_{33} = I_{Ker(y)^{\perp}} + \overline{y}^{-1}x_{11}\overline{y}$. In particular, $Sp(x_{33}, Ker(y)^{\perp}) = Sp(x_{11}, R(y)) + 1$. Then, as $Sp(\overline{x}, H/R(y)) = Sp(x_{22}, M) \cup Sp(x_{33}, Ker(y)^{\perp})$, where $M = R(y)^{\perp} \cap Ker(y)$,we have that $Sp(\overline{x}, H/R(y))$ $(S_1 + 2) \cup S_2$.

On the other hand, it is clear that $Sp(x - 1, Ker(y)) = S_1 \cup (S_2 - 1)$. Thus, by Theorem 1, we conclude the proof.

∎

Finally, we consider the case $R(y)$ closed, $y^2 = 0$, and H finite dimensional. If $r = dim(R(y))$ and $k = dim(Ker(y))$, let us chose a basis of $Ker(y)$ such that the first r-vectors of it are a basis of $R(y)$, and in this basis, x has an upper triangular form, with diagonal entries λ_{ii}, $1 \leq i \leq k$. Then we have the following corollary.

Corollary 1. *Let H, L and the operator y be as in Proposition 2. If H is finite dimensional, and if we consider a basis of $Ker(y)$ with the above conditions, $Sp(L,H)$, in terms of the basis of L and L^* considered in Proposition 2, is the following set,*

$$Sp((y,x),H) = \{0\} \times \{(\lambda_{ii}-1)_{(1\leq i\leq k)} \cup (\lambda_{ii})_{(m\leq i\leq k)} \cup (\lambda_{ii}+1)_{(1\leq i\leq m)}\}.$$

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PARAMETER CONTINUITY OF THE SOLUTIONS OF A MATHEMATICAL MODEL OF THERMOVISCOELASTICITY

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Abstract: In this paper the continuity of the solutions of a mathematical model of thermoviscoelasticity with respect to the model parameters is proved. This was an open problem conjectured in [27] and [28]. The nonlinear partial differential equations under consideration arise from the conservation laws of linear momentum and energy and describe structural phase transitions in solids with non-convex Landau-Ginzburg free energy potentials. The theories of analytic semigroups and real interpolation spaces for maximal accretive operators are used to show that the solutions of the model depend continuously on the admissible parameters, in particular, on those defining the free energy. More precisely, it is shown that if $\{q_n\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to q, then the corresponding solutions $z(t;q_n)$ converge to $z(t;q)$ in the norm of the graph of a fractional power of the operator associated to the linear part of the system.

1. INTRODUCTION

The conservation laws governing the thermomechanical processes in a one-dimensional

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shape memory solid $\Omega = (0,1)$ with Landau-Ginzburg free energy potential Ψ give rise to the following initial-boundary value problem.

$$
(1.1)\begin{cases}
\rho u_{tt} - \beta\rho u_{xxt} + \gamma u_{xxxx} = f(x,t) + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial\epsilon}\Psi(u_x, u_{xx}, \theta)\right], & x\in\Omega, 0\le t\le T,\\
C_v\theta_t - k\theta_{xx} = g(x,t) + 2\alpha_2\theta u_x u_{xt} + \beta\rho u_{xt}^2, & x\in\Omega, 0\le t\le T,\\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), & x\in\Omega,\\
u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, & 0\le t\le T,\\
\theta_x(0,t) = 0, \qquad k\theta_x(1,t) = k_1\left(\theta_\Gamma(t) - \theta(1,t)\right), & 0\le t\le T.
\end{cases}
$$

The functions, variables and parameters involved in (1.1) have the following physical meaning: $u(x,t)$ = displacement; $\theta(x,t)$ = absolute temperature; ρ = mass density; k = thermal conductivity coefficient; C_v = specific heat; β = viscosity coefficient; $f(x,t)$ = distributed forces acting on the body (input); $g(x,t)$ = distributed heat sources (input); $u_0(x)$ = initial displacement; $u_1(x)$ = initial velocity; $\theta_0(x)$ = initial temperature; $\theta_\Gamma(t)$ = temperature of the surrounding medium (input); k_1 = positive constant, proportional to the rate of thermal exchange at the right boundary, and T is a prescribed final time. The function Ψ, which represents the free energy density of the system, is assumed to be a function of the linearized shear strain $\epsilon = u_x$, the spatial derivative of the strain $\epsilon_x = u_{xx}$ and the temperature θ, and is taken in the Landau-Ginzburg form

$$
\begin{aligned}
\Psi(\epsilon,\epsilon_x,\theta) &= \Psi_0(\theta) + \alpha_2(\theta-\theta_1)\epsilon^2 - \alpha_4\epsilon^4 + \alpha_6\epsilon^6 + \frac{\gamma}{2}\epsilon_x^2,\\
\Psi_0(\theta) &= -C_v\theta\log\left(\frac{\theta}{\theta_2}\right) + C_v\theta + C,
\end{aligned}
\tag{1.2}
$$

where θ_1, θ_2 are two critical temperatures and α_2, α_4, α_6, γ are positive constants, all depending on the material being considered. Note that for values of θ close to θ_1 and ϵ_x fixed, the function $\Psi(\epsilon,\epsilon_x,\theta)$ is a nonconvex function of ϵ. This property is related to the hysteresis phenomenon which caracterizes this type of materials in the low and intermediate temperature ranges. The stress-strain relations are strongly temperature-dependent. The behavior goes from elastic, ideally-plastic at low temperatures, to pseudoelastic or superelastic at intermediate temperatures, to almost linearly elastic in the high temperature range. Shape memory and solid-solid phase transitions (martensitic transformations) are other peculiar characteristics of these materials whose dynamical behavior is described by system (1.1). For a detailed review of these and other properties and the derivations of the equations in (1.1) we refer the reader to [25] and the references therein.

The boundary conditions mean that the body is clamped at both ends, thermally insulated at the left end and, at the right end, the rate of thermal exchange is prescribed. The nonlinear coupled equations in (1.1) are sometimes referred to as the equations of thermo-visco-elasto-plasticity. In particular, the first equation in (1.1) can be regarded as a nonlinear beam equation in u, while the second is a nonlinear heat equation in θ.

Initial boundary value problems of the type (1.1) have been studied by several authors ([15], [16], [21], [27], [28], [32], etc.; see [25] for a review). Initial efforts to prove existence of solutions for this type of systems considered the heat flux in the form $q = -k\theta_x - \alpha k\theta_{xt}$, with $\alpha > 0$, instead of the classical Fourier law ($\alpha = 0$). This assumption introduces the additional term $-\alpha k\theta_{xxt}$ on the left hand side of the second equation in (1.1). Although this was done merely for mathematical reasons so that existence theorems could be proved ([15], [16], [21], [22]), it turns out that the second law of thermodynamics is not satisfied if $\alpha > 0$, as it can be easily verified by checking the Clausius-Duhem inequality for the entropy production. Therefore, the case $\alpha > 0$ has no physical meaning. The first results on existence of solutions for the case $\alpha = 0$ are due to Sprekels ([27]). However, he imposed very strong growth conditions on the free energy Ψ. In particular, those conditions excluded the physically relevant case in which Ψ is given in the Landau-Ginzburg form (1.2). Later on, Zheng ([32]) derived certain apriori estimates from which he concluded that, if the initial data is smooth enough, then any local solution of (1.1) with Ψ as in (1.2) can be extended globally in time. This result was later generalized by Sprekels and Zheng ([28]) to include more general free energy functionals. More recently, using a state-space approach ([25]) it was shown that system (1.1)-(1.2) has a local solution for a much broader set of initial data than the one considered in [28] and [32].

From a practical point of view it would be very important to find the values of all the parameters in (1.1)-(1.2) that "best fit" experimental data for a given material. This is called the parameter identification problem (ID problem in the sequel). Once this problem is solved, the next step is to determine how well this model can predict the dynamics of a given shape memory material which is subjected to certain external inputs. This is called the model validation problem. Although numerical experiments performed with system (1.1) have shown that physically reasonable results can be obtained for certain values of the parameters (see [4] and [19]), the ID problem still remains open.

In order to establish the convergence of computational algorithms for parameter identification, one needs to show first that the solutions depend continuously on the parameters that one wants to estimate. As we shall see in the following section, system (1.1)-(1.2) can be written as a semilinear Cauchy problem of the form $\dot{z}(t) = A(q)z(t)+F(q,t,z)$, $z(0) = z_0$, in an appropriate Hilbert space Z_q, where q is a vector of admissible parameters, $A(q)$ is a certain differential operator associated with the linear part of the partial differential equations in (1.1) and $F(q,t,z)$ corresponds to the nonlinear part of the system. In [26] it was shown that the nonlinear term $F(q,t,z)$ is locally Lipschitz continuous in the state variable z in the topology of the graph of $(-A(q))^\delta$, for any $\delta > \frac{3}{4}$. Although this result is necessary to show the continuous dependence of the solutions of (1.1) with respect to the parameter q, it is not sufficient. In fact, it turns out that a key step in achieving this result involves proving that if $\{q_n\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to q, then the associated analytic semigroups $T(t;q_n)$ converge strongly to $T(t;q)$ in the norm of the graph of $(-A(q))^\delta$. This is a much stronger result than the one obtained by using the well known Trotter-Kato Theorem (see [25], Theorem 4.1).

2. PRELIMINARIES AND STATE-SPACE FORMULATION

In the sequel, an *isomorphism* will be understood to denote a bounded invertible operator from a Banach space onto another.

Let X be a Banach space and X^* its topological dual. We denote with $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ the value of x^* at x. For each $x \in X$ we define the *duality set* $S(x) \doteq \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$. The Hahn-Banach theorem implies that $S(x)$ is nonempty for every $x \in X$. If A is a linear operator in X with domain $D(A)$, we say that A is *dissipative* if for every $x \in D(A)$ there exists $x^* \in S(x)$ such that $\mathrm{Re}\langle Ax, x^*\rangle \leq 0$. We say that A is *strictly dissipative* if A is dissipative and the condition $\mathrm{Re}\langle Ax, x^*\rangle = 0$ for all $x^* \in S(x)$ implies that $x = 0$. If X is a Hilbert space then $S(x) = \{x\}$ and therefore A is dissipative iff $\mathrm{Re}\langle Ax, x\rangle \leq 0$ for every $x \in D(A)$. We say that the operator A is *maximal dissipative* if A is dissipative and it has no proper dissipative extension. We say that the operator A is *(maximal) accretive* if $-A$ is (maximal) dissipative. If the operator A is strictly dissipative and maximal dissipative, we will simply say that A is *strictly maximal dissipative.*

If A generates a strongly continuous semigroup $T(t)$ on X then the *type of* T is defined to be the real number $w_0(T) \doteq \inf\limits_{t>0} \frac{1}{t} \log \|T(t)\|$. It can be shown that the type of a semigroup is either finite or equals $-\infty$. Moreover, $w_0(T) = \lim\limits_{t\to\infty} \frac{1}{t} \log \|T(t)\|$. Also, the semigroup $T(t)$ is of negative type iff $T(t)$ is exponentially stable, i.e., $w_0(T) < 0$ iff $\exists M \geq 1$, $\alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t > 0$ (see [1, pp 17-21]). If the semigroup $T(t)$ generated by A is analytic and $\sigma(A)$ denotes the spectrum of A, then $w_0(T) = \sup\limits_{\lambda\in\sigma(A)} \mathrm{Re}\ \lambda$ provided that $\sigma(A) \neq \emptyset$ and $w_0(T) = -\infty$ if $\sigma(A) = \emptyset$ (see [1]).

Let us return now to our original problem (1.1)-(1.2). We define the function $L(x,t) \doteq \theta_\Gamma(t)\cos(2\pi x)$ and the transformation $\tilde{\theta}(x,t) = \theta(x,t) - L(x,t)$. We also define the *state space* $Z \doteq H_0^1(0,1) \cap H^2(0,1) \times L^2(0,1) \times L^2(0,1)$, $z \doteq \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z$ and the *admissible parameter set*

$$\mathcal{Q} \doteq \left\{q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma) \mid q \in \mathbb{R}^8_{>0}\right\}.$$

Next, we define in Z an inner product $\langle\cdot,\cdot\rangle_q$ depending on the parameter q as follows

$$\left\langle \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{pmatrix} \right\rangle_q \doteq \gamma \int_0^1 u''(x)\hat{u}''(x)\,dx + \rho \int_0^1 v(x)\hat{v}(x)\,dx + \frac{C_v}{k} \int_0^1 w(x)\hat{w}(x)\,dx$$

and we denote by Z_q the Hilbert space Z endowed with the inner product $\langle\cdot,\cdot\rangle_q$. The norm induced by $\langle\cdot,\cdot\rangle_q$ in Z_q will be denoted by $\|\cdot\|_q$. Note that these norms are all equivalent and, moreover, they are uniformly equivalent on compact subsets of $\mathcal{Q}$. Then the initial boundary value problem (1.1) with Ψ as in (1.2) can be formally written as an abstract semilinear Cauchy problem in Z_q as follows

$$\begin{cases} \dot z(t) = A(q)z(t) + F(q,t,z(t)), & 0 \le t \le T \\ z(0) = z_0, \end{cases} \tag{2.1}$$

where $z(t)(x) = \begin{pmatrix} u(x,t) \\ u_t(x,t) \\ \tilde\theta(x,t) \end{pmatrix}$,

$$D(A(q)) \doteq \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q \;\middle|\; \begin{array}{l} u \in H^4(0,1),\ u(0) = u(1) = 0 = u''(0) = u''(1), \\ v \in H_0^1(0,1) \cap H^2(0,1), \\ w \in H^2(0,1), \quad w'(0) = 0, \quad kw'(1) = -k_1 w(1) \end{array} \right\}, \tag{2.2}$$

and for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(A(q))$,

$$A(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \doteq \begin{pmatrix} v \\ \beta v'' - \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ -\frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{2.3}$$

The element z_0 is defined by

$$z_0(x) = \begin{pmatrix} u_0(x) \\ u_1(x) \\ \theta_0(x) - \theta_\Gamma(0) cos(2\pi x) \end{pmatrix}$$

and the nonlinear mapping $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_q \to Z_q$ is defined by

$$F(q,t,z) = F\left(q, t, \begin{pmatrix} u \\ v \\ w \end{pmatrix}\right) \doteq \begin{pmatrix} 0 \\ f_2(q,t,z) \\ f_3(q,t,z) \end{pmatrix}, \tag{2.4}$$

where

$$\begin{aligned} \rho f_2(q,t,z)(x) &= f(x,t) \\ &\quad + \frac{\partial}{\partial x}\left[2\alpha_2(w(x) + L(x,t) - \theta_1)u'(x) - 4\alpha_4 u'(x)^3 + 6\alpha_6 u'(x)^5\right], \\ C_v f_3(q,t,z)(x) &= g(x,t) + 2\alpha_2\left(w(x) + L(x,t)\right) u'(x) v'(x) \\ &\quad + \beta\rho v'(x)^2 - C_v \theta_\Gamma'(t)\cos(2\pi x) \\ &\quad - 4k\pi^2 L(x,t). \end{aligned}$$

The following results can be found in [25] and [26].

Theorem 2.1. ([25]) *Let $q \in \mathcal{Q}$ and the operator $A(q) : D(A(q)) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3). Then*
i) *$A(q)$ is strictly maximal dissipative;*
ii) *The adjoint $A^*(q)$ is also strictly maximal dissipative and is given by $D(A^*(q)) = D(A(q))$, and for $\begin{pmatrix} u \\ v \\ w \end{pmatrix} \in D(A^*(q))$*

$$A^*(q)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -v \\ \beta v'' + \frac{\gamma}{\rho} u'''' \\ \frac{k}{C_v} w'' \end{pmatrix} = \begin{pmatrix} 0 & -I & 0 \\ \frac{\gamma}{\rho}\frac{\partial^4}{\partial x^4} & \beta \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 & \frac{k}{C_v}\frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix};$$

iii) $0 \in \rho(A(q))$, *the resolvent set of* $A(q)$;
iv) *The spectrum* $\sigma(A(q))$ *of* $A(q)$ *consists only of eigenvalues,* $\sigma(A(q)) = \sigma_p(A(q)) = \{\lambda_n^{+,-}, \alpha_n\}_{n=1}^{\infty}$ *where* $\lambda_n^{+,-} = \sqrt{\mu_n}\left(-r(q) \pm \sqrt{r^2(q)-1}\right)$, $\alpha_n = -\frac{k\tau_n^2}{C_v}$, *with* $\mu_n = \dfrac{\gamma n^4 \pi^4}{\rho}$, $r(q) = \dfrac{\beta\sqrt{\rho}}{2\sqrt{\gamma}}$ *and* $\{\tau_n\}_{n=1}^{\infty}$ *are all the positive solutions of the equation* $\tan\tau = \dfrac{k_1}{k\tau}$. *The corresponding set of normalized eigenvectors in* Z_q *is given by*

$$\left\{ \begin{pmatrix} e_n \\ \lambda_n^+ e_n \\ 0 \end{pmatrix}, \begin{pmatrix} k_n e_n \\ k_n \lambda_n^- e_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \chi_n \end{pmatrix} \right\}_{n=1}^{\infty},$$

where $e_n(x) = \left(\dfrac{2}{\rho\left(\mu_n + |\lambda_n^+|^2\right)}\right)^{\frac{1}{2}} \sin(\pi n x)$, $\chi_n(x) = \left(\dfrac{k\tau_n}{C_v \int_0^{\tau_n} \cos^2(\xi)\, d\xi}\right)^{\frac{1}{2}} \cos(\tau_n x)$ *and* $k_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}$.
v) *The operator* $A(q)$ *generates an analytic semigroup* $T(t;q)$ *of negative type which satisfies* $\|T(t;q)\|_{\mathcal{L}(Z_q)} \le e^{-\omega(q)t}$, *for* $t \ge 0$, *where* $\omega(q)$ *is given by*

$$\omega(q) = \begin{cases} \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2}\right), & \text{if } \beta^2\rho \le 4\gamma \\ \min\left(\frac{k\tau_1^2}{C_v}, \frac{\beta\pi^2}{2} - \frac{\pi^2}{2\sqrt{\rho}}\sqrt{\beta^2\rho - 4\gamma}\right), & \text{if } \beta^2\rho > 4\gamma. \end{cases}$$

It will be useful to introduce some notation for certain interpolation spaces. If X is a Banach space and $p \ge 1$, $L_*^p(X)$ will denote the Banach space of all Bochner measurable mappings $u : [0,\infty) \to X$ such that $\|u\|_{L_*^p(X)}^p \doteq \int_0^\infty \|u(t)\|_X^p \frac{dt}{t} < \infty$. Let X_0, X_1 be two Banach spaces with X_0 continuously and densely embedded in X_1, $p \ge 1$ and $\theta \in (0,1)$. We shall denote by $(X_0, X_1)_{\theta,p}$ the space of averages (or "real" interpolation space)

$$(X_0, X_1)_{\theta,p} \doteq \left\{ x \in X_1 \;\middle|\; \begin{array}{l} \exists u_i : [0,\infty) \to X_i, i = 0,1, \quad t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and } x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\}.$$

Endowed with the norm

$$\|x\|_{(X_0,X_1)_{\theta,p}} \doteq \inf\left\{ \|t^{-\theta}u_0\|_{L_*^p(X_0)} + \|t^{1-\theta}u_1\|_{L_*^p(X_1)} \;\middle|\; \begin{array}{l} t^{-\theta}u_0 \in L_*^p(X_0), \\ t^{1-\theta}u_1 \in L_*^p(X_1) \text{ and} \\ x = u_0(t) + u_1(t) \text{ a.e.} \end{array} \right\},$$

$(X_0, X_1)_{\theta,p}$ is a Banach space. In the particular case when $p = 2$ and X_0, X_1 are Hilbert spaces, we shall denote $(X_0, X_1)_{\theta,2} = [X_0, X_1]_\theta$.

Since $0 \in \rho(A(q))$ and $A(q)$ generates an analytic semigroup $T(t;q)$, the fractional δ-powers $(-A(q))^\delta$ of $-A(q)$ are well defined, closed, linear, invertible operators for any $\delta \ge 0$ (see [23, pp 69-75]). Moreover, $(-A(q))^{-\delta}$ has the representation

$$(-A(q))^{-\delta} = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} T(t;q)\, dt,$$

where the integral converges in the uniform operator topology for every $\delta > 0$. Since $A(q)$ is closed and $0 \in \rho(A(q))$, the operator $(-A(q))^{\delta}$ is also closed and invertible for each $\delta > 0$. Therefore, $D\left((-A(q))^{\delta}\right)$ endowed with the topology of the graph norm is a Hilbert space. Since $((-A(q))^{\delta}$ is boundedly invertible, the norm of the graph of $((-A(q))^{\delta}$ is equivalent to the norm $\|z\|_{q,\delta} \doteq \left\|(-A(q))^{\delta} z\right\|_q$. We shall denote by $Z_{q,\delta}$ the Hilbert space $D\left((-A(q))^{\delta}\right)$ endowed with the $\|\cdot\|_{q,\delta}$-norm.

Theorem 2.2. ([26]) *Let $q \in \mathcal{Q}$, $A(q) : D(A(q)) \subset Z_q \to Z_q$ as defined by (2.2)-(2.3), $0 < \delta < 1$ and $Z_{q,\delta}$ as defined above. Then*

i) $Z_{q,\delta} = [D(A(q)), Z_q]_{1-\delta}$, *in the sense of an isomorphism;*

ii) *The norms $\|z\|_{q,\delta}$, $\|z\|_{(D(A(q)),Z_q)_{1-\delta,2}}$ and $\|z\|_q + \left\|t^{1-\delta}A(q)T(t;q)z\right\|_{L^2_*(Z_q)}$ are all equivalent in $D\left((-A(q))^{\delta}\right)$.*

The next lemma shows some relations between the spaces $Z_{q,\delta}$ for different q's.

Lemma 2.3. ([26]) *Let $\delta \in (0,1)$. Then,*

i) *For any pair q, $q^* \in \mathcal{Q}$ the spaces $Z_{q,\delta}$ and $Z_{q^*,\delta}$ are isomorphic.*

ii) *Moreover, for any compact subset $\mathcal{Q}_C$ of $\mathcal{Q}$ the norms $\{\|\cdot\|_{q,\delta} : q \in \mathcal{Q}_C\}$ are uniformly equivalent, i.e., there exist positive constants m, M such that $m\|z\|_{q,\delta} \le \|z\|_{q^*,\delta} \le M\|z\|_{q,\delta}$ for every q, $q^* \in \mathcal{Q}_C$ and all $z \in D\left((-A(q))^{\delta}\right) \cap D\left((-A(q^*))^{\delta}\right)$.*

Consider the following standing hypotheses.

(H1) There exist functions K_f, $K_g \in L^2(0,1)$, $K_f(x) \ge 0$ a.e., $K_g(x) \ge 0$ a.e., such that

$$|f(x,t_1) - f(x,t_2)| \le K_f(x)\,|t_1 - t_2| \quad \text{and} \quad |g(x,t_1) - g(x,t_2)| \le K_g(x)\,|t_1 - t_2|$$

for a.e. $x \in (0,1)$ and all t_1, $t_2 \in [0,T]$.

(H2) $\theta_\Gamma \in H^1(0,T)$ and θ_Γ' is locally Lipschitz continuous in $(0,T)$.

Theorem 2.4. ([26]) *Let $q \in \mathcal{Q}$, $0 < \epsilon < \frac{1}{4}$ and assume that the hypotheses (H1) and (H2) hold. Then,*

i) *for any bounded subset U of $[0,T] \times Z_{q,\frac{3}{4}+\epsilon}$ there exists a constant $L = L(q,U,\theta_\Gamma,f,g)$ such that*

$$\|F(q,t_1,z_1) - F(q,t_2,z_2)\|_q \le L\left(|t_1 - t_2| + \|z_1 - z_2\|_{q,\frac{3}{4}+\epsilon}\right)$$

for all (t_1,z_1), $(t_2,z_2) \in U$, i.e., the function $F(q,t,z) : \mathcal{Q} \times [0,T] \times Z_{q,\frac{3}{4}+\epsilon} \to Z_q$ is locally Lipschitz continuous in t and z. Moreover the constant L can be chosen independent of q on any compact subset of $\mathcal{Q}$;

ii) *for any initial data $z_0 \in D\left((-A(q))^{\frac{3}{4}+\epsilon}\right)$, there exists $t_1 = t_1(q,z_0) > 0$ such that the initial value problem (2.1) has a unique strong solution $z(t;q) \in C([0,t_1) : Z_q) \cap C^1((0,t_1) : Z_q)$. Moreover $\frac{d}{dt}z(t;q) \in C^{\frac{1}{4}-\epsilon}_{\mathrm{loc}}((0,t_1] : Z_q)$, i.e., $\frac{d}{dt}z(t;q)$ is locally Hölder continuous on $(0,t_1]$ with exponent $\frac{1}{4} - \epsilon$.*

Finally, we state the following theorem proved in [26], which states that for any compact subset $\mathcal{Q}_C$ of the admissible parameter set $\mathcal{Q}$, it is possible to find a nontrivial common interval of existence for all solutions $z(t,q)$, $q \in \mathcal{Q}_C$.

Theorem 2.5. ([26]) *Let $\mathcal{Q}_C$ be a compact subset of the admissible parameter set $\mathcal{Q}$, $q_0 \in \mathcal{Q}_C$, $z_0 \in Z_{q_0,\delta}$, where $\frac{3}{4} < \delta < 1$. Let $[0, t^M(q)) = [0, t^M(q, z_0))$ denote the maximum interval of existence of the solution $z(t;q)$ with initial condition $z(0;q) = z_0$. Then*

$$t^M(\mathcal{Q}_C) \doteq \inf_{q \in \mathcal{Q}_C} t^M(q) > 0$$

3. CONTINUOUS DEPENDENCE ON THE MODEL PARAMETERS

In this section we show that the mapping $q \to z(\cdot\,;q)$ from the space of admissible parameters $\mathcal{Q}$ into the space of solutions is continuous. More precisely, we shall show that if $\{q_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{Q}$ converging to $q \in \mathcal{Q}$, then the sequence $\{z(t;q_n)\}_{n=1}^{\infty}$ converges to $z(t;q)$ in some appropriate sense.

Throughout this section, to simplify the notation we will denote with $A_n = A(q_n)$, $A = A(q)$, $T_n(t) = T(t;q_n)$, $T(t) = T(t;q)$, $z_n(t) = z(t;q_n)$ and $z(t) = z(t;q)$.

We shall need the following lemmas.

Lemma 3.1. *Let $\{q_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{Q}$, $q_n \to q \in \mathcal{Q}$, and let A, A_n, T, T_n be as above. Then*

$$\|A_n T_n(t)z - AT(t)z\|_q \to 0 \qquad \text{as } n \to \infty$$

for every $z \in Z_q$ and $t > 0$.

Proof. Let $z \in Z_q$. Since $T_n(t)$, $T(t)$ are analytic semigroups, $T_n(t)z$, $T(t)z$, are in $D(A_n)$, $D(A)$, respectively $\forall t > 0$. From Theorem 3.5 in [25] it follows that there exists an angle θ, $0 < \theta < \frac{\pi}{2}$, such that the angular sector

$$\Sigma_\theta = \{0\} \cup \{\lambda \in \mathbb{C} : |\arg\,\lambda| < \frac{\pi}{2} + \theta\} \subset \rho(A) \cap \bigcap_{n=1}^{\infty} \rho(A_n).$$

Now, let $\frac{\pi}{2} < \theta_1 < \frac{\pi}{2} + \theta$ and let Γ be the path composed of the two rays $re^{-i\theta_1}$, $re^{i\theta_1}$, $0 \le r < \infty$, Γ oriented so that $\mathrm{Im}(\lambda)$ increases along Γ. We have the following expresions (see [23])

$$AT(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A) z\, d\lambda,$$

$$A_n T_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t} R(\lambda; A_n) z\, d\lambda,$$

for every $z \in Z_q$, $t > 0$, where $R(\lambda; A) = (\lambda I - A)^{-1}$, $R(\lambda; A_n) = (\lambda I - A_n)^{-1}$.

Then

$$AT(t)z - A_nT_n(t)z = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}\left(R(\lambda;A) - R(\lambda;A_n)\right) z\, d\lambda. \tag{3.1}$$

But

$$\begin{aligned}\|\lambda e^{\lambda t}\left(R(\lambda;A) - R(\lambda;A_n)\right) z\|_q &\le |\lambda| e^{\mathrm{Re}(\lambda)t}\left(\frac{1}{|\lambda|} + \frac{C}{|\lambda|}\right)\|z\|_q \\ &\le (1+C)e^{\mathrm{Re}(\lambda)t}\|z\|_q \in L^1(\Gamma),\end{aligned}$$

where the constant C appears because of the uniform equivalence of the norms $\|\cdot\|_{q_n}$ and $\|\cdot\|_q$. Also, for any fixed $\lambda \in \Gamma$

$$\|\left(R(\lambda;A) - R(\lambda;A_n)\right) z\|_q \to 0 \qquad \text{as } n \to \infty.$$

In fact,

$$\begin{aligned}\|\left(R(\lambda;A) - R(\lambda;A_n)\right) z\|_q &= \|R(\lambda;A_n)\left[(\lambda I - A_n)R(\lambda;A) - I\right] z\|_q \\ &= \|R(\lambda;A_n)(A - A_n)R(\lambda;A)z\|_q \\ &\le \|R(\lambda;A_n)\|_{\mathcal{L}(Z_q)}\|(A - A_n)R(\lambda;A)z\|_q\end{aligned}$$

which converges to zero as n goes to infinity by virtue of the uniform boundedness of $\|R(\lambda;A_n)\|_{\mathcal{L}(Z_q)}$ and the strong convergence of A_n to A (which follows immediately from the definition of A_n and A, and the convergence of q_n to q).

The lemma then follows from (3.1) and the Dominated Convergence Theorem. ∎

Lemma 3.2. *Under the same hypotheses of Lemma 3.1*

$$\left\|(-A)^\delta(T(t) - T_n(t))z\right\|_q \to 0 \qquad \text{as } n \to \infty$$

for every $z \in Z_q$, $\delta \in [0,1]$ *and* $t \ge 0$.

Remark. We note here that the assertion of Lemma 3.2 could be obtained immediately if $(-A)^\delta$ commuted with $T_n(t)$. However, this is not true since A_n does not commute with A, as it can be easily verified.

Proof of Lemma 3.2. It suffices to show the result for $\delta = 1$. We can write

$$\begin{aligned}\|A(T(t) - T_n(t))z\| &= \left\|[AT(t) - A_nT_n(t) + (I - AA_n^{-1})A_nT_n(t)]z\right\|_q \\ &\le \|(AT(t) - A_nT_n(t))z\|_q + \left\|I - AA_n^{-1}\right\|_{\mathcal{L}(Z_q)}\|A_nT_n(t)z\|_q.\end{aligned}$$

As a consequence of Lemma 3.1 the first term on the right of the above inequality tends to zero as n goes to infinity and the sequence $\{\|A_nT_n(t)z\|_q\}_{n=1}^\infty$ is bounded. A straightforward calculation using the definition of $A(q)$ shows that for any pair of admissible parameters $q = (\rho, C_v, \beta, \alpha_2, \alpha_4, \alpha_6, \theta_1, \gamma)$, $\tilde{q} = (\tilde{\rho}, \tilde{C}_v, \tilde{\beta}, \tilde{\alpha}_2, \tilde{\alpha}_4, \tilde{\alpha}_6, \tilde{\theta}_1, \tilde{\gamma})$ $\in \mathcal{Q}$ and any $z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in Z_q$

$$A(\tilde{q})A^{-1}(q)z = \begin{pmatrix} u \\ \left(\tilde{\beta} - \beta\frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma}\right)u'' + \frac{\rho\tilde{\gamma}}{\tilde{\rho}\gamma}v \\ \left(\frac{C_v}{\tilde{C}_v}\right)w \end{pmatrix}, \tag{3.2}$$

from which it follows immediately that $\|I - AA_n^{-1}\|_{\mathcal{L}(Z_q)} \to 0$ as $n \to \infty$. The theorem then follows. ∎

Lemma 3.3. *Let $\mathcal{Q}_C$ be a compact subset of $\mathcal{Q}$. Then for any $\delta \in [0,1]$ there exists a constant C depending only on δ and $\mathcal{Q}_C$ such that*

$$\|(-A(q_1))^\delta(-A(q_2))^{-\delta}\|_{\mathcal{L}(Z_{q_3})} \leq C$$

for every q_1, q_2, $q_3 \in \mathcal{Q}_C$.

Proof. Since the operator $A(q)$ is maximal dissipative (Theorem 2.1), the space $Z_{q,\delta}$ is isomorphic to the real interpolation space $[D(A(q)), Z_q]_{1-\delta}$, of order $1-\delta$ between Z_q and $D(A(q))$ (see [1]), i.e.

$$\left(D\left((-A(q))^\delta\right), \|\cdot\|_{q,\delta}\right) \cong [D(A(q)), Z_q]_{1-\delta}. \tag{3.3}$$

From (3.2) it follows that there exists a constant C depending only on $\mathcal{Q}_C$ such that $\|A(\tilde{q})A^{-1}(q)z\|_{\tilde{q}} \leq C\|z\|_{\tilde{q}}$ for every $q, \tilde{q} \in \mathcal{Q}_C$, $z \in Z_q$. Letting $\eta = A^{-1}(q)z$ we obtain

$$\|A(\tilde{q})\eta\|_{\tilde{q}} \leq C\|A(q)\eta\|_{\tilde{q}} \qquad \text{for all } q, \tilde{q} \in \mathcal{Q}_C,\ \eta \in D(A(q)). \tag{3.4}$$

Since the $\|\cdot\|_q$-norms are uniformly equivalent for $q \in \mathcal{Q}_C$, it follows from (3.4) and (3.3) that the norms $\|\cdot\|_{q,\delta}$ are also uniformly equivalent for $q \in \mathcal{Q}_C$. Thus, for any $q_1, q_2, q_3 \in \mathcal{Q}_C$

$$\begin{aligned}
\|(-A(q_1))^\delta(-A(q_2))^{-\delta}z\|_{q_3} &\leq C_1\|(-A(q_1))^\delta(-A(q_2))^{-\delta}z\|_{q_1} \\
&= C_1\|(-A(q_2))^{-\delta}z\|_{q_1,\delta} \\
&\leq C_1\, C_2\|(-A(q_2))^{-\delta}z\|_{q_2,\delta} \\
&= C_1\, C_2\|z\|_{q_2} \\
&\leq C_1\, C_2\, C_3\|z\|_{q_3},
\end{aligned}$$

where the constants C_i, $i = 1, 2, 3$, depend only on $\mathcal{Q}_C$ and δ. ■

Remark. Since $T_n(t)$ is an analytic semigroup of contractions, by a well known result on semigroup theory ([23]), for any $\delta \in (0,1]$, there exists a constant C_δ independent of n such that

$$\|(-A_n)^\delta T_n(t)\|_{\mathcal{L}(Z_{q_n})} \leq \frac{C_\delta}{t^\delta|\cos\nu_n|}$$

where ν_n is any angle in $(\frac{\pi}{2}, \pi)$ for which

$$\rho(A_n) \supset \{0\} \cup \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \nu_n\}.$$

As we mentioned in Lemma 3.1, in this case the angle ν_n above can be chosen independent of n. Hence, there exists a constant $\tilde{C}_\delta$ depending only on δ such that

$$\|(-A_n)^\delta T_n(t)\|_{\mathcal{L}(Z_{q_n})} \leq \frac{\tilde{C}_\delta}{t^\delta} \quad \forall n = 1, 2, \cdots.$$

Next, we state a lemma whose proof can be found in [14] (Lemma 7.1.1).

Lemma 3.4. *Suppose $L \geq 0$, $0 < \delta < 1$ and $a(t)$ is a nonnegative, locally integrable function on $0 \leq t \leq T$. Let $u(t)$ be a real valued function defined on $[0, T]$ satisfying*

$$u(t) \leq a(t) + L \int_0^t \frac{1}{(t-s)^\delta} u(s)\, ds$$

on this interval. Then, there exists a constant $K = K(\delta)$ such that

$$u(t) \leq a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta}\, ds \quad \text{for } 0 \leq t < T.$$

The following theorem will be essential for our main result.

Theorem 3.5. *Let $\delta \in \left(\frac{3}{4}, 1\right)$, $\{q_n\}_{n=1}^\infty \subset \mathcal{Q}$, $q_n \to q \in \mathcal{Q}$, and $z_n(t)$, $z(t)$ be the solutions of the IVP (2.1) with initial datum $z_0 \in D\left((-A)^\delta\right)$ corresponding to the parameters q_n and q, respectively, and let $[0, t_1)$ be the maximal interval of existence of $z(t)$. Then, for any $t_1' < t_1$ there exists a constant N_0 such that $z_n(t)$ exists on $[0, t_1']$ for every $n \geq N_0$ and a constant D such that*

$$\|z_n(t)\|_{q,\delta} \leq D, \quad \forall n \geq N_0,\ \forall t \in [0, t_1'].$$

Proof. Let $\delta \in \left(\frac{3}{4}, 1\right)$, $0 < t_1' < t_1$, and $t_1^n > 0$ be such that $z_n(t)$ exists on $[0, t_1^n)$ for each $n \in \mathbb{N}$. Then, for $t \in [0, \min\{t_1', t_1^n\})$

$$z(t) = T(t)z_0 + \int_0^t T(t-s)F(s, z(s))\, ds$$

$$z_n(t) = T_n(t)z_0 + \int_0^t T_n(t-s)F_n(s, z_n(s))\, ds,$$

which imply

$$\begin{aligned}
\|z(t) - z_n(t)\|_{q,\delta} &= \|(-A)^\delta z(t) - (-A)^\delta z_n(t)\|_q \\
&\leq \left\|(-A)^\delta \left(T(t) - T_n(t)\right) z_0\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T(t-s)F(q, s, z(s)) - (-A)^\delta T_n(t-s)F(q_n, s, z_n(s))\, ds\right\|_q \\
&\leq \left\|(-A)^\delta \left(T(t) - T_n(t)\right) z_0\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T(t-s)F(q, s, z(s)) - (-A)^\delta T_n(t-s)F(q, s, z(s))\, ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T_n(t-s)\left[F(q, s, z(s)) - F(q_n, s, z(s))\right] ds\right\|_q \\
&\quad + \left\|\int_0^t (-A)^\delta T_n(t-s)\left[F(q_n, s, z(s)) - F(q_n, s, z_n(s))\right] ds\right\|_q \\
&\doteq I_1^n(t) + I_2^n(t) + I_3^n(t) + I_4^n(t).
\end{aligned}$$

Note that, even when this last inequality is true on $[0, \min\{t_1', t_1^n\})$, $I_1^n(t)$, $I_2^n(t)$ and $I_3^n(t)$ are well defined on $[0, t_1']$.

We have the following estimates

$$\begin{aligned}
I_3^n(t) &\leq \int_0^t \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \, ds \\
&\leq C_1 \int_0^t \|(-A_n)^\delta T_n(t-s)\|_{\mathcal{L}(Z_{q_n})} \|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \, ds \\
&\leq C_1 \int_0^t \frac{C_\delta}{(t-s)^\delta} \|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \, ds.
\end{aligned}$$

The second and third inequality follow from Lemma 3.3 and the Remark preceding Lemma 3.4, respectively. Now, for any $s \in [0, t_1']$, $\|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \to 0$ as $n \to \infty$. Also, there exists a constant C_2 independent of n such that $\|F(q,s,z(s)) - F(q_n,s,z(s))\|_q \leq C_2$ for every $s \in [0, t_1']$, which follows easily from the continuity of $z(s)$ and the definition of F. Therefore, $I_3^n(t) \to 0$ as $n \to \infty$ on $[0, t_1']$ by the Dominated Convergence Theorem and $I_3^n(t) \leq \dfrac{C_1 C_2 C_\delta}{1-\delta} t^{1-\delta}$, $\forall n \in \mathbb{N}$, $\forall t \in [0, t_1']$.

To estimate $I_2^n(t)$, observe that

$$I_2^n(t) \leq \int_0^t \|(-A)^\delta \left(T(t-s) - T_n(t-s)\right) F(q,s,z(s))\|_q \, ds.$$

Now, $\|F(q,s,z(s))\|_q$ is uniformly bounded on $[0, t_1']$, say $\|F(q,s,z(s))\|_q \leq C_3$, $\forall t \in [0, t_1']$ and

$$\begin{aligned}
&\|(-A)^\delta (T(t-s) - T_n(t-s))\|_{\mathcal{L}(Z_q)} \\
&\qquad \leq \|(-A)^\delta T(t-s)\|_{\mathcal{L}(Z_q)} + \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \\
&\qquad \leq \|(-A)^\delta T(t-s)\|_{\mathcal{L}(Z_q)} + C\|(-A_n)^\delta T_n(t-s)\|_{\mathcal{L}(Z_{q_n})} \\
&\qquad \leq \frac{C_\delta}{(t-s)^\delta} + \frac{C\, C_\delta}{(t-s)^\delta} = \frac{C_4}{(t-s)^\delta}.
\end{aligned}$$

On the other hand, for any $s \in [0, t_1']$ we have

$$\|(-A)^\delta \left(T(t-s) - T_n(t-s)\right) F(q,s,z(s))\|_q \to 0 \quad \text{as} \quad n \to \infty$$

by Lemma 3.2. Therefore $I_2^n(t) \to 0$ as $n \to \infty$ by the Dominated Convergence Theorem, and also $I_2^n(t) \leq \dfrac{C_3 C_4}{1-\delta} t^{1-\delta}$, $\forall n$, $\forall t \in [0, t_1']$.

In regard to $I_1^n(t)$ observe that

$$\begin{aligned}
I_1^n(t) &= \left\|(-A)^\delta \left(T_n(t) - T(t)\right) z_0\right\|_q \\
&= \left\|(-A)^\delta (-A_n)^{-\delta} (-A_n)^\delta T_n(t) z_0 - (-A)^\delta T(t) z_0\right\|_q \\
&\leq C \left\|T_n(t)(-A_n)^\delta z_0\right\|_q + \left\|T(t)(-A)^\delta z_0\right\|_q \\
&\leq C \|T_n(t)\|_{\mathcal{L}(Z_q)} C \left\|(-A)^\delta z_0\right\|_q + \|T(t)\|_{\mathcal{L}(Z_q)} \left\|(-A)^\delta z_0\right\|_q \\
&\leq C_5 \left\|(-A)^\delta z_0\right\|_q,
\end{aligned}$$

where we have used that $z_0 \in D\left((-A)^\delta\right)$ and the semigroups are contractive. Also, by Lemma 3.2 $I_1^n(t) \to 0$ as $n \to \infty$.

Similarly,

$$\begin{aligned} I_4^n(t) &\leq \int_0^t \|(-A)^\delta T_n(t-s)\|_{\mathcal{L}(Z_q)} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds \\ &\leq C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds. \end{aligned}$$

From the above estimates on $I_1^n(t)$, $I_2^n(t)$, $I_3^n(t)$ and $I_4^n(t)$, there follows

$$\|z(t) - z_n(t)\|_{q,\delta} \leq \epsilon_n(t) + C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds \quad (3.5)$$

where, for all $t \in [0, t_1']$, $\epsilon_n(t) \doteq I_1^n(t) + I_2^n(t) + I_3^n(t)$ satisfies $0 \leq \epsilon_n(t) \leq C_7$ for all $n \in \mathbb{N}$ and $\epsilon_n(t) \to 0$ as $n \to \infty$. In particular, these conditions imply $\int_0^{t_1'} \epsilon_n(t)\, dt \to 0$ as $n \to \infty$.

Let $K = K(\delta)$ be as in Lemma 3.4 and define $\tilde{K} \doteq C_7 + C_6 C_7 K$ and $M \doteq \sup_{0 \leq t \leq t_1'} \|z(t)\|_{q,\delta}$. From the continuity of $z(t)$ it follows that $M < \infty$. Let $n \in \mathbb{N}$. Since $z(0) = z_n(0) = z_0$, there exists $\delta_n > 0$ such that $\|z_n(t)\|_{q,\delta} \leq M + 2\tilde{K}$ for all $t \in [0, \delta_n]$. Let L be a Lipschitz constant for F on the set $U \doteq [0, t_1'] \times \left\{ \|z\|_\delta \leq M + 2\tilde{K} \right\}$, valid for q and all the q_n's. Then, from (3.5) and Lemma 3.4, we have

$$\|z_n(t) - z(t)\|_{q,\delta} \leq f_n(t) \quad \text{on } 0 \leq t \leq \delta_n, \quad (3.6)$$

where $f_n(t) \doteq \epsilon_n(t) + C_6 L K \displaystyle\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds$, for $t \in [0, t_1']$.

Now,

$$\begin{aligned} \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds &\leq \int_0^t \frac{C_7}{(t-s)^\delta}\, ds \\ &= C_7 \int_0^t \frac{1}{s^\delta}\, ds \\ &= \frac{C_7}{1-\delta} t^{1-\delta}. \end{aligned}$$

Choosing $\eta = \eta(L) > 0$ sufficiently small so that $t^{1-\delta} \leq \dfrac{1-\delta}{2L}$ for every $t \in [0, \eta]$, it follows that

$$\int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta}\, ds \leq \frac{C_7}{2L} \quad \text{for every } t \in [0, \eta]. \quad (3.7)$$

On the other hand, if $\eta < t \leq t_1'$

$$\begin{aligned}\int_0^t \frac{\epsilon_n(t)}{(t-s)^\delta}\,ds &= \int_0^t \frac{\epsilon_n(t-s)}{s^\delta}\,ds \\ &= \int_0^\eta \frac{\epsilon_n(t-s)}{s^\delta}\,ds + \int_\eta^t \frac{\epsilon_n(t-s)}{s^\delta}\,ds \\ &\leq \frac{C_7}{2L} + \frac{1}{\eta^\delta}\int_0^t \epsilon_n(t-s)\,ds \\ &\leq \frac{C_7}{2L} + \frac{1}{\eta^\delta}\int_0^{t_1'} \epsilon_n(s)\,ds.\end{aligned}$$

Hence, since $\int_0^{t_1'} \epsilon_n(s)\,ds \to 0$, there exists N_0 such that

$$\int_0^t \frac{\epsilon_n(t)}{(t-s)^\delta} \leq \frac{C_7}{2L} + \frac{C_7}{2L} = \frac{C_7}{L} \quad \forall t \in [\eta, t_1'] \text{ and } n \geq N_0. \tag{3.8}$$

From (3.7) and (3.8) it follows that

$$f_n(t) \leq C_7 + C_6 C_7 K \quad \forall t \in [0, t_1'] \text{ and } n \geq N_0. \tag{3.9}$$

Consequently, from (3.6) and (3.9)

$$\|z_n(t) - z(t)\|_{q,\delta} \leq \tilde{K} \quad \forall n \geq N_0 \text{ and } t \in [0, \delta_n],$$

which implies

$$\|z_n(t)\|_{q,\delta} \leq M + \tilde{K} \quad \forall n \geq N_0 \text{ and } t \in [0, \delta_n]. \tag{3.10}$$

Finally, let $n \geq N_0$ be fixed. We claim that $z_n(t)$ exists on $[0, t_1']$ and for $t \in [0, t_1']$, $\|z_n(t)\|_{q,\delta} < M + 2\tilde{K}$. In fact, suppose, on the contrary, that there exists $t^* \leq t_1'$ such that $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$ and $\|z_n(t)\|_{q,\delta} < M + 2\tilde{K}$ for $0 \leq t < t^*$. Then, in (3.6), δ_n can be replaced by t^* and (3.10) follows with $\delta_n = t^*$, i.e. $\|z_n(t)\|_{q,\delta} \leq M + \tilde{K}$ on $[0, t^*]$. This contradicts $\|z_n(t^*)\|_{q,\delta} = M + 2\tilde{K}$. The theorem then follows taking $D = M + 2\tilde{K}$. ∎

Theorem 3.6. *Under the same hypotheses of Theorem 3.5*

$$\|z_n(t) - z(t)\|_{q,\delta} \to 0, \quad \text{as } n \to \infty$$

for every $t \in [0, t_1)$.

Remark. If the initial data is smooth enough, then the results in [28] and [32] imply that $t_1 = \infty$ and therefore, this theorem ensures the $\|\cdot\|_{q,\delta}$-convergence of $z_n(t)$ to $z(t)$ on the whole interval $[0, \infty)$.

Proof of Theorem 3.6. Let $\delta \in \left(\frac{3}{4}, 1\right)$ and $t_1' < t_1$. By Theorem 3.5 there exist $N_0 \in \mathbb{N}$ and $D > 0$ such that $z_n(t)$ exists and $\|z_n(t)\|_{q,\delta} \leq D$ on $[0, t_1']$ for every

$n \geq N_0$. Following the steps of Theorem 3.5 we see that for every $t \in [0, t_1']$ and $n \geq N_0$

$$\begin{aligned}\|z(t) - z_n(t)\|_{q,\delta} &\leq \epsilon_n(t) + C_6 \int_0^t \frac{1}{(t-s)^\delta} \|F(q_n, s, z(s)) - F(q_n, s, z_n(s))\|_q \, ds \\ &\leq \epsilon_n(t) + LC_6 \int_0^t \frac{1}{(t-s)^\delta} \|z(s) - z_n(s)\|_{q,\delta} \, ds\end{aligned}$$

where $0 \leq \epsilon_n(t) \leq C_7$ and $\epsilon_n(t) \to 0$ as $n \to \infty$ for every $t \in [0, t_1']$. In the last inequality we have used the fact that F is locally Lipschitz continuous and $\|z_n(t)\|_{q,\delta} \leq D$, $\forall n \geq N_0$, $\forall t \in [0, t_1']$.

Hence, by Lemma 3.4, there exists $K > 0$ such that

$$\|z(t) - z_n(t)\|_{q,\delta} \leq \epsilon_n(t) + K \int_0^t \frac{\epsilon_n(s)}{(t-s)^\delta} \, ds \longrightarrow 0 \quad \text{as } n \to \infty.$$

Since t_1' is arbitrary, the theorem follows. ∎

4. CONCLUSIONS

In this paper we have shown that the solutions of the IBVP (1.1), with free energy potential Ψ in the Landau-Ginzburg form (1.2), depend continuously on the parameters ρ, C_v, β, α_2, α_4, α_6, θ_1 and γ. In particular, we have shown that if $\{q_n = (\rho_n, C_{v,n}, \beta_n, \alpha_{2,n}, \alpha_{4,n}, \alpha_{6,n}, \theta_{1,n}, \gamma_n)\}_{n=1}^{\infty}$ is a sequence of admissible parameters converging to the admissible parameter q, then not only $z(t; q_n) \to z(t; q)$ in the norm of Z_q, but also in the stronger $\|\cdot\|_{q,\delta}$-norm ($\delta = \frac{3}{4} + \epsilon$). This constitutes an important step towards solving the parameter identifiability and the ID problems for system (1.1). These problems, to which we are already devoting efforts, involve also showing that the mapping $q \to z(\cdot\,; q)$ from the admissible parameter set $\mathcal{Q}$ into the space of solutions is locally one-to-one. Results on this issue will be published in a forthcoming article.

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THE α-CONCENTRATION OF PROCACCIA OF INFINITE WORDS IN FINITELY GENERATED FUCHSIAN GROUPS.

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ABSTRACT. In order to study the spectral decomposition $(\alpha, f(\alpha))$ of Procaccia of the limit set L(G) of a finitely generated Fuchsian group G of rigid movements in the hyperbolic half plane $\mathbb{H}$, it is necessary to calculate the α of each element of L(G). Each such element is an allowed infinite word, each letter a generator of G. In this paper we calculate first the α of the periodic infinite words, and use this result in order to calculate the α of the non-periodic irrational words.

SECTION 1. INTRODUCTION.

In 1993, a method [1] was proposed to generate fractals Ω such that their multifractal decomposition $(\alpha, f(\alpha))$ of Procaccia modelled all $(\alpha, f(\alpha))$ curves in the Tel classification [2].

The importance of the curves $(\alpha, f(\alpha))$ in the Tel classification and their relevance to the study of a variety of physical phenomena is described in [1]. The fractal sets Ω generated in [1] are the limit sets Ω=L(G) of minimally generated groups G, all generators being rigid movements in $\mathbb{H}$ and having zero trace.

The importance of expressing the elements of Ω=L(G) by means of an infinite word code —each letter a generator of G— is reviewed in [3].

Let us deal then with the α – *concentration* of Procaccia of infinite words coding for elements in Ω=L(G), when G is minimally generated by zero-trace generators (three generators).

Generators A, B, and C have zero trace; then no two letters can be repeated in an allowed word, i.e. a word with correct spelling. Words $W_1 = ABABAB...$ and $W_2 = ABCABC...$ are allowed words denoting two different points in the fractal Ω=L(G), whereas word AABBAABBAABB... denotes no point in L(G), and does not have a correct spelling.

The transformations S=AB and T=ABC have $|trace| > 2$, i.e. they are hyperbolic transformations. Therefore words W_1 and W_2 can be written as infinite words $W_1 = SSSS...$ and $W_2 = TTTT...$ with hyperbolic letters.

This paper deals with the α of infinite words written with hyperbolic letters; specifically, we will calculate the α of infinite words in L(G), here G is a group generated by two hyperbolic operators: two rigid movements in $\mathbb{H}$.

The results can be easily extended to groups with any finite member of generators.

SECTION 2. CONSTRUCTION OF THE LIMIT SET $\mathbb{F}$ OF A FUCHSIAN SEMIGROUP GENERATED BY TWO HYPERBOLIC 2×2 MATRICES.

SECTION 2.1. GENERALITIES AND NOTATION.

Let $T(z) = \dfrac{az+b}{cz+d}$ be an element of the unimodular group U, i.e. $a, b, c,$ and d are integers, and $ad - bc = 1$. The transformation T(z) operates on the values

$$z \in \mathbb{H} = \{x + iy/y > o\}, \quad T : \mathbb{H} \to \mathbb{H}$$

Let us recall that the set $\{z \in \mathbb{H}/|cz+d| \leq 1\} = \{z/|T'(z)| \geq 1\}$ is the isometric circle of $T = T(z)$. With C_T, g_T, and r_T we will denote the isometric circle of T, its centre, and its radius, respectively.

We have $g = \dfrac{-d}{c}$ and $r = \dfrac{1}{|c|}$.

Let us also recall that every hyperbolic $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (i.e. $|traceT| = |a+d| > 2$) has two real fixed numbers, one an attractor, the other a repeller. The repeller belongs to C_T, and the attractor, hereafter denoted as ξ_T, is always inside $C_{T^{-1}}$. Let us recall that if A is hyperbolic then $\mathbb{H} - C_A$ is mapped, by A, onto $Int.C_{A^{-1}}$, and that ∂C_A is mapped onto $\partial C_{A^{-1}}$.

From now on, A and B will be hyperbolic elements of U such that $C_A, C_{A^{-1}}, C_B$ and $C_{B^{-1}}$, are disjoint (see Fig.1)

Let $S(A, B)$ denote the semigroup generated by A and B. Let $x \in \mathbb{H} - (C_A \cup C_B)$. Let $\mathbb{F}(x)$ denote the limit set of $\{T(x)/T \in S(A, B)\}$. It is not hard to prove that, if $y \in \mathbb{H} - (C_A \cup C_B)$, $y \neq x$, we have $\mathbb{F}(x) = \mathbb{F}(y)$. Hence, with $\mathbb{F}$ we will denote $\mathbb{F}(x)$ (for any x in $\mathbb{H}$), and we will call it the limit set of $S(A, B)$.

SECTION 2.2.

Let us now construct a fractal F associated with $S(A, B)$. We will construct it in stages, following an iterative process similar to the one that yields the Cantor ternary. Let us write

$$R = \{x \in \mathbb{R}/x \notin C_A\} \text{ and } S = \{x \in \mathbb{R}/x \notin C_B\}$$

STEP 1. We have, then, $cl.A(R) = C_{A^{-1}} \cap \mathbb{R}$ and $cl.B(S) = C_{B^{-1}} \cap \mathbb{R}$. These two segments, disjoint by our assumption on the transformations A and B, will be the analogue of the two segments $[0, 1/3]$ and $[2/3, 1]$ which constitute the first step in the construction of the Cantor ternary. *Par abus de langage*, and only when there is no danger of confusion, we will denote with the letters A and B (the same letters that denote the hyperbolic generators), these two sets A(R) and B(S), which are the two segments of the first step; see Fig.2.

STEP 2. In strict analogy to the construction of the Cantor set, we continue with the second step of our iterative process, as shown in Fig.3.

STEP 3. The third step is shown in Fig 4.

...and so on ad infinitum. The fractal F is obtained like the Cantor ternary, i.e. it is the intersection of all these steps.

Note. Hereafter, with a word of two letters A and B, of length N, we will refer indistinctly to the corresponding transformation in $S(A, B)$, and to the corresponding segment in step N in the construction of F just described. **Notice that F is well constructed**: all segments in step N are disjoint and contained in some segment in step N-1:

They are disjoint, since $C_{A^{-1}} \cap C_{B^{-1}} = \emptyset$ by the hypothesis, and since both A and B are one-to-one.

They are contained in some segment in step N-1: let us prove, e.g., that segment ABA is contained in segment AB:

$$ABA = AB[A(R)] = AB(C_{A^{-1}} \cap \mathbb{R}) \subset AB(S) = A(C_{B^{-1}} \cap \mathbb{R}) = AB$$

The same reasoning holds for every case, as we only use $C_{B^{-1}} \cap \mathbb{R} \subset R$ and $C_{A^{-1}} \cap \mathbb{R} \subset S$.

Thus, the 2^N disjoint segments in step N are a covering of F.

SECTION 2.3.

We will prove now that $\mathbb{F} = \mathrm{F}$.

1) $\mathbb{F} \subset \mathrm{F}$. The proof is quite easy: Let us first notice that we can associate a semicircle to each segment in any step N of the construction of F, as shown in Fig.5.

Par un tres grand abus de langage indeed, we will denote, with a word of N letters $A\&B$, **three** things now: the corresponding transformation, the corresponding segment in the step N of the construction of F, and the corresponding associated semicircle, and we will make sure that there will be no danger of confusion.

Let us now consider $\xi \in \mathbb{F}$. ξ is, then, a point in $\mathbb{R}$, approximated by elements of a convergent sequence $\{T_N(x)\}_{N \in \mathbb{N}}$, where T_N is a transformation of N letters A and B, and x is, as before, in $\mathbb{H} - (\mathrm{C_A} \cup \mathrm{C_B})$. The reader can infer that ξ is in F by pondering on the following facts:

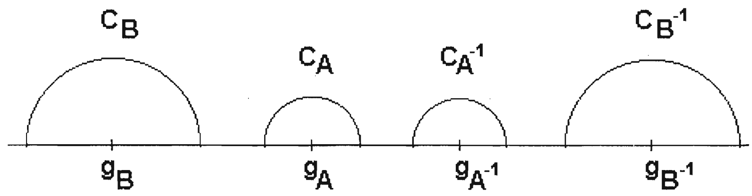

Fig.1

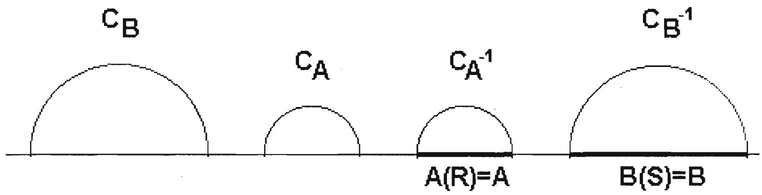

Fig.2

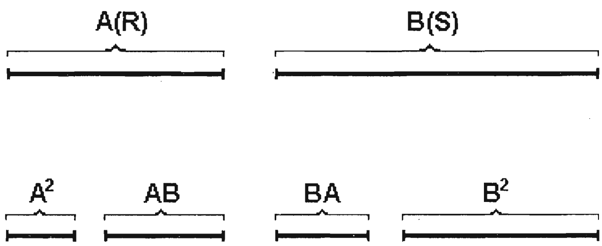

Fig.3

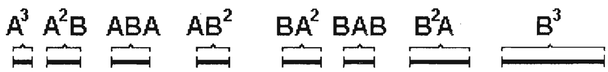

Fig.4

a) $\xi \in \mathbb{R}$,

b) $T_N(x) \longrightarrow \xi$ as $N \longrightarrow \infty$,

c) $T_N(x)$ belong to smaller and smaller semicircles T_N, like the ones in Fig.5, which have to be —for big values of N— one inside the other, due to the convergence of $\{T_N(x)\}$, $N \in \mathbb{N}$.

d) The closeness of the segments in step N of the construction of F, and the inclusion of the boundary of the semicircles referred to in c) completes what we need to prove that $\xi \in F$.

2) $F \subset \mathbb{F}$ is an easy excercise, left to the reader.

SECTION 3. THE INFINITE WORDS IN F AND THEIR α-CONCENTRATION OF PROCACCIA.

SECTION 3.1. INFINITE WORDS.

Let us recall that the finite words of length N made up of two letters A and B are a covering of F by disjoint closed segments; with C_N we will denote this covering. Each $\xi \in F$ will belong to just one such segment $I_N(\xi)$ in C_N. For growing values of N, there is a unique sequence of such intervals of decreasing size, one inside the other, associated with a growing-in-length word in letters A and B. Therefore, ξ is represented by a unique infinite word.

Such an infinite word in two letters can have a structure analogous to that of a rational number written in a binary way, that is, it can have a period, indefinitely repeated, preceded by a finite number of letters which do not necessarily show a periodic arrangement. When such is the case, we will say that ξ is represented by a "rational word".

Observation: if the finite word T is the period of a rational infinite word ξ, then ξ, as a point, is the fixed point atractor ξ_T of the corresponding transformation T.

Lemma 1: *The set of rational word points in F is dense in F with the usual topology of $\mathbb{R}$. This density is also valid if the topology of $\mathbb{R}$ is replaced by the one associated with the Hausdorff measure corresponding to the Hausdorff dimension of the fractal set F.*

The proof is left to the interested reader.

SECTION 3.2. THE α-CONCENTRATION OF PROCACCIA $\alpha(\xi)$ ASSOCIATED WITH A POINT $\xi \in F$.

Following Procaccia, Hensen and others [4], we consider the set F endowed with a probability measure P, and let us recall that the concentration of Procaccia relates lengths of intervals I_N —in the covering by intervals C_N— to the corresponding probabilities $P(I_N \cap F)$ associated with each $F \cap I_N$, in the following way:

$$P(I_N \cap F) = [\mu(I_N)]^{\alpha(I_N)},$$

where μ is the usual measure in $\mathbb{R}^1$.

Hereafter, we will consider all such intervals I_N in C_N as equiprobable, so that $P(I_N \cap F) = \frac{1}{2^N}$ for any of the 2^N intervals in the N^{th} step of the construction of F.

If $\xi \in F$, then there is a unique $I_N = I_N(\xi)$ to which ξ belongs. We will define the "N-aproximated α -- *concentration* of ξ" —abbreviated as $\alpha^N(\xi)$— by the quotient

$$\alpha^N(\xi) = \frac{ln(1/2^N)}{ln(\mu(I_N(\xi)))}$$

We know that [4]

$$\alpha(\xi) = \lim_{N\to\infty} \alpha^N(\xi)$$

when the limit exists.

SECTION 3.3. THE CONCENTRATION $\alpha(\xi)$ OF POINTS ξ ASSOCIATED WITH AN INFINITE RATIONAL WORD.

We will prove

Theorem 1: *Let ξ be a point associated with an infinite rational word, in letters A and B. Let $m \in \mathbb{N}$ be the number of letters in the period of this rational word. Let T be the period itself, a finite word of m letters. Then*

$$\alpha(\xi) = \frac{m\ ln2}{2\ ln|autT|},$$

where autT indicates the largest eigenvalue of T, in absolute value.

Proof: The author has proved this lemma in [3].

SECTION 3.4. THE CONCENTRATION $\alpha(\xi)$ OF POINTS ξ ASSOCIATED WITH ANY INFINITE WORD, RATIONAL OR NOT.

The following theorem expresses the concentration $\alpha(\xi)$, ξ an irrational word, in terms of the α – *concentration* of different rational words.

Theorem 2: *Let $\xi \in F$ and $N \in \mathbb{N}$. Let $I_N(\xi)$ be the only interval in C_N to which ξ belongs. Let T_N be the word of N letters A and B associated with the interval $I_N(\xi)$. Let us consider the corresponding transformation T_N, and let us denote by ξ_N its fixed point (attractor).*
Then we have:

$$\lim_{N\to\infty} \alpha(\xi_N) = \alpha(\xi)$$

Proof: We need a

Lemma: *Under all hypothesis of theorem 2 we have*

$$|\alpha^N(\xi_N) - \alpha(\xi_N)| \longrightarrow 0 \text{ as } N \longrightarrow \infty .$$

Let us suppose the lemma already proved. Let us consider the infinite rational word of period T_N. Since we saw that the corresponding associated point is precisely ξ_N (see the observation in section 3.1), we will think of ξ_N also as an infinite rational word, with a period of N letters.

We will show that

$$\alpha(\xi_N) \longrightarrow \alpha(\xi) \text{ when } N \longrightarrow \infty,$$

that is, the concentration of ξ will be approximated by concentrations of rational words.

Now:

$$|\alpha(\xi_N) - \alpha(\xi)| \leq |\alpha(\xi_N) - \alpha^N(\xi_N)| + |\alpha^N(\xi_N) - \alpha^N(\xi)| + |\alpha^N(\xi) - \alpha(\xi)|.$$

Let $\epsilon > 0$ be arbitrary and fixed. By our lemma, there exists $N_0 \in \mathbb{N}$ such that $N \geq N_0$ implies

$$|\alpha(\xi_N) - \alpha^N(\xi_N)| < \frac{\epsilon}{2}.$$

Next, we observe that $I^N(\xi) = I^N(\xi_N)$ for every $N \in \mathbb{N}$. Therefore, $\alpha^N(\xi_N) - \alpha^N(\xi) = 0$.

Since, by definition,

$$\alpha(\xi) = \lim_{N\to\infty} \alpha^N(\xi),$$

there exists $N_1 \in \mathbb{N}$ such that $N \geq N_1$ implies

$$|\alpha^N(\xi) - \alpha(\xi)| < \epsilon/2.$$

The theorem is proved.

SECTION 3.5. PROOF OF THE LEMMA IN SECTION 3.4.

We know that

$$|\alpha^N(\xi_N) - \alpha(\xi_N)| = |\frac{ln[1/2^N]}{ln[\mu(I^N(\xi_N))]} - \frac{N ln2}{2ln|autT_N|}| =$$

$$= N ln2 |\frac{2ln|autT_N| + ln[\mu(I^N(\xi_N))]}{2ln[\mu(I^N(\xi_N))] ln|autT_N|}| = N\frac{ln2}{2}|\frac{ln[|autT_N|^2 \mu(I^N(\xi_N))]}{ln[\mu(I^N(\xi_N))] ln|autT_N|}| \qquad (1)$$

Let us follow the three steps shown below:

Claim I

$$|ln[\mu(I^N(\xi_N))]| \geq ln|\frac{c|p-q|}{L^{2N}\delta^2}|,$$

where c, p, q, δ, and L are constants depending only on A and B, and $L > 1$.

Claim II

$$|ln|autT_N|| \geq ln|L^N\frac{\delta}{c}|,$$

where L, δ, and c are the same constants in Claim I.

Claim III

$$|ln[|autT_N|^2\mu(I^N(\xi_N))]| \leq K,$$

where K is a constant not depending on N.

Proof of Claim I.

Let us first work with $\mu(I^N(\xi_N))$.

Let p and q be the extremes of the interval $C_{A^{-1}}$ if T_N ends in A; otherwise they are the extremes of the segment $C_{B^{-1}}$.

Let $T_N(z) = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$, where $a_N d_N - b_N c_N = 1$.

Then we have that

$$\mu(I^N(\xi_N)) = |T_N(p) - T_N(q)| = |\frac{a_N p + b_N}{c_N p + d_N} - \frac{a_N q + b_N}{c_N q + d_N}| =$$

$$= \frac{|a_N d_N(p-q) - b_N c_N(p-q)|}{|c_N p + d_N||c_N q + d_N|} = \frac{|p-q|}{|c_N^2||p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} \tag{2}$$

a) Let us deal next with $|p+\frac{d_N}{c_N}|$ and $|q+\frac{d_N}{c_N}|$. In order to fix ideas let us suppose that T_N ends in B. We can write

$$|p+\frac{d_N}{c_N}| = |p-(-\frac{d_N}{c_N})|,$$

and let us recall that $-\frac{d_N}{c_N} = g_{T_N}$ is the centre of isometric circle of T_N.

We have that $C_{T_N} \subset C_A$ since T_N ends in B, and therefore $g_{T_N} \in C_A$ —see Fig 6.

Let us define

$$\delta = min\{\text{distances between all extremes of the segments } C_A,\ C_B,\ C_{A^{-1}},\ C_{B^{-1}}\}$$

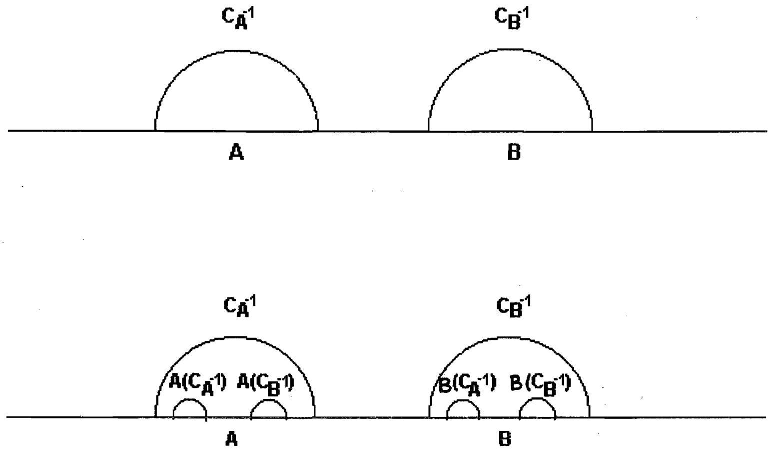
C_A^{-1}
C_B^{-1}
A
B
$A(C_A^{-1})$
$A(C_B^{-1})$
$B(C_A^{-1})$
$B(C_B^{-1})$

Fig.5

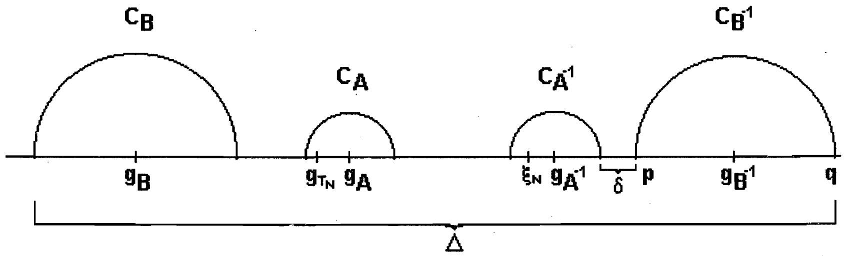
C_B
C_A
C_A^{-1}
C_B^{-1}
g_B
g_{T_N}
g_A
ξ_N
g_A^{-1}
δ
p
g_B^{-1}
q
Δ

Fig.6

as shown in the same figure. We can clearly observe that

$$|p-(-\frac{d_N}{c_N})| \geq \delta \text{ and } |q-(-\frac{d_N}{c_N})| \geq \delta$$

b) Let us deal now with $\frac{1}{|c_N|^2}$. In order to fix ideas, let us suppose that T_N ends in A, i.e. let us write

$$T_N = T_{N-1}A.$$

Let us recall that $\frac{1}{|c_N|}$ is the radius of the isometric circle of T_N. Therefore

$$\frac{1}{|c_N|} = r_{T_N} = \frac{r_{T_{N-1}} r_A}{|g_{A^{-1}} - g_{T_{N-1}}|}, \tag{3}$$

where $g_{A^{-1}}$ and $g_{T_{N-1}}$ are the centres of the isometric circles of A^{-1} and T_{N-1} respectively.

Now $g_{A^{-1}} \in C_{A^{-1}}$, but $g_{T_{N-1}} \in C_A$ *or* C_B, hence

$$|g_{T_{N-1}} - g_{A^{-1}}| > r_{A^{-1}} = r_A.$$

We can strengthen this fact by observing Fig.6 carefully and deducing that there exists $L > 1$ depending solely on the value of δ, such that

$$|g_{T_{N-1}} - g_{A^{-1}}| \geq L r_A.$$

Therefore

$$\frac{1}{|c_N|} = r_{T_N} \leq \frac{r_{T_{N-1}} r_A}{L r_A} = \frac{r_{T_N}}{L}.$$

Iterating this procedure we can write

$$\frac{1}{|c_N|} = r_{T_N} \leq \frac{r_{T_{N-1}}}{L} \leq \frac{r_{T_{N-2}}}{L^2} \leq \ .... \leq \frac{r_{T_1}}{L^{N-1}} = \frac{C}{L^N},$$

where C is a constant which does not depend on N. Therefore

$$\frac{1}{|c_N|} \leq \frac{C}{L^N} \quad L \geq 1, \ or \ |c_N| \geq \frac{L^N}{C}. \tag{4}$$

From equations (2), (3), and (4) we obtain

$$\mu(I^N(\xi)) = \frac{|p-q|}{|c_N|^2|p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} \leq \frac{1}{|c_N|^2}\frac{|p-q|}{\delta^2} \leq \frac{C}{L^{2N}}\frac{|p-q|}{\delta^2}, \text{ where } L \geq 1.$$

Since $\mu(I^N(\xi)) \longrightarrow o \; as \; N \longrightarrow \infty$, we can safely work with values $\mu(I^N(\xi)) < 1$ and $\frac{C|p-q|}{L^{2N}\delta^2} < 1$. But $|lnx|$ is a decreasing function for $o < x < 1$, therefore

$$|ln|\mu(I^N(\xi))| \geq |ln|\frac{C|p-q|}{L^{2n}\delta^2}||,$$

which is claim I.

Proof of claim II

Let us suppose that $\dfrac{a_N + d_N + \sqrt{(a_N+d_N)^2 - 4}}{2}$ is the largest eigenvalue of T_N in absolute value. Then

$$\xi_N = \frac{a_N - d_N + \sqrt{(a_N+d_N)^2-4}}{2c_N}$$

is the fixed point attractor of T_N. Then we can write

$$aut(T_N) = \frac{a_N + d_N + \sqrt{(a_N+d_N)^2-4}}{2} =$$

$$= c_N(\frac{a_N - d_N + \sqrt{(a_N+d_N)^2-4} + 2d_N}{2c_N}) =$$

$$= c_N(\frac{a_N - d_N + \sqrt{(a_N+d_N)^2-4}}{2c_N} - (-\frac{d_N}{c_N})) = c_N(\xi_N - g_{T_N}). \tag{5}$$

The same argument holds if $\dfrac{a_N + d_N - \sqrt{(a_N+d_N)^2-4}}{2}$ is the largest eingevalue of T_N in absolute value.

Now $\xi_N \in C_{A^{-1}}$ or to $C_{B^{-1}}$, and $-\frac{d_N}{c_N} \in C_A$ or to C_B, respectively; therefore, we again have

$$|\xi_N - g_{T_N}| \geq \delta. \tag{6}$$

We will next use Eqs. (5), (4), and (6), in order to obtain

$$|autT_N| = |c_N||\xi_N - g_{T_N}| \geq \frac{L^{2N}}{C}|\xi_N - g_{T_N}| \geq \frac{L^{2N}}{C}\delta\;.$$

Therefore

$$|autT_N| \longrightarrow \infty \; as \; N \longrightarrow \infty,$$

and we can assume that

$$|autT_N| \geq L^N\frac{\delta}{C} > 1,$$

from which

$$|ln|autT_N|| \geq ln(L^N\frac{\delta}{C}),$$

which is claim II.

From Eqs.(5) and (2) we have

$$|autT_N|^2\mu(I_N(\xi)) = |c_N|^2||\xi_N - g_{T_N}|^2 \frac{|p-q|}{|c_N|^2|p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} =$$

$$= \frac{|p-q||\xi_N - g_{T_N}|^2}{|p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} \tag{7}$$

Next, a glance at Fig.6 shows that Eqs. (3) and (6) can be rewritten as

$$(3') \quad \triangle \geq |p-(-\frac{d_N}{c_N})|; \quad \triangle \geq |q-(-\frac{d_N}{c_N})|;$$

and

$$(6') \quad \triangle \geq |\xi_N - g_{T_N}|,$$

and from (3) and (6') we have

$$\frac{|p-q||\xi_N - g_{T_N}|^2}{|p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} \leq \frac{|p-q||\xi_N - g_{T_N}|^2}{\delta^2} \leq \frac{|p-q|\triangle^2}{\delta^2}$$

Also, from Eqs.(3') and (6) we obtain

$$\frac{|p-q||\xi_N - g_{T_N}|^2}{|p+\frac{d_N}{c_N}||q+\frac{d_N}{c_N}|} \geq \frac{|p-q||\xi_N - g_{T_N}|^2}{\triangle^2} \geq \frac{|p-q|\delta^2}{\triangle^2}.$$

We have estimated the quotient (7) above and below.

Therefore

$$|ln|autT_N|\mu(I_N(\xi))| \leq \ max\{|ln[\frac{|p-q|\triangle^2}{\delta^2}]| \text{ and } |ln[\frac{|p-q|\delta^2}{\triangle^2}]\} = K,$$

Which is claim III.

Claims I, II, III, yield

$$|\alpha^N(\xi_N) - \alpha(\xi_N)| = Nln2/2\ |\frac{ln[|autT_N|^2\mu(I^N(\xi))]}{ln[|autT_N|]ln[\mu(I^N(\xi))]}| \leq$$

$$\leq Nln2/2\ |\frac{ln[|autT_N|^2\mu(I^N(\xi)]}{ln[|autT_N|ln[\frac{C|p-q|}{L^{2N}\delta^{2N}}]}| \leq Nln2/2\ |\frac{ln[|autT_N|\mu(I^N(\xi))]}{ln[L^N\delta/C]ln[\frac{C|p-q|}{L^{2N}\delta^{2N}}]}| \leq$$

$$Nln2/2\ |\frac{K}{ln[L^N\delta/C]ln[\frac{C|p-q|}{L^{2N}\delta^2}]}| =$$

$$= Nln2/2\ |\frac{K}{(NlnL + ln\delta/C)(-2NlnL + ln|p-q|C/\delta^2)}| \longrightarrow 0 \text{ as } N \longrightarrow \infty \quad \text{q.e.d.}$$

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A MODIFICATION OF THE ERA AND A DETERMINANTAL APPROACH TO THE STABILITY OF COMPLEX SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

Recently, a new stability criterion for systems of differential equations with complex coefficients has been advanced. It is based on a sequence of polynomials associated with the system. This criterion known as the Extended Routh Array (ERA) suffers the defect that it gets cumbersome and highly complicated as the dimension of the system gets large. In this paper, we propose a modification of the ERA which reduces significantly the burden of computations. The modified array requires only computations of a set of second order determinants. The new algorithm is then applied to produce a determinantal criterion for the stability of the above systems.

A M S Subject Classification: 34E05.

Key Words and Phrases: Stability Criteria, Extended Routh Array, Systems of Differential Equations.

1. Introduction

Tests of stability of systems of differential equations are crucial in many areas of mathematical analysis. In the case of real coefficients, the classical Routh-Hurwitz criterion gives a quite complete solution, among many others see [1,2,3,5]. The case of complex coefficients has recently become an active area of research. Different approaches to this interesting problem are recorded in the literature, see for example [4,6,7,8]. The Extended Routh Array (ERA) introduced in [7] settles the stability of systems with complex coefficients.

However, a large amount of computations will be involved to produce the ERA when the dimension of the system becomes high. Therefore, there is a need to work towards more simplified versions of the ERA. The establishment of simpler and more easily realizable criteria in practice will also further the theoretical development of the subject.

In this paper we address this problem and we propose a modification of the ERA which we call the Modified Extended Routh Array (MERA), where much simpler arithmetic is performed. At each step of the MERA only the calculation of a second order determinant is required. Furthermore, we exploit the MERA towards a new determinantal criterion for the asymptotic stability of a system of differential equations with complex coefficients.

In section 2 we give a quick reminder of the ERA and the way it is constructed. In section 3, we introduce the MERA and we prove that it is in fact another algorithm for testing the stability of complex systems, from which the equivalence of the MERA and the ERA follows. A determinantal approach to the stability problem is introduced in section 4. We end up in section 5 with some concluding remarks.

2. The Extended Routh Array

All the terminology of this section is taken from [7]. Suppose

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-2} z^2 + a_{n-1} z + a_n \tag{1}$$

is the characteristic polynomial of a system of differential equations with complex coefficients and of arbitrary dimension. Consider the rational function:

$$h(z) = \frac{z^n + i\,\mathrm{Im}\,a_1 z^{n-1} + \mathrm{Re}\,a_2 z^{n-2} + i\,\mathrm{Im}\,a_3 z^{n-3} + \mathrm{Re}\,a_4 z^{n-4} + \ldots.}{\mathrm{Re}\,a_1 z^{n-1} + i\,\mathrm{Im}\,a_2 z^{n-2} + \mathrm{Re}\,a_3 z^{n-3} + i\,\mathrm{Im}\,a_4 z^{n-4} + \ldots.}$$

The function h(z) is sometimes referred to as the test fraction associated with the system [8].

Let f_1 be the numerator and f_2 the denominator of h. Suppose $\mathrm{Re}\,a_1 \neq 0$, and call f_3 the remainder of the division of f_1 by f_2. By induction, define the polynomial f_j to be the remainder of the division of f_{j-2} by f_{j-1} for j = 3, ..., n+1. Lemma 4.1 of [7] expresses explicitly the coefficients of f_j in terms of those of f_{j-1} and f_{j-2}. The ERA is the following array in which the j-th row represents the coefficients of f_j for j = 1, 2, 3, ..., n+1 and where each row is completed by zeros to the size of the first row. We assume no zeros exist in the first column:

$$
\begin{array}{cccccccccc}
1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \mathrm{Re}\,a_4 & i\,\mathrm{Im}\,a_5 & . & . & . \\
\mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \mathrm{Re}\,a_5 & & . & . & . \\
b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & . & & . & . & . \\
b_{4,1} & b_{4,2} & b_{4,3} & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
. & . & . & . & . & & . & . & . \\
b_{n,1} & b_{n,2} & 0 & . & . & & . & . & . \\
b_{n+1,1} & 0 & . & . & . & & . & . & .
\end{array}
$$

where

$$b_{3,1} = \frac{1}{\mathrm{Re}\,a_1}(\mathrm{Re}\,a_1.\mathrm{Re}\,a_2 - \mathrm{Re}\,a_3) - \frac{i\,\mathrm{Im}\,a_2}{(\mathrm{Re}\,a_1)^2}(i\,\mathrm{Re}\,a_1.\mathrm{Im}\,a_1 - i\,\mathrm{Im}\,a_2)\ ,$$

$$b_{3,2} = \frac{1}{\mathrm{Re}\,a_1}(i\,\mathrm{Re}\,a_1.\mathrm{Im}\,a_3 - i\,\mathrm{Im}\,a_4) - \frac{\mathrm{Re}\,a_3}{(\mathrm{Re}\,a_1)^2}(i\,\mathrm{Re}\,a_1.\mathrm{Im}\,a_1 - i\,\mathrm{Im}\,a_2)\ ,$$

$$b_{4,1} = \frac{1}{b_{3,1}}(b_{3,1}.\mathrm{Re}\,a_3 - \mathrm{Re}\,a_1.b_{3,3}) - \frac{b_{3,2}}{b_{3,1}^2}(ib_{3,1}.\mathrm{Im}\,a_2 - \mathrm{Re}\,a_1.b_{3,2})\ ,$$

$$b_{4,2} = \frac{1}{b_{3,1}}(ib_{3,1}.\mathrm{Im}\,a_4 - \mathrm{Re}\,a_1.b_{3,4}) - \frac{b_{3,3}}{b_{3,1}^2}(ib_{3,1}.\mathrm{Im}\,a_2 - \mathrm{Re}\,a_1.b_{3,2})\ ,$$

and so on.

Theorem 4.1 of [7] states the following:

Theorem 1. The system with characteristic polynomial (1) is asymptotically stable if and only if each term of the first column of the ERA is positive, where asymptotic stability is as defined in [7].

3. The Modified Extended Routh Array

Consider the following array in which the first and second row are the same as in the ERA. We call the new array the Modified Extended Routh Array (MERA) for reasons to become clear later. The c's and the d's along with their respective subscripts have been so chosen for technical purposes.

$$
\begin{array}{cccccccc}
d_{01} & d_{02} & d_{03} & d_{04} & d_{05} & . & . \\
d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & . & . \\
c_{11} & c_{12} & c_{13} & c_{14} & . & . & . \\
d_{21} & d_{22} & d_{23} & d_{24} & . & . & . \\
c_{21} & c_{22} & c_{23} & . & . & . & . \\
d_{31} & d_{32} & d_{33} & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & .
\end{array}
$$

where

$$d_{01} = 1, \quad d_{0k} = \begin{cases} \mathrm{Re}\, a_{k-1} & , \; k \ge 3 \;\; and \;\; odd \\ i\,\mathrm{Im}\, a_{k-1} & , \; k \;\; even \end{cases}, \quad d_{1k} = \begin{cases} \mathrm{Re}\, a_k & , \; k \;\; odd \\ i\,\mathrm{Im}\, a_k & , \; k \;\; even \end{cases}$$

and

$$c_{11} = \frac{d_{11}.d_{02} - d_{01}.d_{12}}{d_{11}}, \quad c_{12} = \frac{d_{11}.d_{03} - d_{01}.d_{13}}{d_{11}}, \quad c_{13} = \frac{d_{11}.d_{04} - d_{01}.d_{14}}{d_{11}}, \ldots$$

$$d_{21} = \frac{d_{11}.c_{12} - c_{11}.d_{12}}{d_{11}}, \quad d_{22} = \frac{d_{11}.c_{13} - c_{11}.d_{13}}{d_{11}}, \quad d_{23} = \frac{d_{11}.c_{14} - c_{11}.d_{14}}{d_{11}}, \ldots$$

$$c_{21} = \frac{d_{21}.d_{12} - d_{11}.d_{22}}{d_{21}}, \quad c_{22} = \frac{d_{21}.d_{13} - d_{11}.d_{23}}{d_{21}}, \quad c_{23} = \frac{d_{21}.d_{14} - d_{11}.d_{24}}{d_{21}}, \ldots$$

$$d_{31} = \frac{d_{21}.c_{22} - c_{21}.d_{22}}{d_{21}}, \quad d_{32} = \frac{d_{21}.c_{23} - c_{21}.d_{23}}{d_{21}}, \quad d_{33} = \frac{d_{21}.c_{24} - c_{21}.d_{24}}{d_{21}}, \ldots$$

The following theorem implies the equivalence between the ERA and the MERA.

Theorem 2. The system with characteristic polynomial (1) is asymptotically stable if and only if $d_{k1} > 0$ for all k = 1, ..., n.

Proof. Suppose the system is asymptotically stable, then by [7, theorem 3.2] the test fraction h(z) can be expanded in the following continued fraction expansion:

$$h(z) = a_0 + b_0 z + \cfrac{1}{a_1 + b_1 z + \cfrac{1}{a_2 + b_2 z + \ldots + \cfrac{1}{a_{n-2} + b_{n-2} z + \cfrac{1}{a_{n-1} + b_{n-1} z}}}}$$

where $\mathrm{Re}\, a_k = 0$ and $b_k > 0$ for k = 0, ..., n-1.

The proof of theorem 4.1 of [7] makes it clear how the coefficients b_k in the above expansion relate to the first column of the ERA, namely

$$b_k = \frac{b_{k+1,1}}{b_{k+2,1}},$$

for k = 0, ..., n-1, where we suppose that $b_{1,1} = 1$ and $b_{2,1} = \mathrm{Re}\, a_1$.

The polynomials $f_1, \; f_2, \; \ldots, \; f_{n+1}$ forming the ERA are related by the recurrence relations:

$f_{k+1} = (a_k + b_k z) f_{k+2} + f_{k+3}$, k = 0, ..., n-1 ,

$f_{n+2} = 0$.

These recurrence relations are simply another version of lemma 4.1 of [7]. Upon checking these relations, we see that the terms that arise are exactly those contained in the MERA. Therefore, the following continued fraction expansion arises out of the terms of the MERA,

$$h(z) = c_0 + d_0 z + \cfrac{1}{c_1 + d_1 z + \cfrac{1}{c_2 + d_2 z + \ldots + \cfrac{1}{c_{n-1} + d_{n-1} z}}}$$

where $d_k = \dfrac{d_{k1}}{d_{(k+1)1}}$ for k = 0, 1, ..., n-1.

From the uniqueness of the continued fraction expansion of h(z) [7, section 3], we conclude that $b_k = d_k$ for k = 0, 1, ..., n-1.

We claim that

$$d_{k1} = b_{k+1,1}$$

for all k = 0, 1, ...,n.

It is clear that $b_0 = d_0 = \dfrac{1}{\mathrm{Re}\,a_1}$, and the relation $b_1 = d_1$ leads to $\dfrac{b_{2,1}}{b_{3,1}} = \dfrac{d_{11}}{d_{21}}$. Since $b_{2,1} = d_{11} = \mathrm{Re}\,a_1$, we conclude that $d_{21} = b_{3,1}$.

By induction suppose that $d_{(k-1)1} = b_{k,1}$ for some k, $3 \le k \le n$, then

$$d_{k-1} = \frac{d_{(k-1)1}}{d_{k1}} \text{ and } b_{k-1} = \frac{b_{k,1}}{b_{k+1,1}}.$$

By combining the relations $d_{k-1} = b_{k-1}$ and $d_{(k-1)1} = b_{k,1}$ we get $d_{k1} = b_{k+1,1}$ which proves our claim.

Since $b_{k+1,1} > 0$ for all k = 1, ..., n we conclude that $d_{k1} > 0$ for k = 1, ..., n.

4. Determinantal approach

In this section we exploit the results of section 3 to advance a new determinant-type algorithm for the stability of the systems described above.

Theorem 3. The system with characteristic polynomial (1) is asymptotically stable if and only if

$$(-1)^{k(k-1)/2}\Delta_k > 0$$

for k = 1, ..., n and where $\Delta_1, \Delta_2, \dots, \Delta_n$ are the first n principal minors indicated in the arrangement

$$\begin{array}{c|cc|cc|cc}
\mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \mathrm{Re}\,a_5 & i\,\mathrm{Im}\,a_6 & \cdot\ \cdot \\
1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \mathrm{Re}\,a_4 & i\,\mathrm{Im}\,a_5 & \cdot\ \cdot \\
0 & \mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \mathrm{Re}\,a_5 & \cdot\ \cdot \\
0 & 1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \mathrm{Re}\,a_4 & \cdot\ \cdot \\
0 & 0 & \mathrm{Re}\,a_1 & i\,\mathrm{Im}\,a_2 & \mathrm{Re}\,a_3 & i\,\mathrm{Im}\,a_4 & \cdot\ \cdot \\
0 & 0 & 1 & i\,\mathrm{Im}\,a_1 & \mathrm{Re}\,a_2 & i\,\mathrm{Im}\,a_3 & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot
\end{array}$$

where each row is completed by zeros to the size of the first row.

Proof. Suppose the system is asymptotically stable, then by theorem 2 $d_{k1} > 0$ for k = 0, 1, ..., n.

Consider the determinant Δ_k of order 2k-1 for $2 \le k \le n$. Subtract $1/\mathrm{Re}\,a_1$ times the (2j-1)-st row from the 2j-th row, for j = 1, 2, ..., k-1, and with the use of the MERA, we find that

$$\Delta_k = d_{11}.\det\begin{bmatrix} c_{11} & c_{12} & c_{13} & . & . \\ d_{11} & d_{12} & d_{13} & . & . \\ 0 & c_{11} & c_{12} & . & . \\ 0 & d_{11} & d_{12} & . & . \\ 0 & 0 & c_{11} & . & . \\ . & . & . & . & . \end{bmatrix}$$

where obviously $d_{11} = \mathrm{Re}\,a_1$, $d_{12} = i\,\mathrm{Im}\,a_2$, $d_{13} = \mathrm{Re}\,a_3$ and so on. Clearly the new determinant is of order 2k-2.

Now subtract c_{11}/d_{11} times the 2j-th row from the (2j-1)-st row for j = 1, 2,..., k-1 and again with the help of the MERA, we obtain

$$\Delta_k = (-1)^{k-1} d_{11}^2 \Delta_{k-1}^{(1)}$$

for k = 2, 3, ..., n, where $\Delta_r^{(j)}$ denotes the determinant Δ_r with both the subscripts of all its elements increased by j. From the last relation we find immediately that

$$\Delta_k = (-1)^{k(k-1)/2} d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$$

or

$$(-1)^{k(k-1)/2} \Delta_k = d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$$

for k = 2, 3, ..., n. From theorem 2 it follows that $d_{k1} > 0$ for k = 0, 1, ..., n, therefore

$$(-1)^{k(k-1)/2} \Delta_k > 0$$

for k = 1, ..., n.

Conversely, suppose that $(-1)^{k(k-1)/2} \Delta_k > 0$ for k = 1, ..., n. If k = 1, then $\Delta_1 = d_{11} > 0$. In the relation $(-1)^{k(k-1)/2} \Delta_k = d_{11}^2 d_{21}^2 \ldots d_{(k-1)1}^2 d_{k1}$ for k = 2, 3, ..., n, put k = 2, then $d_{21} > 0$.

Continuing by induction we get $d_{k1} > 0$ for k = 1, ..., n, and by theorem 2 the system is asymptotically stable and that completes the proof.

5. Concluding Remarks

The complexity of computation in the ERA stability test has been reduced by exploiting special features of the continued fraction expansion of the test fraction associated with the system. With the introduction of the MERA, this paper contributes to ongoing research to finding algorithms which are computationally attractive, numerically simple and accurate for assessing the stability of a system of differential equations. However, the search for tests with reduced computational efforts is still continuing. We have also applied the new MERA to obtain a determinantal criterion for the stability of the system.

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GEOMETRIA DIFERENCIAL DE SISTEMAS DINAMICOS SOBRE C*-ALGEBRAS

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ABSTRACT.

Let's call *SD(G,M)* the space of dynamical systems from an abelian locally compact group G over an injective W*-algebra M. Let's consider the natural action from *Aut(M)* over *SD(G,M)*. The first objective of this work consists of, under suitable conditions, defining in *SD(G,M)* a structure of homogeneous reductive space.

The set $U(G, M)$ containing the unitaries representations of G on M admits a bijection with the space of $\star$-representations of $C^\star$(G) on M. This last space will be called R$(C^\star(G), M)$. The second objective of this work consists of answering the following question, which it was asked in[**ACS 2**]: which topology does this bijection induce in $U(G, M)$? The answer will let us define in $U(G, M)$ a structure of reductive homogeneous space.

INTRODUCCION.

Un sistema dinámico es una terna (M,G,α), donde M es una C*-álgebra, G es un grupo localmente compacto (que consideraremos abeliano) y α es una representación de G en el grupo *Aut(M)* de $\star$-automorfismos de M tal que para cada $x \in M$ la aplicación $g \rightarrow \alpha_g(x)$ es continua. Si M es una W*-álgebra entonces la función $g \rightarrow \alpha_g(x)$ deberá ser σ-débil continua. Indicaremos con *SD(G,M)* al conjunto de los sistemas dinámicos del grupo G en el álgebra M.

Los sistemas dinámicos aparecen de manera natural en el estudio de diversas ramas de la Matemática y la Física; en particular, por ejemplo, en Mecánica Cuántica son estudiados sistemas dinámicos sobre C*-álgebras. Paralelamente G. Corach, H. Porta

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y L. Recht desarrollaron con éxito una teoría geométrica inspirada en el espacio de proyectores en una $C^\star$-álgebra. Tal teoría, consistente en el estudio de la estrucura de espacio homogéneo de dimensión infinita, se aplica a una extensa lista de espacios, como las medidas espectrales, representaciones de grupos compactos, operadores simétricos, operadores positivos y muchos otros (véase **[CPR 1]**, **[CPR 2]**, **[ARS]**, **[MR]**, **[ACS 1]** y **[ACS 2]**).

Es nuestro interés incorporar a esta extensa lista de espacios donde es definible una estructura homogénea reductiva al conjunto $SD(G, M)$ de los sistemas dinámicos de un grupo abeliano localmente compacto G en un álgebra de von Neumann inyectiva M. Con este fin vamos a estudiar la acción de *Aut(M)* en *SD(G,M)* definida por $T\star\alpha_k = T\alpha_k T^{-1}$ si $T \in Aut(M)$, $\alpha \in SD(G, M)$ y $k \in G$, o también esta misma acción restringida al conjunto $In(M)$ de los $\star$-automorfismos interiores de M.

La pregunta básica que nos hacemos, entonces, es bajo qué condiciones es posible definir en $SD(G, M)$ una estructura de espacio homogéneo. Si pudiésemos ver a $SD(G, M)$ en el contexto de un espacio de Banach, entonces habremos dado un paso importante en el camino hacia obtener una respuesta a la pregunta; pues los espacios de Banach son el hábitat natural de los objetos diferenciables.

Un tal primer paso es llevado a cabo en la primera sección, donde se establece la inclusión de $SD(G, M)$ en un espacio de Banach conveniente. En particular este paso determinará la topología a considerar en $SD(G, M)$.

Nuestra pregunta básica será respondida para el caso de grupos finitos. Este ejemplo, a primera vista puede parecer de poco interés, sin embargo será la clave que nos permitirá estudiar los $\star$-automorfismos de M de orden finito. Una consecuencia de este estudio será una demostración de que los $\star$-automorfismos de orden 2 admiten una estructura de espacio homogéneo y que además constituyen un subconjunto abierto de $Aut(M)$.

Algunos de los ejemplos más importantes entre los sistemas dinámicos ocurren cuando el grupo G es el grupo $\mathbb{Z}$ de los números enteros. En la segunda sección nos ocupamos de este ejemplo. Estudiamos la acción dada por *Aut(M)*. En esta sección se demuestra que si $C_1, C_2 \in Aut(M)$ son automorfismos centarles distintos entonces las órbitas de los sistemas dinámicos inducidos por C_1 y C_2 distan en más de $\frac{1}{4}$.

La última sección no está dedicada a los sistemas dinámicos sino a un breve estudio de las representaciones unitarias de un grupo G localmente compacto y abeliano en un álgebra de von Neumann M. El objetivo es responder a una pregunta que había quedado planteada en **[ACS 2]**. Explicaremos brevemente la naturaleza de la pregunta. El conjunto $U(G,M)$ de las representaciones unitarias de G en M admite una biyección natural con el conjunto $R(C^\star(G),M)$ de las $\star$-representaciones de $C^\star(G)$ en M. La pregunta se refiere a cuál es la topología inducida por esta biyección en $U(G,M)$; su respuesta permitirá considerar en $U(G,M)$ una estructura de espacio homogéneo.

Este trabajo consta de cinco secciones; en la primera, dados una C*-álgebra M y un grupo G localmente compacto y abeliano, consideraremos el conjunto *SD(G,M)*. El objetivo será determinar para el mismo un contexto adecuado, que permita definir en él una estructura de fibrado principal. En particular, responderemos a la pregunta de cuál es la topología que corresponde considerar en el conjunto.

En la segunda parte tomamos el caso particular en que $G = \mathbb{Z}$ y, aplicando a $SD(\mathbb{Z},M)$ los resultados de la sección previa estudiamos las órbitas de los sistemas dinámicos de la forma $n \to C^n$ donde $n \in \mathbb{Z}$ y $C \in Aut(M)$ es un automorfismo central. Para ello se define en *Aut(M)* una nueva métrica d' y se estudia el homeomorfismo resultante entre $SD(\mathbb{Z},M)$ y *(Aut(M),d')*.

La tercera sección está dedicada a estudio de *SD(G,M)* en el caso en que G es un grupo finito. Fijado $\alpha \in SD(G,M)$ consideramos la aplicación $\Pi_\alpha : Aut(M) \to SD(G,M)$, definida por $\Pi_\alpha(T) = T\star\alpha$ y estudiamos los objetos diderenciales inducidos por ella (esta sección está inspirada en [**MR**], sección 12).

En la cuarta parte se aplican los resultados obtenidos en la sección previa para efectuar un estudio de la estructura de los automorfismos de *Aut(M)* de orden 2; se discutirá particularmente la existencia de secciones locales continuas para la acción Π_α. Una conclusión resultante de este setudio será que el conjunto de $\star$-automorfismos de orden 2 es abierto en *Aut(M)*.

La última sección responde a la pregunta planteada en [**ACS 2**] acerca de la topología a considerar en el conjunto de representaciones unitariaas de un grupo localmente compacto y abeliano G en una $W^\star$-álgebra M. El objetivo de obtener un homeomorfismo con el espacio $R(C^\star(G),M)$ de las $\star$-representaciones de $C^\star(G)$ en M. Este conjunto tiene, en virtud de [**ACS 2**] una estructura de espacio homogéneo; el homeomorfismo indicado permite llevar esa estructura a $U(G,M)$.

1. SISTEMAS DINAMICOS.

Sean M una W*-álgebra con predual separable, G un grupo abeliano localmente compacto. Llamaremos *SD(G,M)* al conjunto de los sistemas dinámicos de G en M; sea

$\alpha \in SD(G, M)$ un sistema dinámico. Notaremos *M(G)* al conjunto de las medidas complejas definidas en G y *L(M)* al conjunto de los operadores continuos de M.

Proposición 1.1:

Si α es un sistema dinámico de G en M y $\mu \in M(G)$ entonces para cada $x \in M$ existe un único elemento de M, que llamaremos $\tilde{\alpha}_\mu(x)$ caracterizado por la siguiente propiedad:

$$\Phi(\tilde{\alpha}_\mu(x)) = \int_G \Phi(\alpha_g(x))d\mu(g) \text{ para todo } \mu \in M(G), \Phi \in M_\star$$

Queda así definida una aplicación $\tilde{\alpha}_\mu : M \rightarrow M$ que verifica las siguientes propiedades:

i) Para todo $\mu \in M(G)$, $\tilde{\alpha}_\mu$ es un operador σ-débil continuo de M.

ii) $\tilde{\alpha}_{\mu\star\nu} = \tilde{\alpha}_\mu\tilde{\alpha}_\nu \quad \forall\nu, \mu \in M(G)$.

iii) $\| \tilde{\alpha}_\mu \| \leq \| \mu \| \quad \forall\mu \in M(G)$.

iv) $\tilde{\alpha}_\mu(x^\star) = \tilde{\alpha}_{\overline{\mu}}(x)^\star$ donde $\overline{\mu}(\Delta) = \overline{\mu(\Delta)}\ \forall\Delta \subset G$.

v) $\tilde{\alpha}_{\delta_{g+h}} = \tilde{\alpha}_{\delta_g}\tilde{\alpha}_{\delta_h}$ para todo $g, h \in G$.

vi) $\tilde{\alpha}_{\delta_g}(xy) = \tilde{\alpha}_{\delta_g}(x)\tilde{\alpha}_{\delta_g}(y)$ para todo $g \in G; x, y \in M$.

vii) $\int_G \Phi(\tilde{\alpha}_{\delta_g}(x))d\mu(g) = \Phi(\tilde{\alpha}_\mu(x))$ para todo $\mu \in M(G)$.

Los dos primeros puntos pueden resumirse diciendo que $\tilde{\alpha}$ es un homomorfismo de álgebras de $M(G)$ en el álgebra $L_\sigma(M)$ de operadores σ-débiles continuos de M.

Recíprocamente si $\tilde{\alpha} : M(G) \rightarrow L(M)$ verifica las siete propiedades anteriores y definimos $\alpha : G \rightarrow Aut(M)$ por $\alpha_g = \tilde{\alpha}_{\delta_g}$; entonces α es un sistema dinámico.

Demostración:

Sea α un sistema dinámico de G en M, μ una medida compleja en G y $x \in M$; la demostración de la existencia de $\tilde{\alpha}_\mu(x)$ y de que se verifican las tres primeras propiedades, puede encontrarse en **[S]**, Proposition 3.2.2.

Para demostrar iv basta probar que $\Phi(\tilde{\alpha}_\mu(x^\star)) = \Phi(\tilde{\alpha}_{\overline{\mu}}(x)^\star)$ cualquiera sea $\Phi \in M_\star$. En efecto:

$$\Phi(\tilde{\alpha}_\mu(x^\star)) = \int_G \Phi(\alpha_g(x^\star))d\mu(g) = \int_G \Phi(\alpha_g(x)^\star)d\mu(g) = \int_G \overline{\Phi_c(\alpha_g(x))}d\mu(g)$$
$$= \overline{\int_G \Phi_c(\alpha_g(x))d\overline{\mu}(g)} = \overline{\Phi_c(\tilde{\alpha}_{\overline{\mu}}(x))} = \Phi(\tilde{\alpha}_{\overline{\mu}}(x)^\star)$$

Donde $\Phi_c(y) = \overline{\Phi(y^\star)}\ \forall y \in M$.

La propiedad v es consecuencia inmediata de iii. Las dos restantes propiedades son consecuencia de:

$$\Phi(\tilde{\alpha}_{\delta_g}(x)) = \int_G \Phi(\alpha_h(x))d\delta_g(h) = \Phi(\alpha_g(x))$$

De donde se deduce que $\tilde{\alpha}_{\delta_g} = \alpha_g$.

Recíprocamente; sea $\tilde{\alpha} : M(G) \to L(M)$ que verifique i – vii, si se define $\alpha_g = \tilde{\alpha}_{\delta_g}$ $\forall g \in G$; es inmediato que α_g es un sistema dinámico. •

A partir de la proposición 1.1 podemos definir una aplicación Θ que a cada sistema dinámico α le asigne un operador $\Theta(\alpha) = \tilde{\alpha} \in L(M(G).L_\sigma(M))$, que verifica i – vii, según la siguiente fórmula:

$$\Phi(\Theta(\alpha)_\mu(x)) = \int_G \Phi(\alpha_g(x))d\mu(g) \text{ para todo } \mu \in M(G), \Phi \in M_\star$$

Recíprocamente, dada una aplicación $\tilde{\alpha} \in L(M(G).L_\sigma(M))$ que verifique i – vii de la proposición podemos definir un sistema dinámico $\tilde{\Theta}(\tilde{\alpha}) = \alpha$ según la siguiente fórmula: $\tilde{\Theta}(\tilde{\alpha})_g = \tilde{\alpha}_{\delta_g}$.

Veamos que las funciones Θ y $\tilde{\Theta}$ son una la inversa de la otra. En efecto, si α un sistema dinámico, entonces probemos que: $\tilde{\Theta}\Theta(\alpha)_g = \alpha_g$.

Para ello hay que verificar que $\Theta(\alpha)_{\delta_g} = \alpha_g$, esto a su vez se deduce de lo siguiente:

$$\Phi(\Theta(\alpha)_{\delta_g}(x)) = \int_G \Phi(\alpha_h(x))d\delta_g(h) = \Phi(\alpha_g(x)) \ \forall x \in M, \forall \Phi \in M_\star$$

Para completar la demostración hay que probar que: $\Theta\tilde{\Theta}(\tilde{\alpha})_\mu = \tilde{\alpha}_\mu$.

Es decir, queremos probar que $\Phi(\Theta\tilde{\Theta}(\tilde{\alpha})_\mu(x)) = \Phi(\tilde{\alpha}_\mu(x))$ $\forall x \in M, \forall \Phi \in M_\star$. En efecto:

$$\Phi(\Theta\tilde{\Theta}(\tilde{\alpha})_\mu(x)) = \int_G \Phi(\tilde{\Theta}(\tilde{\alpha})_g(x))d\mu(g) = \int_G \Phi(\tilde{\alpha}_{\delta_g}(x))d\mu(g) = \Phi(\tilde{\alpha}_\mu(x))$$

Definición 1.2: *Si $\alpha_n, \alpha \in SD(G,M)$ entonces diremos que $\alpha_n \to \alpha$ uniformemente si y sólo si $\forall \epsilon > 0$ $\exists m \in \mathbb{N}$ tal que $\| \alpha_n(g) - \alpha(g) \| < \epsilon$ para todo $n \geq m$, $g \in G$ donde $\alpha(g) = \alpha_g$ y la norma se entiende tomada en $Aut(M) \subset L(M)$.*

Veremos a continuación que la topología inducida por esta convergencia es la que debemos considerar en *SD(G,M)* para que la biyección sea un homeomorfismo.

Proposición 1.3:

Con las notaciones anteriores; $\tilde{\alpha}_n \to \tilde{\alpha}$ *en norma de* $L(M(G), L(M))$ *si y sólo si* $\alpha_n \to \alpha$ *uniformemente.*

Demostración:

Supongamos que $\alpha_n \to \alpha$ uniformemente y sean $x \in M$ y $\Phi \in M_\star$ tales que $\| x \| \leq 1$ y $\| \Phi \| \leq 1$. Sea $m \in I\!N$ tal que $\| \alpha_n(g) - \alpha(g) \| < \epsilon$ para todo $n \geq m$, $g \in G$. Entonces $|\Phi((\tilde{\alpha}_n(\mu) - \tilde{\alpha}(\mu))x)| \leq \int_G \| \alpha_n(g)x - \alpha(g)x \| \, d|\mu|(g) \leq \epsilon \| \mu \| \ \ \forall m \leq n$. Recíprocamente si $\tilde{\alpha}_n \to \tilde{\alpha}$ según la norma de *L(M(G),L(M))* entonces:

$$\| \alpha_n(g) - \alpha(g) \| = \| \tilde{\alpha}_n(\delta_g) - \tilde{\alpha}(\delta_g) \| \leq \| \tilde{\alpha}_n - \tilde{\alpha} \| \| \delta_g \| \leq \| \tilde{\alpha}_n - \tilde{\alpha} \| .$$

De donde se deduce que $\alpha_n \to \alpha$ uniformemente. •

La conclusión que se extrae de ambas proposiciones es que *SD(G,M)* es homeomorfo al subconjunto de los operadores de *L(M(G),L(M))* que verifican las propiedades i – vii de la proposición 1.1. Esto nos permite de manera natural considerar a *SD(G,M)* como subconjunto de *L(M(G),L(M))*.

De esta manera hemos colocado a *SD(G,M)* en el contexto de un espacio de Banach (que posee una estructura natural de variedad diferencial).

Queda pendiente el estudio de bajo qué condiciones el subconjunto de *L(M(G),L(M))* que verifica i – vii es una subvariedad en la que puede definirse una estructura de fibrado principal.

La acción natural a considerar en *SD(G,M)* es la acción dada por los unitarios de M del siguiente modo: $u{\star}\alpha_g = Ad(u)\alpha_g Ad(u^\star)$. Donde $u \in M$ es unitario y $Ad(u)x = uxu^\star$. Es posible también considerar una acción desde *Aut(M)*; $T{\star}\alpha_g = T\alpha_g T^{-1} \ \forall T \in Aut(M)$.

Cualquiera de ambas acciones se extiende al subconjunto de *L(M(G),L(M))* que, según la proposición 1.1, es homeomorfo a *SD(G,M)*.

2. SISTEMAS DINAMICOS ENTEROS.

Según las notaciones de la sección anterior consideremos el conjunto $SD(\mathbb{Z}, M)$, donde $\mathbb{Z}$ indica el conjunto de los números enteros y M un álgebra de von Neumann inyectiva con predual separable actuando en un espacio de Hilbert H. Consideremos en $SD(\mathbb{Z}, M)$ la métrica $\mathrm{d}(\alpha, \beta) = \sup_{n \in \mathbb{Z}} \| \alpha_n - \beta_n \|_{L(M)}$.

Esta métrica está bien definida ya que, por ejemplo, por la propiedad iii de la proposición 1.1 vale que $\| \rho_n \| = \| \tilde{\rho}_{\delta_n} \| \leq \| \delta_n \| \leq 1 \ \forall \rho \in SD(\mathbb{Z}, M)$; además induce en $SD(\mathbb{Z}, M)$ la misma topología que la considerada en la sección anterior para los sistemas dinámicos definidos en grupos localmente compactos y abelianos.

Definamos ahora en *Aut(M)* una nueva métrica d'(A,B) $= \sup_{n\in\mathbb{Z}} \| A^n - B^n \|$.

Existe una aplicación natural $\Gamma : SD(\mathbb{Z}, M) \to Aut(M)$ definida como $\Gamma(\alpha) = \alpha_1$.

Lema 2.1:

Adoptemos las notaciones anteriores y consideremos en Aut(M) la métrica d'. Entonces la aplicación $\Gamma : SD(\mathbb{Z}, M) \to Aut(M)$ *es una isometría suryectiva.*

Demostración:

Si $\alpha \in SD(\mathbb{Z}, M)$ entonces $\alpha_n = (\alpha_1)^n$, por lo tanto d'$(\Gamma\alpha, \Gamma\beta) = \sup_{n\in\mathbb{Z}} \| \alpha_1^n - \beta_1^n \| =$ d(α, β). La suryectividad resulta de que, dado $A \in Aut(M)$, si definimos $\alpha_n = A^n$, entonces $\alpha \in SD(\mathbb{Z}, M)$ y $\Gamma(\alpha) = A$.•

De las dos acciones que pueden considerarse en $SD(\mathbb{Z}, M)$ queremos considerar aquella definida desde *Aut(M)*; $T\star\alpha_n = T\alpha_n T^{-1}$ $\forall T \in Aut(M)$.

A fin de dotar a *Aut(M)* de una estructura diferencial, considerémoslo como el espacio de las $\star$-representaciones de M sobre sí mismo. Puesto que M es una W*-álgebra inyectiva, podemos aplicar los resultados de [**ACS 1**] y [**ACS 2**], que nos permiten definir en $Aut(M)$ una estructura de espacio homogéneo.

Además, como $(T\star\alpha)_n = T\alpha_n T^{-1} = (T\alpha_1 T^{-1})^n$ $\forall T \in Aut(M), \alpha \in SD(\mathbb{Z}, M), n \in \mathbb{Z}$; entonces la acción de *Aut(M)* en $SD(\mathbb{Z}, M)$ se traduce, vía Γ, en la acción de *Aut(M)* sobre símismo dada por la conjugación; $T\star A = TAT^{-1}$ $\forall T, A \in Aut(M)$. Se trata entonces de estudiar la estructura de *Aut(M)* con la métrica d' y la acción recién indicada.

En esta sección vamos a dar un primer paso para el estudio de esta estructura, estableciendo un hecho y una conjetura acerca de las órbitas inducidas por la acción de conjugación. Recordemos las notaciones de la sección anterior; si $u \in M$ es unitario, llamaremos $Ad(u) \in Aut(M)$ a la aplicación definida por $Ad(u)x = uxu^\star$ $\forall x \in M$.

Lema 2.2:

Sean $i = id_{L(H)}$, $u \in L(H)$ *unitario y* $Ad(u) \in L(L(H))$ *tal que* $\| (Adu)^n - i \| \leq \frac{1}{2}$ $\forall n \in \mathbb{Z}$. *Entonces* $Ad(u) = i$.

Demostración:

Si $\gamma = Ad(u)$ y c pertenece a la cápsula convexa cerrada del espectro de u, que indicaremos $\overline{co}(sp(u))$, entonces $|c| \geq \frac{1}{2}(4- \| \gamma - i \|^2)^{\frac{1}{2}}$ (véase [**KR**], 10.5.67 - 10.5.68 - 10.5.69). Entonces, como $\| \gamma^n - i \| \leq \frac{1}{2}$ $\forall n \in \mathbb{Z}$ se deduce que $|c| \geq \frac{\sqrt{5}}{4} > 0,96$ $\forall c \in \overline{co}(sp(u^n))$ $\forall n \in \mathbb{Z}$.

Multiplicando por un número complejo de módulo 1 conveniente, podemos suponer que $1 \in sp(u)$. La condición $|c| > 0,96\ \forall c \in \overline{co}(sp(u^n))\ \forall n \in \mathbb{Z}$ dice primeramente que para todo $n \in \mathbb{Z}$ el espectro de u^n está contenido en un arco de circunferencia de longitud 0,57 simétrico alrededor de 1.

Supongamos que $sp(u) \neq \{1\}$ y sea $\lambda \in sp(u) - \{1\}$. Vale que $\{\lambda^n, 1\} \subset sp(u^n)\ \forall n \in \mathbb{Z}$; pero, tomando una potencia de λ conveniente, $\{\lambda^n, 1\}$ quedará fuera del arco de circunferencia antes indicado; llegamos así a una contradicción. Luego $sp(u) = \{1\}$.

Como u es un operador normal de $L(H)$ entonces para toda función f continua en el espectro de u vale que $\| f(u) \| = \sup_{\lambda \in sp(u)} |f(\lambda)|$. Tomando $f(t) := t - 1$ resulta que $u = I$ y entonces $Ad(u) = i$.•

Observación 2.3: El lema 2.2 es válido aún, si reemplazamos $L(H)$ por M.

Corolario 2.4:

La aplicación identidad de M, $i \in Aut(M)$, es un punto aislado en $(Aut(M), d')$.

Demostración:

Sea $A \in Aut(M)$ tal que $d'(A, i) < \frac{1}{4}$; luego $\| A^n - i \| < \frac{1}{4}\ \forall n \in \mathbb{Z}$. Como en particular $\| A - i \| < 2$, entonces, por **[KR]**, 10.5.73 o **[P]**, Proposition 8.7.9, existe $u \in M$ unitario tal que $A = Ad(u)$. Por el lema 2.2 se deduce que $A = i$. Luego

$$\left\{ A \in Aut(M) : d'(A, i) < \frac{1}{4} \right\} = \{i\} \bullet$$

Definición 2.5: *Diremos que $C \in Aut(M)$ es central si $AC = CA\ \forall A \in Aut(M)$.*

Teorema 2.6

Si $C \in Aut(M)$ es central entonces C es punto aislado de $(Aut(M), d')$. Más aún

$$\left\{ A \in Aut(M) : d'(A, C) < \frac{1}{4} \right\} = \{C\}$$

Demostración:

Sea $A \in Aut(M)$ tal que $d'(A, C) < \frac{1}{4}$, entonces $\forall n \in \mathbb{Z}$:

$$\| A^n - C^n \| = \| C^n(C^{-n}A^n - i) \| = \| C^{-n}A^n - i \| = \| (C^{-1}A)^n - i \| < \frac{1}{4}$$

Aplicando el corolario 2.4 se deduce que $C^{-1}A = i$.•

Observemos que si $C \in Aut(M)$ es central entonces para todo $T \in Aut(M)$ vale que $TCT^{-1} = C$, es decir la órbita de C es exactamente $\{C\}$. Si $A \in Aut(M)$ llamemos $Or(A)$ a la órbita de A; $Or(A) = \{TAT^{-1} : T \in Aut(M)\}$.

El teorema 2.6 puede refrasearse diciendo que si C es central y $A \in Aut(M), A \neq C$ entonces la distancia (según d') entre $Or(C)$ y $Or(A)$ es mayor o igual que $\frac{1}{4}$. Nuestra conjetura es que existe un número constante $k_0 > 0$ tal que si $A, B \in Aut(M)$ son tales que $Or(A) \neq Or(B)$, entonces la distancia entre $Or(A)$ y $Or(B)$ es mayor o igual que un número constante k_0.

3. SISTEMAS DINAMICOS EN GRUPOS FINITOS.

Sea G un grupo abeliano finito, indicaremos por $|G|$ al cardinal de G. Sea M un álgebra de von Neumann inyectiva actuando en un espacio de Hilbert separable H y sea $\alpha : G \to Aut(M)$ un sistema dinámico. En particular $\alpha_{k+j} = \alpha_k \alpha_j \; \forall k, j \in G$.

Notación: Indicaremos con $C(G)$ al conjunto $\mathbb{C}^G$, de todas las funciones de G al conjunto $\mathbb{C}$ de los números complejos.

Observación 3.1: Puesto que G es finito entonces es obvio que $C(G) = L^1(G)$ tomando en G su medida de Haar. En este contexto la convolución de dos funciones $f, g \in C(G)$ queda definida por $f \star g(j) = \frac{1}{|G|} \sum_{k \in G} f(k) g(j-k) \; \forall j \in G$.

Tomaremos como norma en C(G) la siguiente: $\| f \|_{C(G)} := \sum_{k \in G} |f(k)|$.

Por razones puramente de comodidad en la escritura (que serán evidentes en el teorma 3.8) y sin que esto represente una diferencia esencial omitimos en la definición de $\| f \|_{C(G)}$ el factor $\frac{1}{|G|}$, que era dado a esperar delante de la sumatoria.

Vamos ahora a seguir una línea argumental similar a aquella desarrollada en la sección 1, con el fin de dotar a $SD(G,M)$ de una estructura diferencial. Esencialmente vamos a dar una versión finita de las proposiciones 1.1 y 1.2. Dado $\alpha \in SD(G, M)$ queda definida una aplicación $\tilde{\alpha} : C(G) \to L(M)$ dada por la siguiente fórmula: $\tilde{\alpha}_f = \frac{1}{|G|} \sum_{k \in G} f(k) \alpha_k \; \forall f \in C(G)$.

Proposición 3.2:

Si $\alpha \in SD(G, M)$ y $\tilde{\alpha} : C(G) \to L(M)$ es la aplicación antes definida, entonces $\tilde{\alpha}$ verifica:

i) $\tilde{\alpha}_{f \star g} = \tilde{\alpha}_f \tilde{\alpha}_g$.

ii) $\tilde{\alpha}_f(x^\star) = \tilde{\alpha}_{\overline{f}}(x)^\star \; \forall x \in M$, *donde* $\overline{f}(k) = \overline{f(k)}$.

iii) $\tilde{\alpha}_{\delta_k} \in Aut(M) \; \forall k \in G$.

iv) $\frac{1}{|G|} \sum_{k \in G} f(k) \tilde{\alpha}_{\delta_k} = \tilde{\alpha}_f$.

Recíprocamente si $\beta \in L(C(G), L(M))$ verifica $i - iv$ entonces existe $\alpha \in SD(G, M)$ tal que $\tilde{\alpha} = \beta$, explícitamente α está dada por la fórmula $\alpha_k = \beta_{\delta_k}$.

La demostración de la proposición es completamente elemental por lo que la omitimos.

Queda definida una biyección entre *SD(G,M)* y el subconjunto de $L(C(G), L(M))$ dado por las propiedades i – iv. La proposición 1.3 nos dice cual es la topología a considerar en *SD(G,M)* para que la biyección resulte un homeomorfismo. Esta topología es aquella inducida por la métrica $d(\alpha, \beta) = \max_{k \in G} \| \alpha_k - \beta_k \|$.

Notación:

Indicaremos por *In(M)* al conjunto de los automorfismos interiores de *M*, es decir:

$$In(M) = \{T \in Aut(M) : T = Ad(u) \text{ con } u \in M \text{ unitario } \}.$$

E indicaremos por *Der(M)* al $\mathbb{R}$- espacio vectorial de las $\star$-derivaciones de *M*, esto es, $\Delta \in Der(M)$ si y sólo si es lineal y para todo $x, y \in M$ vale $\Delta(xy) = x\Delta(y) + \Delta(x)y$ y $\Delta(x^\star) = \Delta(x)^\star$.

Es bien sabido que toda $\star$-derivación de M es acotada.

Consideramos sobre *SD(G,M)* la acción de *In(M)* definida por $(T \star \alpha)_k := T\alpha_k T^{-1}$. La acción correspondiente sobre $\tilde{\alpha}$ es idéntica. Es fácil probar que $T \star \tilde{\alpha}$ verifica i – iv y que $T \star \tilde{\alpha} = \widetilde{T \star \alpha}$.

Como M es inyectiva entonces *In(M)* tiene una estructura diferencial natural. Por [**KR**], 10.5.63; $T \in In(M)$ si y sólo si existe $\Delta \in Der(M)$ tal que $T = e^\Delta$. Luego *Der(M)* es el espacio tangente natural de *In(M)*. En particular existe un proyector continuo $\mathbb{P} : L(M) \to Der(M)$.

Supongamos que $A \in L(M)$ y $\Delta = \mathbb{P}(A)$; digamos $A = \Delta + \overline{\Delta}$. Si $T \in Aut(M)$ entonces $T\Delta T^{-1} \in Der(M)$, de donde se deduce que $\mathbb{P}(TAT^{-1}) = T\mathbb{P}(A)T^{-1}$.

Dado un sistema dinámico α indicaremos a partir de ahora con la misma letra a la aplicación inducida $\alpha \in L(C(G), L(M))$ e identificaremos $\alpha_k = \alpha_{\delta_k}$.

Observación: Los cálculos efectuados en gran parte del resto de esta sección están inspirados en [**MR**].

Definición 3.3: *Dado* $\alpha \in L(C(G), L(M))$ *definimos* $\Pi_\alpha : In(M) \to L(C(G), L(M))$ *por* $\Pi_\alpha(T) = T \star \alpha$.

Notación: Llamaremos I_α al conjunto $I_\alpha := \{\Delta \in Der(M) : \Delta\alpha_k = \alpha_k \Delta \ \forall k \in G\}$.

Proposición 3.4:

Si $E_\alpha : Der(M) \to L(M)$ *se define por* $E_\alpha(\Delta) = \frac{1}{|G|} \sum_{k \in G} \alpha_k \Delta \alpha_{-k}$.

Entonces $E_\alpha(Der(M)) \subset I_\alpha$ *y* $E_\alpha E_\alpha = E_\alpha$.

Compárese este enunciado con [**MR**], 12.2.

Demostración:

Sean $\Delta \in Der(M)$, vamos a probar que $E_\alpha(\Delta) \in I_\alpha$, es decir que $E_\alpha(\Delta) \in Der(M)$ y que $E_\alpha(\Delta)\alpha_g = \alpha_g E_\alpha(\Delta)$ $\forall g \in G$.

Para la primera afirmación basta observar que $\alpha_k \Delta \alpha_{-k} \in Der(M)$ $\forall k \in G$ y que si $\{\Delta_k\}_{k\in G} \subset Der(M)$ entonces $\sum_{k\in G} \Delta_k \in Der(M)$. Veamos la segunda afirmación:

$$\begin{aligned} E_\alpha(\Delta)\alpha_g &= \frac{1}{|G|}(\sum_{k\in G} \alpha_k \Delta \alpha_{-k})\alpha_g = \frac{1}{|G|}\sum_{k\in G} \alpha_k \Delta \alpha_{k+g} = \frac{1}{|G|}\sum_{r\in G} \alpha_{r+g}\Delta\alpha_{-r} \\ &= \frac{1}{|G|}\alpha_g \sum_{r\in G} \alpha_r \Delta \alpha_{-r} = \alpha_g E_\alpha(\Delta). \end{aligned}$$

Probemos finalmente que $E_\alpha E_\alpha(\Delta) = E_\alpha(\Delta)$.

$$E_\alpha E_\alpha(\Delta) = \frac{1}{|G|}\sum_{k\in G} \alpha_k E_\alpha(\Delta)\alpha_{-k} = \frac{1}{|G|}\sum_{k\in G} E_\alpha(\Delta)\alpha_k\alpha_{-k} = E_\alpha(\Delta). \qquad \bullet$$

Llamemos Q al conjunto de los operadores $\beta \in L(C(G), L(M))$ que verifican i - iv de la proposición 3.2; conjunto éste que identificamos con *SD(G,M)*. Fijemos $\alpha \in Q$. Asumamos por el momento que se verifican las siguientes hipótesis; que nos permitirán suponer en Q una estructura de espacio homogéneo.

Hipótesis:

1) La acción Π_α es localmente transitiva y admite secciones locales continuas.

2) El espacio tangente a Q en α es complementado en $L(C(G), L(M))$.

Observación 3.5:

Supongamos que $T \in In(M)$ es tal que $\Pi_\alpha(T) = \alpha$; esto significa que $T\alpha_k = \alpha_k T$ $\forall k \in G$. Derivando respecto de T obtenemos que $\Delta\alpha_k = \alpha_k\Delta$ si Δ pertenece al tangente de $In(M)$ en T.

En otras palabras, si llamamos II_α al conjunto $II_\alpha = \{T \in Aut(M) : \Pi_\alpha(T) = \alpha\}$; entonces vale que $T_T(II_\alpha) = I_\alpha$. por otra parte este último conjunto, por ser imagen del proyector E_α es complementado en $Der(M)$.

Observación 3.6:

Si $\alpha \in Q$ entonces $\alpha_{f\star g} = \alpha_f\alpha_g$, derivando obtenemos que si $X \in T_\alpha(Q)$ luego $X_{f\star g} = \alpha_f X_g + X_f \alpha_g$. Además si identificamos $X_k = X_{\delta_k}$ $(k \in G)$ entonces $X_{j+k} = \alpha_j X_k + X_j \alpha_k$ $\forall k, j \in G$. Por otra parte, siendo $\alpha_k(xy) = \alpha_k(x)\alpha_k(y)$, (donde $k \in G$,

$x, y \in M$), y $\alpha(x^\star) = \alpha(x)^\star$ entonces $X_k(xy) = \alpha_k(x)X_k(y) + X_k(x)\alpha_k(y)$ y además $X_k(x^\star) = X_k(x)^\star$.

Finalmente si $\overline{\Pi}_\alpha$ es la diferencial de Π_α se tiene que $\overline{\Pi}_\alpha : Der(M) \to L(C(G), L(M))$ y $\overline{\Pi}_\alpha(\delta)f = \delta\alpha_f - \alpha_f\delta$.

Las fórmulas que definen a $\overline{\Pi}_\alpha$ y E_α pueden naturalmente extenderse a *L(M)*, indicaremos estas extensiones con las mismas letras $\overline{\Pi}_\alpha$ y E_α y haremos uso de ellas sin mencionarlo explícitamente en el teorema 3.8.

Definición 3.7:

Definimos $K_\alpha : L(C(G), L(M)) \to L(M)$ *por:* $K_\alpha(X) = \frac{1}{|G|}\sum_{r\in G} X_r\alpha_{-r}$.

Teorema 3.8:

De acuerdo con las notaciones anteriores, valen los siguientes hechos:

i) $K_\alpha(X) \in Der(M)\ \forall X \in T_\alpha(Q)$.

ii) $\overline{\Pi}_\alpha(K_\alpha(X)) = X\ \forall X \in T_\alpha(Q)$.

iii) $K_\alpha(\overline{\Pi}_\alpha(A)) = (1 - E_\alpha)(A)\ \forall A \in L(M)$.

Demostración:

Sean $x, y \in M$; demostremos primeramente la propiedad i. Ya que $X_k(x^\star) = X_k(x)^\star\ \forall x \in G$ entonces basta ver que $K_\alpha(X)(xy) = xK_\alpha(X)(y) + K_\alpha(X)(x)y$. En efecto:

$$\begin{aligned}
K_\alpha(X)(xy) &= \frac{1}{|G|}\sum_{k\in G} X_k(\alpha_{-k}(xy)) = \frac{1}{|G|}\sum_{k\in G} X_k(\alpha_{-k}(x)\alpha_{-k}(y)) \\
&= \frac{1}{|G|}\sum_{k\in G}(\alpha_k\alpha_{-k}(x)X_k(\alpha_{-k}(y)) + X_k(\alpha_{-k}(x))\alpha_k\alpha_{-k}(y)) \\
&= xK_\alpha(X)(y) + K_\alpha(X)(x)y.
\end{aligned}$$

Para probar ii veamos que $\overline{\Pi}_\alpha(K_\alpha(X))_j = X_j\ \forall j \in G$.

$$\begin{aligned}
\overline{\Pi}_\alpha(K_\alpha(X))_j &= K_\alpha(X)\alpha_j - \alpha_j K_\alpha(X) = \frac{1}{|G|}\left(\sum_{k\in G} X_k\alpha_{-k+j} - \sum_{k\in G}\alpha_j X_k\alpha_{-k}\right) \\
&= \frac{1}{|G|}\left(\sum_{k\in G} X_k\alpha_{-k+j} - \sum_{k\in G}(X_{j+k} - X_j\alpha_k)\alpha_{-k}\right) \\
&= \frac{1}{|G|}\sum_{k\in G} X_j = X_j.
\end{aligned}$$

Finalmente probemos el punto iii.

$$K_\alpha(\overline{\Pi}_\alpha(A)) = \frac{1}{|G|}\sum_{k\in G}\overline{\Pi}_\alpha(A)|_k\alpha_{-k} = \frac{1}{|G|}\sum_{k\in G}(A\alpha_k - \alpha_k A)\alpha_{-k}$$
$$= A - \frac{1}{|G|}\sum_{k\in G}\alpha_k A\alpha_{-k} = (1-E_\alpha)(A) \bullet$$

Nota:

Estudiaremos a continuación la validez de la hipótesis 2 que afirma que el espacio tangente $T_\alpha(Q)$ es complementado en *L(C(G),L(M))*.

Proposición 3.9:

L(C(G),L(M)) es isométricamente isomorfo a $L(M)^{|G|} := L(M)\oplus\cdots\oplus L(M)$ *(*$|G|$ *sumandos).*

Demostración:

Si $\underline{A}\in L(M)^{|G|}$; $\underline{A}=(A_1,\ldots,A_{|G|})$ entonces definimos: $\| \underline{A} \| := \max_{k\in G} \| A_k \|$.

Sea $\Gamma : L(M)^{|G|} \to L(C(G), L(M))$ definida por: $\Gamma(\underline{A})f := \sum_{k\in G} f(k)A_k$.

Entonces $\| \Gamma(\underline{A})f \| = \| \sum_{k\in G} f(k)A_k \| \leq \| \underline{A} \| \sum_{k\in G} |f(k)| = \| \underline{A} \| \| f \|_{C(G)}$.

Se deduce que $\| \Gamma(\underline{A}) \| \leq \| \underline{A} \|$. La igualdad de las normas resulta de considerar que $\| \delta_k \| = 1$ y que $\Gamma(\underline{A})\delta_k = A_k$. Asimismo esta observación dice que si $X\in L(C(G), L(M))$ entonces $\Gamma^{-1}(X) = (X_{\delta_k})_{k\in G}$. $\bullet$

Observación 3.10:

Existe una inclusión natural de *L(M)* en $L(M)^{|G|}$; por lo que podemos considerar a $Der(M) \subset L(M) \subset L(M)^{|G|}$. Además el teorema 3.8 nos permite asumir que $T_\alpha(Q) \subset L(M)^{|G|}$. De este modo hemos podido colocar a los espacios tangentes de *In(M)* y *SD(G,M)* en el contexto de un mismo espacio de Banach.

Observación 3.11:

Diremos que $X \in L(C(G), L(M))$ verifica la propiedad P si valen las tres siguientes afirmaciones:

i) $X_{j+k} = \alpha_j X_k + X_j \alpha_k \ \forall k, j \in G$;

ii) $X_k(xy) = \alpha_k(x)X_k(y) + X_k(x)\alpha_k(y) \ \forall k \in G \ \forall x, y \in M$;

iii) $X_k(x^\star) = X_k(x)^\star \ \forall x \in M$.

La observación 3.5 nos dice que Si $X \in T_\alpha(Q)$ entonces X verifica la propiedad P. Veremos que también vale la recíproca.

Proposición 3.12:

Sea $X \in L(C(G), L(M))$; entonces $X \in T_\alpha(Q)$ si y sólo si X verifica la propiedad P. Además si $X \in T_\alpha(Q)$ entonces existe $\Delta \in Der(M)$ tal que $X_k = \Delta\alpha_k - \alpha_k\Delta$.

Demostración:

Sea $X \in L(C(G), L(M))$ que verifica la propiedad P. Es fácil probar que en ese caso $K_\alpha(X) \in Der(M)$ y en consecuencia $e^{tK_\alpha(X)} \in In(M)$ cualquiera sea $t \in \mathbb{R}$.

Tomemos la curva $c : \mathbb{R} \to Q$ definida por $c(t) = e^{tK_\alpha(X)} \star \alpha$. Entonces $\frac{d}{dt}\big|_{t=0} c(t)_k = K_\alpha(X)\alpha_k - \alpha_k K_\alpha(X)$.

Como X verifica la propiedad P entonces es fácil ver que $\overline{\Pi}_\alpha(K_\alpha(X)) = X$; de donde se deduce que $X \in T_\alpha(Q)$ y que $X_k = K_\alpha(X)\alpha_k - \alpha_k K_\alpha(X)$ $\forall k \in G$. •

Corolario 3.13:

$T_\alpha(Q) = \{\Delta\alpha - \alpha\Delta : \Delta \in Der(M)\}$. *Donde* $(\Delta\alpha - \alpha\Delta)_k = \Delta\alpha_k - \alpha_k\Delta$.

Consideremos el caso $G = \mathbb{Z}_2 = \{0, 1\}$. Combinando la proposición 3.9 con el corolario 3.13 tenemos que $T_\alpha(Q)$ se identifica con el subespacio de $L(M)^2$ caracterizado por $T_\alpha(Q) = \{(0, \Delta\alpha_1 - \alpha_1\Delta) : \Delta \in Der(M)\}$.

Si $A \in L(M)$ entonces

$$A = A\alpha_1\alpha_1 = \frac{1}{2}(A\alpha_1\alpha_1 - \alpha_1 A\alpha_1 + \alpha_1 A\alpha_1 + A\alpha_1\alpha_1) = \frac{1}{2}(A - \alpha_1 A\alpha_1) + E_\alpha(A).$$

De donde se deduce que $\frac{1}{2}(A - \alpha_1 A\alpha_1) = (1 - E_\alpha)(A)$.

Por otra parte como *Der(M)* es complementado en *L(M)*, con proyector asociado $\mathbb{P}$, entonces $Der(M)\alpha_1 := \{\Delta\alpha_1 : \Delta \in Der(M)\}$ también es complementado, con proyector asociado $\overline{\mathbb{P}}(A) := \mathbb{P}(A\alpha_1)\alpha_1$.

Afirmamos además que $\mathbb{P}$ conmuta con $1 - E_\alpha$. En efecto:

$$\begin{aligned}\mathbb{P}(1 - E_\alpha(A)) = \mathbb{P}\left(A - \frac{1}{2}(A + \alpha_1 A\alpha_1)\right) &= \mathbb{P}(A) - \frac{1}{2}(\mathbb{P}(A) + \mathbb{P}(\alpha_1 A\alpha_1)) \\ &= \mathbb{P}(A) - \frac{1}{2}(\mathbb{P}(A) + \alpha_1\mathbb{P}(A)\alpha_1) = (1 - E_\alpha)\mathbb{P}(A)\end{aligned}$$

Análogamente se prueba que $\overline{\mathbb{P}}(1 - E_\alpha) = (1 - E_\alpha)\overline{\mathbb{P}}$ y en consecuencia $\overline{\mathbb{P}}(1 - E_\alpha)$ es un proyector.

Además si $A \in L(M)$ entonces $\mathbb{P}(A) = \Delta_0\alpha_1$ para algún $\Delta_0 \in Der(M)$ y entonces $\overline{\mathbb{P}}(1 - E_\alpha)(A) = \mathbb{P}(A) - \alpha_1\mathbb{P}(A)\alpha_1 = \Delta_0\alpha_1 - \alpha_1\Delta_0$.

Definamos $\overline{E}_\alpha \in L(L(M)^2)$ por $\overline{E}_\alpha(A,B) = (0, \mathbb{P}(1 - E_\alpha)(A))$; entonces se deduce que $\overline{E}_\alpha$ es un proyector sobre $T_\alpha(Q)$, que, por lo tanto, resulta complementado.

Observación 3.14:

Hemos visto que en el caso $G = \mathbb{Z}_2$ se verifica la segunda de las hipótesis planteadas (que $T_\alpha(Q)$ sea complementado). Veremos en la siguiente sección que en este caso también se verifica la hipótesis restante.

El proyector $\overline{E}_\alpha$ nos permite definir en Q una conexión, donde los espacios horizontales se definen como $H^\alpha := Ker(\overline{E}_\alpha)$ y cuya exponencial estará dada por: $\Phi_\alpha(X)_f = e^{K_\alpha(X)}\alpha_f e^{-K_\alpha(X)}$. Luego, las geodésicas de la conexión están dadas por: $c_{\alpha,X}(t)_f = e^{tK_\alpha(X)}\alpha_f e^{-tK_\alpha(X)}$.

Observación 3.15: Acerca de las secciones locales.

Si continuásemos con la analogía con [**MR**] podríamos intentar definir una sección local mediante la siguiente fórmula $s_\alpha(\beta) = \frac{1}{|G|}\sum_{k\in G}\alpha_k\beta_{-k}$. Es fácil verificar que si la distancia (positiva) entre α y β es suficientemente pequeña entonces $s_\alpha(\beta)$ es inversible y además $s_\alpha(\beta)\alpha_j s_\alpha(\beta)^{-1} = \beta_j \; \forall j \in G$.

Sin embargo **no** se puede tomar a s_α como sección local pues, en general, $s_\alpha \notin Aut(M)$ (no es multiplicativa). Por ejemplo, en el caso $G = \mathbb{Z}_2$; puede probarse que $s_\alpha \in Aut(M) \iff \alpha = \beta$. Este hecho muestra una diferencia esencial entre la sección previa y [**MR**].

4. AUTOMORFISMOS DE ORDEN 2.

Sea M una W*-álgebra inyectiva. Denotaremos por $Z(M)$ al *centro* de M, $Z(M) = \{x \in M : xy = yx \; \forall y \in M\}$. Llamaremos por otra parte $\mathbb{Z}_2(M)$ al conjunto de los automorfismos de orden 2, es decir $\mathbb{Z}_2(M) = \{\alpha \in Aut(M) : \alpha^2 = id_M\}$. Es evidente que cada $\alpha \in \mathbb{Z}_2(M)$ induce de manera natural una representación $\tilde{\alpha} \in SD(\mathbb{Z}_2, M)$. Además si $\tilde{\alpha} \in SD(\mathbb{Z}_2, M)$ entonces $\tilde{\alpha}_1 \in \mathbb{Z}_2(M)$. Más aún, la aplicación $SD(\mathbb{Z}_2, M) \to \mathbb{Z}_2(M)$ dada por $\tilde{\alpha} \to \tilde{\alpha}_1$ es un homeomorfismo. Podemos aplicar, entonces, a $\mathbb{Z}_2(M)$ todos los resultados expuestos en la sección anterior. Veremos que en este caso la acción $\Pi_\alpha : In(M) \to \mathbb{Z}_2(M)$ admite secciones locales.

Fijemos una rama del logaritmo en $\mathbb{C}$ y sea r un número racional fijo.

La función $\mathbb{C} \to \mathbb{C}$; $z \to z^r = e^{r\log(z)}$ es analítica en 1, luego, para todo z en un entorno U_r de 1 vale $z^r = \sum_{n=0}^{\infty} a_n(r)(z-1)^n$.

Observación 4.1:

Si $u \in M$ unitario es tal que $sp(u) \subset U_r$ entonces el desarrollo en serie antes indicado nos permite definir el elemento unitario $u^r \in M$ por $u^r = \sum_{n=0}^{\infty} a_n(r)(u-1)^n$. Además si $\alpha \in Aut(M)$ entonces $sp(u) = sp(\alpha(u))$ y vale que $\alpha(u^r) = \alpha(u)^r$.

Observación 4.2:

Si M es una W*-álgebra y $\alpha, \beta \in Aut(M)$ verifican que $\| \alpha - \beta \| < 2$ entonces existe $u \in M$ unitario tal que $\alpha(x) = u\beta(x)u^\star$ $\forall x \in M$ y además

$$sp(u) \subset \left\{ z \in \mathbb{C} : |z| = 1 \text{ y } \Re z \geq \frac{1}{2}(4 - \| \alpha - \beta \|^2)^{\frac{1}{2}} \right\}$$

Para una demostración de esta observación puede verse **[P]**, Proposition 8.7.9.

Lema 4.3:

Dado $\epsilon > 0$ existe $\delta > 0$ tal que si $0 < \delta_1 < \delta$ entonces todo $z \in \mathbb{C}$ tal que $|z| = 1$ y $\Re z \geq \frac{1}{2}(4 - \delta_1^2)^{\frac{1}{2}}$ verifica que $|z - 1| < \epsilon$.

Corolario 4.4:

De acuerdo con las notaciones anteriores, fijado un número racional r, existe $\delta = \delta(r) > 0$ que depende sólo de r tal que si $\alpha, \beta \in Aut(M)$ verifican $\| \alpha - \beta \| < \delta(r)$ entonces existe $u \in M$ unitario que cumple las siguientes condiciones

i) $\alpha(x) = u\beta(x)u^\star$ $\forall x \in M$;

ii) $sp(u) \subset U_r$.

Las demostraciones del lema 4.3 y del corolario 4.4 son completamente elementales y por lo tanto se omiten.

Sea ahora $\alpha \in \mathbb{Z}_2(M)$ y, según las notaciones del corolario 4.4, tomemos $\delta = \delta(\frac{1}{2})$. Si $\| \alpha - \beta \| < \delta$ entonces existe $u \in M$ unitario tal que $\alpha(x) = u\beta(x)u^\star$ $\forall x \in M$ y además $sp(u) \subset U_{\frac{1}{2}}$.

Proposición 4.5:

En las condiciones antes descriptas, si $v = u^{\frac{1}{2}}$ entonces $Ad(v)^{-1}\alpha Ad(v) = \beta$.

Demostración:

Considerando que $\alpha(x) = u\beta(x)u^\star$ y que $\alpha, \beta \in \mathbb{Z}_2(M)$ entonces $\forall x \in M$:

$$x = \beta(\beta(x)) = u^\star\alpha(u^\star\alpha(x)u)u = u^\star\alpha(u^\star)\alpha^2(x)\alpha(u)u = u^\star\alpha(u^\star)x\alpha(u)u.$$

Se deduce que $\alpha(u) = u^\star c$ donde $c \in Z(M)$. Sea $v = u^{\frac{1}{2}}$. Entonces:

$$\alpha(v)^\star v = \alpha\left(u^{\frac{1}{2}}\right)^{-1} u^{\frac{1}{2}} = (u^\star c)^{-\frac{1}{2}} u^{\frac{1}{2}} = u^{\frac{1}{2}} c^{-\frac{1}{2}} u^{\frac{1}{2}} = uc^{-\frac{1}{2}}.$$

En consecuencia $\alpha(x) = u\beta(x)u^\star = uc^{-\frac{1}{2}}\beta(x)c^{\frac{1}{2}}u^\star = \alpha(v)^\star v\beta(x)v^\star\alpha(v)$. Y entonces $\alpha(v)\alpha(x)\alpha(v)^\star = v\beta(x)v^\star$; de donde se deduce que $\alpha Ad(v)(x) = Ad(v)\beta(x)$. •

La proposición 4.5 implica la existencia de secciones locales continuas para la acción $\Pi_\alpha : In(M) \to \mathbb{Z}_2(M)$. De la demostración anterior se deduce además el siguiente hecho.

Observación 4.6: Acerca de las secciones locales

Supongamos que $\alpha, \beta \in Aut(M)$ verifican que existe $u \in M$ unitario tal que $\alpha(x) = u\beta(x)u^\star$. Si existen $v \in M$ unitario y $c \in Z(M)$ tales que $\alpha(v)^\star v = uc$ entonces el tal v verifica que $Ad(v)^{-1}\alpha Ad(v) = \beta$.

Corolario 4.7:

$\mathbb{Z}_2(M)$ es un subconjunto abierto de $Aut(M)$.

5. REPRESENTACIONES UNITARIAS.

Sea G un grupo localmente compacto y abeliano; y sea M un álgebra de von Neumann actuando en un espacio de Hilbert H. Si $\Pi : G \to L(H)$ es una representación unitaria y $\pi : C^\star(G) \to L(H)$ es la $\star$-representación no degenerada asociada a ella, entonces el rango de Π está contenido en M si y sólo si el rango de π lo está.

Recordemos que dada $\Pi : G \to L(H)$ la representación π asociada queda caracterizada por la fórmula:

$$\pi(f) = \int_G \Pi(g)f(g)dg \ \forall f \in L^1(G).$$

Recíprocamente dada $\pi : C^\star(G) \to L(H)$, la representación Π está definida por:

$$< \Pi(g)\psi, \eta > = lim_\lambda < \pi(\delta_g \star \Phi_\lambda)\psi, \eta > \quad \forall \psi, \eta \in H.$$

Donde $(\Phi_\lambda)_{\lambda\in\Lambda}$ es una aproximación acotada de la identidad en $L^1(G)$; $\| \Phi_\lambda \|_1 \leq K \ \forall\lambda \in \Lambda$. Para mayores detalles véase [**ACS 2**], sección 4.2 y [**P**], capítulo 7.

Si $R(C^\star(G), M)$ es el conjunto de $\star$-representaciones de $C^\star$(G) en M y *S(G,M)* es el conjunto de representaciones unitarias de G en M; sea ρ la biyección recién definida. El conjunto $R(C^\star(G), M)$ tiene una topología natural dada por la norma. La pregunta que queda planteada es qué topología hay que considerar en *S(G,M)* para que ρ resulte un homeomorfismo (cf. [**ACS 2**]). Veremos a continuación la respuesta.

Definición 5.1: *Si Π_n, Π son representaciones unitarias de G en $L(M)$ entonces diremos que $\Pi_n \to \Pi$ uniformemente si y sólo si $\forall\epsilon > 0 \ \exists n_0 \in I\!N$ tal que $\| \Pi_n(g) - \Pi(g) \| < \epsilon \ \forall n \geq n_0 \ \forall g \in G$.*

Proposición 5.2:

Según las notaciones anteriores; sean Π_n, $\Pi : G \to L(H)$ *representaciones unitarias y* π_n, π *las representaciones asociadas. Entonces* $\pi_n \to \pi$ *en norma de* $L(C^\star(G), L(H))$ *si y sólo si* $\Pi_n \to \Pi$ *uniformemente.*

Demostración:

Supongamos que $\pi_n \to \pi$ y sea $(\Phi_\lambda)_{\lambda\in\Lambda}$ una aproximación acotada de la identidad, donde Λ es un conjunto dirigido y $\| \Phi_\lambda \|_1 \leq K \ \forall \lambda \in \Lambda$.

Entonces para cada $\psi, \eta \in H$, $g \in G$ y $\forall n \in I\!N$, $< \Pi(g)\psi, \eta > = lim_\lambda < \pi(\delta_g \star \Phi_\lambda)\psi, \eta >$ y $< \Pi_n(g)\psi, \eta > = lim_\lambda < \pi_n(\delta_g \star \Phi_\lambda)\psi, \eta >$.

Como $\pi_n \to \pi$ en norma, entonces para cada $\psi, \eta \in H$ y para cada $\lambda \in \Lambda$:

$$lim_n < \pi_n(\delta_g \star \Phi_\lambda)\psi, \eta > = < \pi(\delta_g \star \Phi_\lambda)\psi, \eta > .$$

Tenemos además que si $\psi, \eta \in H$, $\| \psi \| \leq 1, \| \eta \| \leq 1$ entonces

$$| < \pi_n(\delta_g \star \Phi_\lambda)\psi, \eta > - < \pi(\delta_g \star \Phi_\lambda)\psi, \eta > | = | < (\pi_n - \pi)(\delta_g \star \Phi_\lambda)\psi, \eta > |$$

$$\leq \| (\pi_n - \pi)(\delta_g \star \Phi_\lambda) \| \ \| \psi \| \ \| \eta \| \leq \| \pi_n - \pi \| \ \| \delta_g \star \Phi_\lambda \| \ \| \psi \| \ \| \eta \| \leq K \| \pi_n - \pi \| .$$

Nótese que la acotación anterior es independiente de λ. Por lo tanto si $\psi, \eta \in H$, $\| \psi \| \leq 1, \| \eta \| \leq 1$; $| < (\Pi_n(g) - \Pi(g))\psi, \eta > | = lim_\lambda | < \pi_n(\delta_g \star \Phi_\lambda)\psi, \eta > - < \pi(\delta_g \star \Phi_\lambda)\psi, \eta > | \leq K \| \pi_n - \pi \|$.

La conclusión es que $\forall \epsilon > 0 \ \exists n_0 \in I\!N$ tal que si $n \geq n_0$ entonces

$$| < (\Pi_n(g) - \Pi(g))\psi, \eta > | < \epsilon \ \forall g \in G \ \forall \| \eta \|, \| \psi \| \leq 1.$$

Recordemos que si $B \in L(H)$ es tal que $| < B\psi, \eta > | < \epsilon \ \forall \| \psi \|, \| \eta \| \leq 1$ entonces $\| B \|_{L(H)} \leq \epsilon$.

Aplicando esta observación a $B := \Pi_n(g) - \Pi(g)$ obtenemos que:

$$\forall \epsilon > 0 \ \exists n_0 \in I\!N \text{ tal que } \| \Pi_n(g) - \Pi(g) \| < \epsilon \ \forall n \geq n_0 \ \forall g \in G.$$

Para probar la recíproca, supongamos que $\Pi_n \to \Pi$ uniformemente. Dado $\epsilon > 0$ sea $n_0 \in I\!N$ tal que $\| \Pi_n(g) - \Pi(g) \| < \epsilon \ \forall g \in G$, $n \geq n_0$. Sea $f \in L^1(G)$, $\| f \|_1 \leq 1$. Si $n \geq n_0$ entonces $\| \pi_n(f) - \pi(f) \| \leq \int_G \| \Pi_n(g) - \Pi(g) \| \, |f(g)| dg \leq \epsilon \int_G |f(g)| dg = \epsilon$. Por la densidad de $L^1(G)$ se deduce que $\| \pi_n - \pi \| \leq \epsilon \ \forall n \geq n_0$. •

Observación 5.3:

A partir de lo demostardo en [**ACS 2**], la proposición 5.2 nos permite definir en el conjunto de representaciones unitarias de G en M una estructura de espacio homogéneo.

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SL(2,R)-MODULE STRUCTURE OF THE EIGENSPACES OF THE CASIMIR OPERATOR

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ABSTRACT. In this paper, on the space of smooth sections of a $SL(2,R)$−homogeneous vector bundle over the upper half plane we study the $SL(2,R)$ structure for the eigenspaces of the Casimir operator. That is, we determine its Jordan-Hölder sequence and the socle filtration. We compute a suitable generalized principal series having as a quotient a given eigenspace. We also give an integral equation which characterizes the elements of a given eigenspace. Finally, we study the eigenspaces of twisted Dirac operators.

§1. *Introduction*

Let $G = SL(2,\mathbf{R})$ and K be a fixed maximal compact subgroup K of G. Let (τ, V) be a representation of K, we denote

$$C^{\infty}(G/K,V) = \{\, f : G \to V \ /\, f \text{ is } C^{\infty} \text{ and } f(gk) = \tau(k)^{-1} f(g) \quad \text{for all } k \in K \,\}$$
$$L^{2}(G/K,V) = \{\, f : G \to V \ /\, f(gk) = \tau(k)^{-1} f(g) \quad \text{for all } k \in K,\ \|f\|_2^2 < \infty \,\}$$

where $\| \ \|_2$ is computed with respect to Haar measure. On $L^2(G/K,V)$ we fix the obvious topology. On $C^{\infty}(G/K,V)$ we fix the topology of uniform convergence on compacts of the functions and their derivatives. Both spaces are representations of G under the left regular action $L_g f(x) = f(g^{-1}x)$ for all $g, x \in G$.

Let Ω the Casimir element of the universal algebra $\mathcal{U}(g_o)$ of the Lie algebra g_o of G, Ω define a G-left invariant operator on $C^{\infty}(G/K,V)$. Here, we obtain the G-module structure of each eigenspace of the Casimir operator

$$\Omega : C^{\infty}(G/K,V) \quad \to \quad C^{\infty}(G/K,V)$$

whenever V is an irreducible representation of K. Actually, we prove that whenever an eigenspace is irreducible, then it is infinitesimally equivalent to a principal series representation, and when an eigenspace is reducible then we have an exact sequence $0 \to W \to A_{\lambda}^{n} \to M \to 0$, where A_{λ}^{n} is the λ−eigenspace of Ω in $C^{\infty}(G/K,V)$, W is a Verma module and M an irreducible Verma module.

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As a corollary we obtain the eigenvalues and eigenspaces of

$$\tilde{\Omega}\colon L^2(G/K,V) \to L^2(G/K,V)$$

From this, it results that if λ is an eigenvalue of $\tilde{\Omega}$ the corresponding eigenspace is a proper subset of the respective one of Ω. We also compute the L^2-eigenspaces of the Dirac operator $\mathbf{D}$.

Knapp-Wallach [K-W] obtained an integral operator which sends an adjusted principal series onto the K−finite vector of the L^2−kernel of the Dirac operator $\mathbf{D}$. In this work we obtain a similar result for each L^2-eigenspace of $\mathbf{D}$ (c.f §4).

Let $\phi_{\lambda,n}$ be the Eisenstein function (cf. ***) in $C^\infty(G/K,V)$ that belongs to the λ−eigenspace of Ω, we prove:

(i) a continuous function that satisfies the integral equation

$$\int_K f(gkx)dk = f(g)\phi_{\lambda,n} \quad \text{for all } g, x \in G$$

is smooth and is an eigenfunction of Ω corresponding to the eigenvalue λ.

(ii) Any λ−eigenfunction of Ω satisfies the integral equation in (i).

Now, we stablish some notations,

$$(1.2)\qquad \begin{aligned}
K &= \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad : \quad \theta \in \mathbf{R} \right\} \\
A &= \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad : \quad t \in \mathbf{R}^+ \right\} \\
M &= \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad : \quad x \in \mathbf{R} \right\} \\
A^+ &= \{ a_t \in A \quad : \quad 1 < t \} \\
A^- &= \{ a_t \in A \quad : \quad 0 < t < 1 \}
\end{aligned}$$

We will use the decompositions $G = KAN$ and $G = KAK = K\overline{A^+}K = K\overline{A^-}K$ [K]. If we denote by

$$(1.3)\qquad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the Iwasawa decomposition of the Lie algebra g_o of G is $g_o = k_o \oplus a_o \oplus n_o$ where $k_o = \mathbf{R}X$, $a_o = \mathbf{R}H$, $n_o = \mathbf{R}Y$. We denote by g, k, a, n their complexifications.

The Casimir operator Ω is an element of the universal algebra $\mathcal{U}(g)$ of g, moreover, the center of $\mathcal{U}(g)$ is $\mathbf{C}[\Omega]$ [L]. It is defined by

$$(1.4)\qquad \Omega = \tfrac{1}{2}\left(H^2 - H - YX\right)$$

If

$$W = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad E_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad E_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \tag{1.5}$$

another expression of Casimir operator is

$$\Omega = \tfrac{1}{8}\left(W^2 + 2W + 4E_-E_+\right) \tag{1.6}$$

W, E_+ and E_- satisfy the relations

$$\overline{W} = -W \qquad \overline{E_\pm} = E_\mp \qquad [E_+, E_-] = W \qquad [W, E_\pm] = \pm 2E_\pm$$

Let θ be the usual Cartan involution on g_o. Therefore, k_o is the subspace of fix points of θ. Let p_o be the (-1)–eigenspace of θ.

The Killing form in g_o is

$$B(X,Y) = 4\mathrm{Trace}(XY).$$

Thus $\{\frac{1}{2}E_+, \frac{1}{2}E_-\}$ is an orthonormal base of p with respect to the hermitian form

$$-B(X, \theta\bar{Y})$$

The Iwasawa decomposition for E_+ and E_- is

$$\begin{aligned} \tfrac{1}{2}E_+ &= \tfrac{1}{4}W + \tfrac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \\ \tfrac{1}{2}E_- &= -\tfrac{1}{4}W + \tfrac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{1.7}$$

§2. *Eigenspaces of* Ω

Since K is abelian, the irreducible representations of K are onedimensional. They are (τ_n, V_n) with $n \in \mathbf{Z}$, where

$$\dim V_n = 1 \; and \; \tau_n(k_\theta)v = e^{in\theta}v \qquad \text{for all } v \in V_n$$

Given $n \in \mathbf{Z}$, the elements of the center of the universal enveloping algebra of g will be considered acting on $C^\infty(G/K, V_n)$ as left invariant operators.

For all $\lambda \in \mathbf{C}$ define

$$A^n_\lambda = \left\{ f \in C^\infty(G/K, V_n) \;\Big/\; \Omega f = \frac{\lambda^2 - 1}{8} f \right\} \tag{2.1}$$

Since Ω is a continuous linear operator on $C^\infty(G/K, V_n)$, it follows that A^n_λ is a closed subspace of $C^\infty(G/K, V_n)$. Thus, A^n_λ is a subrepresentation of $C^\infty(G/K, V_n)$ with infinitesimal character χ_{λ_δ} , where δ is the linear functional of a_o such that $\delta(H) = \frac{1}{2}$ and $\chi_{\lambda\delta}$ is the character of $\mathbf{C}$ multiplication by $\frac{\lambda^2-1}{8}$.

We denote by $A^n_\lambda[m]$ the K-type τ_m of A^n_λ.

PROPOSITION 2.1.

Given $n \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, the representation A_λ^n of G is admissible and finitely generated. Moreover,

(i) $\dim A_\lambda^n[m] \leq 1$ *for all* $m \in \mathbf{Z}$

(ii) *If* $A_\lambda^n[m] \neq \{0\}$, *then* n *and* m *have the same parity.*

Remark: The converse of (*ii*) is also true. It follows from proposition 2.4.

We need some results to prove the proposition 2.1

Let $f \in A_\lambda^n[m]$, f is a spherical function of type (m, n) because

$$f(k_\theta g k_\psi) = e^{-im\theta} f(g) e^{-in\psi} \quad \text{for all } g \in G,\ k_\theta,\ k_\psi \in K$$

Since $G = KAK$, the values of f are determined by its values on A. Besides, if $m \neq n$ then $f|_K \equiv 0$. In fact, the equallity $f(k_\theta) = f(k_\theta.1) = e^{-im\theta} f(1)$, implies that $f|_K \neq 0 \Leftrightarrow f(1) \neq 0$, now since f is spherical of type (m, n) we have that $f(k_\theta) = f(1.k_\theta) = f(1)e^{-in\theta} = f(1)e^{-im\theta}$, therefore if $f|_K$ were nonzero we would have that $m = n$.

The subgroup A is Lie isomorphic to $\mathbf{R}^+$ (positive real numbers with the usual product) by the isomorphism $\alpha(a_t) = t^2$.

Lemma 2.2.

If $f \in A_\lambda^n[m]$, *the function* $F : \mathbf{R}^+ \to \mathbf{C}$ *associated to* f *given by* $F(\alpha(a)) = f(a)$ *for all* $a \in A$ *satisfy the differential equation*

$$(2.2) \quad z^2 \frac{d^2}{dz^2} - \frac{2z^3}{1-z^2}\frac{d}{dz} - \frac{z^2}{(1-z^2)^2}(m^2+n^2) + \frac{z(1+z^2)}{(1-z^2)^2} nm - \frac{\lambda^2-1}{4} = 0$$

The equation has regular singularities at the points $0, \pm 1, \infty$.

A proof of this lemma is in [Ca-M].

Proof of the Proposition 2.1. Since Ω is an elliptic operator in $C^\infty(G/K, V_n)$, if $f \in A_\lambda^n$, $f|_A$ is real analytic. Therefore, the function $F : \mathbf{R}^+ \to$ defined in (2.2) is a real analytic function. Hence there is a holomorphic extension of F to a neighborhood of $\mathbf{R}^+$ in the right half plane.

On the other hand by the Frobenius theory for differential equations with regular singular points [C-page 132] the equation (2.2) has an analytic solution on a neighborhood of 1 if and only if m and n have the same parity. Moreover, any holomorphic solution of (2.2) is a multiple of the power series

$$(2.3) \qquad (z-1)^{\frac{1}{2}|m-n|} \sum_{j=0}^{\infty} c_j (z-1)^j \qquad c_0 = 1$$

In fact, the indicial equation of (2.2) is

$$s(s-1)+s-\frac{1}{4}(m-n)^2 = 0$$

and its roots are $\pm\frac{1}{2}(m-n)$. Thus, as the roots differ by an integer, the exponent of the first term of (2.3) is $\frac{1}{2}|m-n|$, if this number were not an integer the function (2.3) would not be analytic on a neighborhood of 1, this forces the same parity for n and m.

As the other singularities of (2.2) are $0, -1, \infty$, there is an extension of the analytic solution on a neighborhood of 1 to an analytic solution on a neighborhood of $\mathbf{R}^+$. So (i) and (ii) holds. $\square$

Remark. Since A_λ^n has infinitesimal character $\chi_{\lambda\delta}$ and A_λ^n is admissible by Proposition 2.1, A_λ^n has finite length by a known rwsult of Harish-Chandra [V,Corollary 5.4.16].

Corollary 2.3.

Given $n \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, the K-type τ_n occurs in any subrepresentation of A_λ^n. Moreover, A_λ^n has a unique irreducible G-submodule.

Proof. Let W be a nontrivial closed submodule of A_λ^n and denote by W_K the set of K-finite elements in W, we consider the map

$$(*)\qquad \begin{array}{ccc} \mathrm{Hom}_G(W, A_\lambda^n) & \longrightarrow & \mathrm{Hom}_K(W_K, V_n) \\ T & \longrightarrow & \left(v \to \tilde{T}v = Tv(1)\right) \end{array}$$

This map is well defined. In fact, if $v \in W_K$,

$$\tilde{T}(kv) = T(kv)(1) = \left(L_k.Tv\right)(1) = Tv(k^{-1}) = \tau_n(k)Tv(1)$$

Moreover, it is inyective. In fact, suppose that $\tilde{T} \equiv 0$, so $Tv(1) = 0$ for all $v \in W_K$. As T is a continuous linear transformation, W_K is a dense subset of W [L-page 24], and there exists a sequence $\{v_m\}$ in W_K such that $v_m \to w$ for each $w \in W$, then

$$Tv_m \to Tw \quad \Longrightarrow \quad 0 = Tv_m(1) \to Tw(1)$$

that is, $Tw(1) = 0$ for all w. Now, for $w \in W$,

$$Tw(g) = \left(L_{g^{-1}}.Tw\right)(1) = T(g^{-1}w)(1) = 0 \quad \text{for all } g \in G,$$

so $T \equiv 0$. If W is a closed submodule of A_λ^n, by (*) $W[n] \neq 0$, and by (i) $W[n] = A_\lambda^n[n]$. This concludes the first statement of the corollary. The second follows from the equality $W[n] = A_\lambda^n[n]$. $\square$

Fix $n \in \mathbf{Z}$, $\lambda \in \mathbf{C}$, let δ be the linear functional on a_o such that $\delta(H) = \frac{1}{2}$, $\log a_t = t\ H$, and denote by $(-1)^n$ the character of M such that $\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \to (-1)^n$. As usual, define

$$(2.4)\quad \mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1) =$$
$$= \{f : G \to \mathbf{C} \quad C^\infty \text{ such that}$$
$$f(xman) = e^{-(\lambda+1)\delta(\log a)}(-1)^n(m^{-1})f(x) \ \text{ for all } x \in G,\ man \in MAN\}$$

the representation of G induced by the representation $(-1)^n \otimes e^{\lambda\delta} \otimes 1$ of MAN. G acts by left translation. Recall that $\mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1)$ has infinitesimal character $\chi_{\lambda\delta}$ and $\mathrm{I}^G_{MAN}((-1)^n \otimes e^{\lambda\delta} \otimes 1)$ is irreducible if and only if $\lambda \not\equiv (n+1)\,mod(2)$ [B].

Define linear transformations

$$(2.5)\quad \begin{array}{ccc} \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right) & \xrightarrow{T} & A^n_\lambda \\ f & \longrightarrow & \left(x \to Tf(x) = \int_K f(xk)\tau_n(k)dk\right) \end{array}$$

Whenever it becomes necessary to sea which is the domain of the operators, we will write $T_\pm$, otherwise we will write T.

The linear transformation T is well defined because

$$Tf(xk') = \int_K f(xk'k)\tau_n(k)\,dk = \tau(k')^{-1}\int_K f(xk)\tau_n(k)\,dk.$$

Besides, since $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right)$ has infinitesimal character $\chi_{\lambda\delta}$, T is a left G-morphism and left infinitesimal translation by Ω agrees with right infinitesimal translation, ($L_\Omega.f = R_\Omega.f$ for all $f \in C^\infty(G/K, V_n)$). Hence the image of T is contained in A^n_λ.

T is not zero because

$$T\tau_{-n}(1) = \int_K \tau_{-n}(k)\tau_n(k)dk = \int_K dk \neq 0$$

Note that A^n_λ and $A^n_{\lambda'}$ is the same eigenspace of Ω if $\lambda^2 = (\lambda')^2$. So, if $\lambda \in \mathbf{Z}$ we will always assume that $\lambda \geq 0$.

PROPOSITION 2.4.

Given $n \in \mathbf{Z}$,

(i) If $\lambda \in \mathbf{C} \setminus \mathbf{Z}$, or $\lambda \in \mathbf{Z}$ and $\lambda \not\equiv (n+1)\,\mathrm{mod}(2)$, A^n_λ is infinitesimally equivalent to $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$.

(ii) If $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda+1 \equiv n\,\mathrm{mod}(2)$ and $\lambda > |n|$, A^n_λ is infinitesimally equivalent to $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$.

(iii) If $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n\,\mathrm{mod}(2)$ and $\lambda < n$, the (g, K)-module structure of A^n_λ is the following

$$E_+A^n_\lambda[m] \neq 0 \quad \textit{for all } m \textit{ such that } A^n_\lambda[m] \neq 0$$
$$E_-A^n_\lambda[m] \neq 0 \quad \textit{for all } m \neq \pm\lambda \textit{ such that } A^n_\lambda[m] \neq 0$$
$$E_-A^n_\lambda[\pm\lambda + 1] = 0$$

(iv) If $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n \bmod(2)$, $n < 0$ and $\lambda < -n$, the (g, K)-module structure of A^n_λ is the following

$$\begin{aligned} &E_- A^n_\lambda[n] \neq 0 \quad \textit{for all } m \textit{ such that } A^n_\lambda[m] \neq 0 \\ &E_+ A^n_\lambda[m] \neq 0 \quad \textit{for all } m \neq \pm\lambda + 1 \textit{ such that } A^n_\lambda[m] \neq 0 \\ &E_+ A^n_\lambda[\pm\lambda + 1] = 0. \end{aligned}$$

Remark 1: Under the hypothesis (*iii*) or (*iv*) we have that A^n_λ is not a quotient of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\pm\lambda\delta} \otimes 1\right)$.
Remark 2: A^n_λ is irreducible if and only if $\lambda \not\equiv (n+1) \bmod(2)$.

We need the following lemma to prove (*iii*) of proposition 2.4.

Lemma 2.5.
Given $n \in \mathbf{Z}$, let $\lambda \in \mathbf{Z}_{\geq 0}$, $\lambda + 1 \equiv n \bmod 2$ and $\lambda < n$, there exist $m \in \mathbf{Z}$, $m < -\lambda$ such that $A^n_\lambda[m]$ is not zero.

Proof of Lemma 2.5. Let m be an integer such that

$$m \equiv n \bmod 2 \qquad m < -\lambda \qquad \tfrac{1}{2}(n-m) \quad \text{is even} \tag{2.6}$$

The conditions on m and n ensure the existence of a smooth solution F of (2.2) on the interval $(0, \infty)$. In fact, using the Frobenius method for differential equations with regular singularities, that (2.2) has a analytic solution in a neighbordhood of 1 if and only if m and n have the same parity. Besides, the singularities of (2.2) are $0, \pm 1, \infty$. Therefore, this solution extends to a solution on the interval $(0, \infty)$. Moreover, any smooth solution of (2.2) in the interval $(0, \infty)$ is a multiple of the power series

$$(z-1)^{\frac{1}{2}|m-n|} \sum_{j=0}^{\infty} c_j (z-1)^j \qquad c_0 = 1$$

Therefore, F has a zero of order $\frac{1}{2}|m-n|$ at 1.
We have to prove that F extends to an element of $A^n_\lambda[m]$. This will take some work.
Let $N_K(A)$ be the normalizer of A on K.
Consider $C^\infty_{\tau_{n-m}}(A)$ to be the set of smooth funtions on A such that

(*j*) $\phi(kak^{-1}) = \tau_{n-m}(k)\,\phi(a)$ for all $a \in A$, $k \in N_K(A)$
(*jj*) $\dfrac{\phi(a)}{\delta(\log a)^{\frac{1}{2}(n-m)}}$ is a smooth function and even on A.

Let $f \colon A \to \mathbf{C}$ given by $f(a) = F(\alpha(a))$, with α the isomorphism between A and $\mathbf{R}^+$ defined in (2.2). Let's prove that the function f is in $C^\infty_{\tau_{n-m}}(A)$. In fact, the normalizer of A on K, is exactly

$$N_K(A) = \{\pm I\} = \left\{k_{\frac{\pi}{2}}, k_{-\frac{\pi}{2}}\right\}$$

As $n-m$ and $\frac{1}{2}(n-m)$ are even numbers,

$$\tau_{n-m}(\pm I) = \tau_{n-m}(k_{\pm\frac{\pi}{2}}) = e^{\pm i(n-m)\frac{\pi}{2}} = 1$$

So, f satisfy (j) if and only if $f(a) = f(a^{-1})$ for all $a \in A$, or equivalently $F(x) = F(x^{-1})$ for all $x \in \mathbf{R}^+$. Let's prove that $F(x) = F(x^{-1})$. Let h be the function given by $h(z) = F(z^{-1})$, we want to prove that $h = F$. We claim that h satisfies the same differential equation that F does. In fact, let $w = z^{-1}$, then

$$\begin{aligned}\frac{dh}{dz}(z) &= \frac{dF}{dw}(w)\,w' \\ &= -w^2\frac{dF}{dw}(w)\end{aligned}$$

$$\begin{aligned}\frac{d^2F}{dz^2}(z) &= -2ww'\frac{dF}{dw}(w) + w^4\frac{d^2F}{dw^2}(w) \\ &= 2w^3\frac{dF}{dw}(w) + w^4\frac{d^2F}{dw^2}(w)\end{aligned}$$

and

$$-\frac{2z^3}{1-z^2} = -\frac{2w^{-3}}{1-w^{-2}} = \frac{2w^{-1}}{1-w^2}$$

$$-\frac{z^2}{(1-z^2)^2} = -\frac{w^{-2}}{(1-w^{-2})^2} = -\frac{w^2}{(1-w^2)^2}$$

$$\frac{z(1+z)}{(1-z^2)^2} = \frac{w^{-1}(1+w^{-2})}{(1-w^{-2})^2} = \frac{w(w^2+1)}{(1-w^2)^2}$$

So,

$$\begin{aligned}&z^2\frac{d^2h}{dz^2}(z) - \frac{2z^3}{1-z^2}\frac{dh}{dz}(z) + \\ &\quad\left(-\frac{z^2}{(1-z^2)^2}(m^2+n^2) + \frac{z(1+z^2)}{(1-z^2)^2}nm - \frac{\lambda^2-1}{4}\right)h(z) =\end{aligned}$$

$$\begin{aligned}&= w^2\frac{d^2F}{dw^2}(w) + \left(2w - \frac{2w^{-1}}{1-w^2}w^2\right)\frac{dF}{dw}(w) + \\ &\quad + \left(-\frac{w^2}{(1-w^2)^2}(m^2+n^2) + \frac{w(1+w^2)}{(1-w^2)^2}nm - \frac{\lambda^2-1}{4}\right)F(w)\end{aligned}$$

The right hand side is exactly the equation(2.2) on F, so it is zero. Both h and F are smooth functions on $(0,\infty)$ and solutions of the differential equation (2.2). So, by (2.6) they are multiple of each other in a neighborhood of 1. Hence, we write,

$$\begin{aligned}h(z) &= (z-1)^{\frac{1}{2}|n-m|}\psi_h(z) \\ F(z) &= (z-1)^{\frac{1}{2}|n-m|}\psi_F(z)\end{aligned}$$

with ψ_h and ψ_F power series, such that $c\psi_h(z) = \psi_F(z)$ for a suitable nonzero complex number. Therefore,

$$h(z) = F(z^{-1}) = (z^{-1} - 1)^{\frac{1}{2}|n-m|}\psi_F(z^{-1}) = (z-1)^{\frac{1}{2}(n-m)}z^{-\frac{1}{2}|n-m|}\psi_F(z^{-1})$$

Thus, $\psi_h(z) = (z-1)^{-\frac{1}{2}(n-m)}\psi_F(z^{-1})$. This imply that

$$c\psi_h(z) = (z-1)^{-\frac{1}{2}(n-m)}\psi_F(z^{-1})$$

Hence, $F(z) = F(z^{-1})$ in a neighborhood of 1. As F is real analytic in $(0,\infty)$, $F(z) = F(z^{-1})$ for all $z \in \mathbf{R}^+$. Equivalently, $f(a) = f(a^{-1})$ for all $a \in A$. Thus, f satisfies (j).

We want to prove that f satisfies (jj). The function $\delta(\log a)^{-\frac{1}{2}(n-m)}$ is even on A because

$$\begin{aligned} \delta(log a_t)^{-\frac{1}{2}(n-m)} &= (t\ \delta(H))^{-\frac{1}{2}(n-m)} \\ &= (-t\ \delta(H))^{-\frac{1}{2}(n-m)} \qquad \text{by (2.6)} \\ &= \delta(\log a_t^{-1})^{-\frac{1}{2}(n-m)} \end{aligned}$$

Thus, the function $f(a)\delta(\log a)^{-\frac{1}{2}(n-m)}$ is even. The function $f(a)\delta(\log a)^{-\frac{1}{2}(n-m)}$ is smooth because f is real analytic and has a zero of order $\frac{1}{2}(n-m)$ at 1. Therefore, we have proved that $f \in C^\infty_{\tau_{n-m}}(A)$. We want to extend f to an element of $A^n_\lambda[m]$

Let $C^\infty(G/K)[\tau_{n-m}]$ be the space of smooth complex valued functions on G/K such that $f(kx) = \tau_{n-m}(k)f(x)$ for all $k \in K$, $x \in G$.

We need to prove:

Sublemma 2.6.

The restriction map from $C^\infty(G/K)[\tau_{n-m}]$ to $C^\infty_{\tau_{n-m}}(A)$ is biyective.

Proof of sublemma 2.6. : The equallity $G = KAK$ implies that the restriction map is inyective. To prove that is suryective we appeal to a theorem of Helgason. Let $\mathcal{H}$ be the set of harmonic polynomial functions on p_o. We consider the usual action of K on $\mathcal{H}$. That is, the one determinated by the isotropy representation of K in p_o. We now set ourselves in §10 of [H-1], with $\delta = \tau_{n-m}$. Since $n \equiv m mod(2)$, we have that $\tau_{n-m} \in \hat{K}_o$. Let $degQ^\delta(\lambda) = p(\delta)$. A formula due to Kostant and cited on pag 203 of [H-1] says that $p(\delta) = d(\delta)$ =degree of the harmonic homogeneous polynomials in the $\delta-$isotypic component of $\mathcal{H}$. To compute $d(\delta)$ we proceed as follow: If e_1, e_2 is an orthonormal basis for p_o, we know that $k(\theta)\dot{e}_1 = \cos(2\theta)e_1 - \sin(2\theta)e_2$, $k(\theta)\dot{e}_2 = \sin(2\theta)e_1 + \cos(2\theta)e_2$. Since $(n-m)/2$ is a whole number the polynomial function on p_o, $(e_1 + ie_2)^{(n-m)/2}$ is harmonic and has degree $(n-m)/2$, moreover $k(\theta)(e_1 + ie_2)^{(n-m)/2} = e^{i(n-m)\theta}(e_1 + ie_2)^{(n-m)/2}$. Thus, we have that $p(\delta) = (n-m)/2$. Therefore, our space $C^\infty_{\tau_{n-m}}(A)$ contains the space $\mathcal{D}^{\tau_{n-m}}(A)$ of page 211 in [H-1]. Hence, lemma 10.1 of [H-1] implies that the restricction map from $\mathcal{D}^{\tau_{n-m}}(G/K)$ into $\mathcal{D}^{\tau_{n-m}}(A)$ is a linear homeomorphism. We remark that $\mathcal{D}^{\tau_{n-m}}(G/K) \subset C^\infty(G/K)[\tau_{n-m}]$. A density argument together with the fact that K is compact imply sublemma 2.6. $\square$

We proceed with the proof of lemma 2.5. For this end, we now have that the function f admits a smooth extension $\tilde{f}\colon \exp p_o \to \mathbf{C}$ which satisfies

$$\begin{aligned} \tilde{f}(kak^{-1}) &= \tau_{n-m}(k)\,\tilde{f}(a) \\ &= \tau_m(k)^{-1}\tilde{f}(a)\tau_n(k) \end{aligned} \tag{2.7}$$

The diffeomorphism between G and $\exp p_o K$ ensures that the function $\hat{f}\colon G \to \mathbf{C}$ given by

$$\hat{f}(pk) = \tilde{f}(p)\tau_n(k)^{-1} \qquad \text{for all } p \in \exp p_o\,,\, k \in K$$

is well defined and it is smooth. Also, $\hat{f}$ is in the K-type τ_m of $C^\infty(G/K, V_n)$. In fact, for $x \in G$ we write $x = k_2 a k_2^{-1} k_1$ with $k_1, k_2 \in K$, and $a \in A$, hence

$$\begin{aligned} (L_k\hat{f})(x) = \hat{f}(k^{-1}k_2ak_2^{-1}k_1) &= \tilde{f}(k^{-1}k_2ak_2^{-1}k)\tau_n(k^{-1}k_1)^{-1} \\ &= \tau_{n-m}(k^{-1}k_2)f(a)\tau_n(k^{-1}k_1)^{-1} \\ &= \tau_{n-m}(k^{-1})\tau_{n-m}(k_2)f(a)\tau_n(k^{-1}k_1)^{-1} \\ &= \tau_{n-m}(k^{-1})\tilde{f}(k_2ak_2^{-1})\tau_n(k^{-1})^{-1}\tau_n(k_1)^{-1} \\ &= \tau_n(k^{-1})\tau_m(k)\tilde{f}(p)\tau_n(k^{-1})^{-1}\tau_n(k_1)^{-1} \\ &= \tau_m(k)\tilde{f}(p)\tau_n(k_1)^{-1} \\ &= \tau_m(k)\hat{f}(x) \qquad \square \end{aligned}$$

A comutation like the one in [Wa] page 280, implies that

$$(\Omega\hat{f})(x) = \tau_m(k_2^{-1})\tau_n(k_2^{-1}k_1)(z^2\frac{d^2F}{d^2z} + \dots) = 0$$

because F satisfies the equation 2.2.

This concludes the proof of lemma 2.5

Proof of the Proposition 2.4. (*i*) As T is not the zero function and since $\lambda \not\equiv n+1 \bmod(2)$ the module $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is irreducible. Thus T is inyective. The K-types τ_m which occur in $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ are indexed by all the m with the same parity as n. Since T is one-to-one they must occur in A^n_λ. By proposition 2.1 (*i*), (*ii*), they are exactly the K-types of A^n_λ. Thus, T is suryective at the level of (g, K)-modules.

(*ii*) Since $\lambda \geq 0$, $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ has only one irreducible submodule F which is finite dimensional and whose K-types are parametrized by $\{m : -(\lambda-1) \leq m \leq \lambda - 1,\ m \equiv n\,(2)\}$. The structure of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ is

$$\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right) \quad \begin{matrix} \supset W_+ \\ \supset W_- \end{matrix} \quad \supset F \quad \supset \quad 0$$

where W_+ is the G-submodule spanned by the K-types $\{-(\lambda-1), -(\lambda-3), \dots, \lambda-1, \lambda+1, \dots\}$ and W_- is the one spanned by the K-types $\{\dots, \lambda-3, \lambda-1\}$. As

$\lambda > |n|$ the K-type τ_n occur in F. On the other hand, we have verified that T maps non trivially the $K-$type τ_n, so F is not a submodule of Ker$T$. Since F is contained in every nonzero submodule of $\mathrm{I}_{MAN}^{G}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$. T is 1:1; by a similar argument to the one used on (i) we get that T is suryective.

(iii) Suppose that $n, \lambda > 0$ $\lambda < n$, $\lambda \not\equiv n+1(2)$. Then the image of T_- is the discrete serie $H_{\lambda\delta}$ of infinitesimal character $\chi_{\lambda\delta}$. We recall that the K-types of $H_{\lambda\delta}$ are parametrized by $\{\lambda+1, \lambda+3, \dots\}$. In fact, the nonzero quotients of $\mathrm{I}_{MAN}^{G}\left((-1)^n \otimes e^{-\lambda\delta} \otimes 1\right)$ are $H_{\lambda\delta}$, $H_{-\lambda\delta}$, $H_{\lambda\delta} \oplus H_{-\lambda\delta}$ or itself. Now, the irreducible finite-dimensional submodule occurs in KerT_-, otherwise $T_-(F)$ would be an irreducible submodule of A_λ^n and do not have the K-type τ_n ($\lambda < |n|!$), that contradicts corollary 2.3. This contradiction ensures that T_- is not inyective. By corollary 2.3, A_λ^n has only one irreducible submodule, Im$T_- \neq H_{\lambda\delta} \oplus H_{-\lambda\delta}$. Furthermore, since the irreducible submodule contains the K-type τ_n ,so Im$T_- = H_{\lambda\delta}$. Therefore $H_{\lambda\delta}$ is the irreducible submodule of A_λ^n.

The structure of $\mathrm{I}_{MAN}^{G}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is the following

$$\mathrm{I}_{MAN}^{G}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right) \supset H_{\lambda\delta} \oplus H_{-\lambda\delta} \quad \begin{matrix} \supset H_{\lambda\delta} \\ \supset H_{-\lambda\delta} \end{matrix} \quad \supset 0$$

T_+ is not inyective; otherwise $T_+(H_{-\lambda\delta})$ is an irreducible submodule of A_λ^n and does not have the K-type τ_n. Also Ker $T_+ \neq H_{\lambda\delta} \oplus H_{-\lambda\delta}$; otherwise, the finite dimensional representation F is a subrepresentation of A_λ^n, contradicting corollary 2.3. Thus,

$$\mathrm{Im}T_+ \cong \mathrm{I}_{MAN}^{G}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)/H_{-\lambda\delta}$$

This implies that

$$(\mathrm{Im}T_+)_K = \bigcup_{\substack{m \geq -(\lambda-1) \\ m \equiv n(2)}} A_\lambda^n[m]$$

which is the Verma module of lowest weight $-(\lambda-1)$. Thus,

$$\begin{aligned} &E_+A_\lambda^n[m] \neq 0 \qquad \text{for all } m \geq -(\lambda-1) \\ &E_-A_\lambda^n[m] \neq 0 \qquad \text{for all } m \geq -(\lambda-1) \text{ and } m \neq -\lambda+1 \end{aligned}$$

By lemma 2.5 there exists a K-type $A_\lambda^n[m] \neq 0$ for some $m < -\lambda$. This ensure that $A_\lambda^n[m] \neq 0$ for all $m < -\lambda$ and $m \equiv n\mathrm{mod}(2)$, on the other hand, A_λ^n would have a lowest weight submodule with lowest weight less than $-\lambda\delta$. The infinitesimal character of this lowest weight submodule would be different from $\chi_{\lambda\delta}$, giving a contradiction. Following the same argument, E_+ acts nontrivially on each $A_\lambda^n[m]$, $m < -\lambda$.

For the case $\lambda = 0$ and $\lambda + 1 \equiv n\mathrm{mod}(2)$ the proof is easier.

(iv) It has the same proof of (iii). This concludes the proof of proposition 2.4. □

Remark 1: Given $n \in \mathbf{Z}$ and $\lambda \in \mathbf{C}$, the K-types $A_\lambda^n[m]$ are not zero for all m with the same parity of n.

Remark 2: In view of [S] , in cases (i) and (ii) A_λ^n is equivalent to the maximal model of I_{MAN}^G which is the induced representation with hiperfunctions coefficients. In case (iii) A_λ^n is a quotient of the maximal model of a generalized principal series.

Remark 3: Given $n \in \mathbf{Z}_{\geq 0}$ and $\lambda \geq 0$ as in (iii) of proposition 2.4 , the G-module structure of A_λ^n is

$$\cdots \qquad \bullet \qquad -(\lambda+1)\bullet \underset{0}{\overset{\neq 0}{\rightleftarrows}} \bullet -(\lambda-1) \qquad \cdots \qquad \lambda-1 \bullet \underset{0}{\overset{\neq 0}{\rightleftarrows}} \bullet \lambda+1 \qquad \bullet \qquad \cdots$$

the right arrows represent the action of E_+ and the left ones the action of E_-. That is, we have proved

Corollary 2.6.
Let $\lambda \in \mathbf{Z}_{\geq 0}$ and $\lambda \equiv n+1 \bmod(2)$. A composition series for A_λ^n is

$$0 \to V \to A_\lambda^n \to M \to 0$$

where V is the Verma module of lowest weight $-(\lambda-1)$ and M is the irreducible Verma module of highest weight $-(\lambda+1)$.

PROPOSITION 2.7.
Given $n \in \mathbf{Z}$ and λ as in (iii) of proposition 2.4 (i.e. $\lambda \equiv n+1 \bmod(2)$ and $\lambda \geq 0$ an integer), then A_λ^n is quotient of a generalized principal series $\mathrm{I}_{MAN}^G(W_0)$ where $W_0 = \mathbf{R}^2$ and the representation of MAN is

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \to (-1)^n \exp t \begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

Proof. For $f = (f_1, f_2) \in \mathrm{I}_{MAN}^G(W_0)$ let

$$S : \mathrm{I}_{MAN}^G(W_0) \to C^\infty(G/K, V_n)$$

defined by

$$(Sf)(x) = \int_K f_1(xk)\tau_n(k)\, dk + \int_K f_2(xk)\tau_n(k)\, dk$$

Since $\mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ is contained in $\mathrm{I}_{MAN}^G(W_0)$ via the map $f \to F = (f, 0)$ and S restricted to $\mathrm{I}_{MAN}^G(W_0)$ is equal to T_+, hence $\mathrm{Im}(S)$ contains $\mathrm{Im}(T_+)$. An easy calculation shows that $\mathrm{Im}(S)$ contains properly $\mathrm{Im}(T_+)$. Now, corollary 2.6 implies that any K-finite vector in A_λ^n outside of $\mathrm{Im}(T_+)$ is cyclic in $A_\lambda^n/\mathrm{Im}(T_+)$. Therefore, S is onto. $\square$

Now, consider the Casimir operator acting on the subspace of compactly supported functions in $C^\infty(G/K, V_n)$. We denote by $\tilde{\Omega}$ the unique essentially selfadjoint extension of Ω to a dense subspace of

$$\mathrm{L}^2(G, V_n) = \left\{ f : G \to \mathbf{C} \quad \Big/ \quad \begin{array}{l} f(xk) = \tau_n(k)^{-1} f(x) \\ \int_G |f(x)|^2 \, dx < \infty \end{array} \right\}$$

(cf [A-S]).

PROPOSITION 2.8.

If $W_\lambda^n = \{f \in L^2(G/K, V_n) \ / \ \tilde{\Omega} f = \frac{\lambda^2 - 1}{8} f\}$, then W_λ^n is non zero if and only if $\lambda \in \mathbf{Z} - \{0\}$, $\lambda + 1 \equiv n \mathrm{mod}(2)$ and $|\lambda| < |n|$. Moreover, $W_\lambda^n = W_{-\lambda}^n$ is isomorphic to the discrete series of Harish-Chandra parameter $\lambda\delta$.

Proof. Suppose that $\lambda \in \mathbf{Z} - \{0\}$, $\lambda + 1 \equiv n \mathrm{mod}(2)$ and $|\lambda| < |n|$. As $\tilde{\Omega}$ is elliptic, a Connes-Moscovici result [C-M] ensure that W_λ^n is a sum of discrete series, actually, it is irreducible by the Frobenius Reciprocity. The K-finite elements of $L^2(G/K, V_n)$ are in the set of K-finite elements of $C^\infty(G/K, V_n)$, so $W_\lambda^n[m] \subset A_\lambda^n[m]$ for all $m \in \mathbf{Z}$. By proposition 2.4, A_λ^n has subspaces infinitesimally equivalent to a discrete series for λ such that

$$\lambda \in \mathbf{Z} \qquad \lambda \equiv n + 1 \bmod(2), \qquad 0 < |\lambda| < |n|$$

This "discrete series" subspaces are really contained in $L^2(G/K, V_n)$. In fact, if $f \in A_\lambda^n[m]$ and it belongs to a "discrete series", then f satisfies the differential equation (2.2) or the one which results from the identification of A^+ with $\mathbf{R}_{>0}$ via $a_t \leftrightarrow t$. Then the theory of leading exponents as in [K] says that $f(a_t)\ e^{-(\lambda-1)t}$ at $t = \infty$. Now, the integral formula for the Cartan decomposition together with $\lambda > 0$ imply that f is square integrable. For negative λ we have a similar proof.

For the converse we use the structure of the discrete series, Frobenius Reciprocity together with proposition 2.4. This concludes proposition 2.8. □

§3.*L^2 and C^∞-eigenspaces of the Dirac operator*

Let $g_o = k_o \oplus p_o$ be the Cartan decomposition of g_o, then p_o is the subspace of symmetric matrix of g_o.

If we fix a minimal left ideal S in the Clifford algebra of p_o, the resulting representation of $so(p_o)$ brakes down in two irreducible representations. Such representation composed with the adjoint representation of k_o restricted to p_o lift up at a representation of K called the spin representation of K. Let $\{X_1, X_2\}$ be an orthonormal base of p_o, let c be the Clifford multiplication and fix an integer n. The Dirac operator

$$\mathbf{D} : C^\infty(G/K, V_{n+1} \otimes S) \quad \to \quad C^\infty(G/K, V_{n+1} \otimes S)$$

is defined by

$$\mathbf{D} = \sum_{i=1}^{2} (1 \otimes c(X_i))\, X_i \tag{3.1}$$

where X_i act as left invariant operators for all i. The spin representation S decompose into a sum of two irreducible subrepresentations $S = S^+ \oplus S^-$ (c.f. 4.2 bellow). If $X \in p_o$, then $c(X)S^{\pm} = S^{\mp}$, so

$$\mathbf{D}^{\pm} : C^{\infty}\left(G/K, V_n \otimes S^{\pm}\right) \quad \rightarrow \quad C^{\infty}\left(G/K, V_n \otimes S^{\mp}\right) \tag{3.2}$$

are well defined.

We also consider

$$\tilde{\mathbf{D}} \colon L^2(G/K, V_{n+1} \otimes S) \quad \rightarrow \quad L^2(G/K, V_{n+1} \otimes S)$$

Some properties of the Dirac operators $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are: both are elliptic G-invariant differential operator. As the Rimannian metric of G/K is complete, $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{D}}^2$ are essentially selfadjoint in $L^2(G/K, V_{n+1} \otimes S)$ [W], that is, the minimal extension is the unique selfadjoint closed extension over the set of smooth compactly supported funtions. Thus, we consider $\tilde{\mathbf{D}}$ equal to this extension which coincides with the maximal one [A]. The eigenvalues of $\tilde{\mathbf{D}}$ are defined as the eigenvalues of the unique selfadjoint extension.

The following proposition is a corollary to proposition 2.8.

PROPOSITION 3.1.

If α is an eigenvalue of $\tilde{\mathbf{D}}$, then the α-eigenspace $\mathrm{W}_{\alpha}(\tilde{\mathbf{D}})$ is irreducible and it is a proper subspace of the α-eigenspace $\mathrm{W}_{\alpha}(\mathbf{D})$ of $\mathbf{D}$. The eigenvalues of $\tilde{\mathbf{D}}$ are $\alpha \in \mathbf{R}$ such that $\alpha^2 = \frac{1}{8}(n+2)^2 - \lambda^2$ with λ integer and $0 < |\lambda| \le n+1$.

Proof. For $G = SL(2,R)$ The Parthasarathy equality [A-S] is

$$\begin{aligned} \mathbf{D}^2 &= -\Omega + \frac{(n+1)^2 - 1}{8}\, Id \\ \tilde{\mathbf{D}}^2 &= -\tilde{\Omega} + \frac{(n+1)^2 - 1}{8}\, Id \end{aligned} \tag{3.3}$$

If α is a non-zero eigenvalue of $\tilde{\mathbf{D}}$,

$$\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2) = \mathrm{W}_{\alpha}(\tilde{\mathbf{D}}) \oplus \mathrm{W}_{-\alpha}(\tilde{\mathbf{D}}) \tag{3.4}$$

(cf [G-V]). Because of (3.3), the left hand side of (3.4) is the $-\alpha^2 + (n+1)^2 - 1 = \frac{1}{8}(\lambda^2 - 1)$ eigenspace of the Casimir operator. Now, since $S = V_{-1} \oplus V_1$,

$$\mathrm{L}^2(G/K, V_{n+1} \otimes S) = \mathrm{L}^2(G/K, V_n) \oplus \mathrm{L}^2(G/K, V_{n+2})$$

Hence proposition 2.8 implies that $0 \leq \lambda \leq n+1$ and

$$\alpha^2 = \frac{(n+1)^2 - \lambda^2}{8}$$

Moreover,

$$\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2) = A_\lambda^n \cap L^2(G/K, V_n) \oplus A_\lambda^{n+1} \cap L^2(G/K, V_{n+2})$$

Thus, $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$ is equal to the sum of two copies of the discrete series $H_{\lambda\delta}$. Since, $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is isomorphic to $H_{\lambda\delta}$ we get that $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is properly contained in $\mathrm{W}_\alpha(\mathbf{D})$. $\square$

Corollary 3.2.
(τ_n, V_n) and (τ_{n+2}, V_{n+2}) are K-types of $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ for every non-zero eigenvalue α of $\tilde{\mathbf{D}}$. For the case $\alpha = 0$, (τ_{n+2}, V_{n+2}) is contained in $\mathrm{Ker}\tilde{\mathbf{D}}$ and (τ_n, V_n) is not.

§4. *Szegö kernels associated to the eigenspaces of* $\tilde{\mathbf{D}}$

In [K-W] Knapp and Wallach gave an integral operator to explicitly obtain a discrete serie as the image of a nonunitary principal serie when the discrete serie is realized as the kernel of Schmid operator. In §3 we have obtained that each eigenspace of the Dirac operator

$$\tilde{\mathbf{D}}\colon L^2(G/K, V_{n+1} \otimes S) \quad \rightarrow \quad L^2(G/K, V_{n+1} \otimes S)$$

is a discrete serie. The purpose of this section is to give an integral operator for each non zero eigenvalue α of $\tilde{\mathbf{D}}$ which will realize the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ as a quotient of an appropiated principal serie. From §3 it is easy to deduce which will be the principal serie corresponding to each eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$, the problem is to obtain the G-invariant integral operator onto $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. Let $G = SL(2, \mathbf{R})$ and K the maximal compact subgroup defined as in (1.2).

Let V_{n+1} be the $n+1$ irreducible representation of K, we assume that $n+1 > 0$. In §3, given an orthonormal base of p_o it was defined the Dirac operator $\tilde{\mathbf{D}}$. If we take $\{X_i\}_{i=1}^2$ an orthonormal base of the complexification p of p_o, another expresion of $\tilde{\mathbf{D}}$ is

$$\tilde{\mathbf{D}} = \sum_{i=1}^{2} (1 \otimes c(X_i))\, \bar{X}_i \tag{4.1}$$

where bar is conjugation with respect to g_o.

One form to obtain the representations $S^{\pm}$ is choosing the left minimal ideals of the Clifford algebra of p,

$$S^+ = \mathbf{C}E_+ \qquad\qquad S^- = \mathbf{C}E_-E_+$$

where the product is Clifford multiplication. In $Cliff(p)$ the following set of relations holds:

$$E_+^2 = E_-^2 = 0 \qquad E_+E_-E_+ = -E_+ \tag{4.2}$$

Hence $S = V_{-1} \oplus V_1$. Thus, we have that

$$V_{n+1} \otimes S = V_n \oplus V_{n+2}$$

The set of K-finite elements of a principal serie $\mathrm{I}_{MAN}^G(\epsilon \otimes e^{\lambda\delta} \otimes 1)$ defined in (2.4), is the representation of K induced by ϵ of M, hence

$$I_M^K(\epsilon) = \bigoplus_{i\in\hat{K}} V_i \otimes \mathrm{Hom}_M(V_i, \epsilon)$$

So, if the representation ϵ occur at V_n and V_{n+2} as M-submodule, then $\epsilon = (-1)^n$. We denote by i_j the inclusions

$$i_j\colon (\epsilon, W_\epsilon) \quad\rightarrow\quad (\tau_j, V_j) \qquad j = n, n+2$$

As W_ϵ and V_j are one dimensional

$$W_\epsilon = \mathbf{C}w \qquad V_j = \mathbf{C}\, v \otimes u$$

where $w \in W_\epsilon$, $v \in V_{n+1}$ and $u \in S^{\pm}$.

Then the inclusions i_j are determined by the constants a_j such that

$$i_j(w) = a_j\, v \otimes u \qquad \text{where } u = \begin{cases} E_+ & j = n \\ E_-E_+ & j = n+2 \end{cases} \tag{4.3}$$

If $sg\,\alpha$ is the sign of the real number α, fix

$$a_n = \left(\frac{\lambda + n + 1}{-\lambda + n + 1}\right)^{\frac{1}{2}} sg\,\alpha \qquad \text{con } 0 \neq \lambda \in \mathbf{Z}, |\lambda| \leq n$$

$$a_{n+2} = 1$$

Let $G = KAN$ be the Iwasawa decomposition of G. According to this decomposition we write an element of G by

$$x = \kappa(x)e^{H(x)}n(x)$$

Let $S(x,t)$ be the function on $G \times K$ defined by

$$S(x,t) = e^{(\lambda-1)\delta H(x^{-1}t)} \left(\tau_n(\kappa(x^{-1}t))i_n + \tau_{n+2}(\kappa(x^{-1}t))i_{n+2}\right) \tag{4.4}$$

Let $\tau = \tau_n + \tau_{n+2}$ on $V_n \oplus V_{n+2}$, so (4.4) implies

$$S(xk,t) = \tau(k)^{-1}S(x,t) \qquad \text{for all } k \in K \tag{4.5}$$

We will call $S(x,t)$ the Szegö kernel associated to the parameters $(\lambda, n+1)$. If $f \in \mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$, the Szegö map associated to the parameters $(\lambda, n+1)$ is

$$\begin{aligned} S(f)(x) &= \int_K S(x,t)\, f(t)\, dt \\ &= \int_K e^{(\lambda-1)\delta H(x^{-1}t)}\tau(\kappa(x^{-1}t))(i_n + i_{n+2})\, f(t)\, dt \end{aligned} \tag{4.6}$$

The equation (4.5) ensure that the image of the Szegö map is in $C^\infty(G/K, V_n \oplus V_{n+2})$.

Let $\tilde{\mathbf{D}}$ defined as in §3

PROPOSITION 4.1.

Given $n \in \mathbf{Z}$, α a non zero eigenvalue of $\tilde{\mathbf{D}}$, and λ a negative integer which satisfies the equality

$$\alpha = \tfrac{1}{8}\left(-\lambda^2 + (n+1)^2\right)^{\frac{1}{2}} \, sg\, \alpha$$

Then, the Szegö map of parameters $(\lambda, n+1)$ is a G-invariant operator onto the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$.

Before proving this result we will see that Szegö map is not the zero map. Let $f \in C^\infty(K/M, W_\epsilon)$ where $\epsilon = (-1)^n$, given by

$$f(k) = i^{-1}\, \tau_n(k)^{-1}\, i_n w$$

Extend f to G so that $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$.

$$\begin{aligned}(S(f)(1), i_n w) &= \int_K \left(\tau(t)(i_n + i_{n+2})\,(i_n^{-1}\, \tau_n(t)^{-1}\, i_n w), i_n w\right)\, dt \\ &= \int_K \left(i_n w + \tau_{n+2}(t) i_{n+2}\,(i^{-1}\, \tau_n(t)^{-1}\, i_n w), i_n w\right)\, dt \\ &= \int_K \|i_n w\|^2\, dt \\ &\neq 0\end{aligned}$$

because $\tau_{n+2}(t) i_{n+2}\,(i^{-1}\, \tau_n(t)^{-1}\, i_n w) \in V_{n+2}$ which is orthogonal to V_n.

To see that the Szegö map is G-invariant we need next lemma

Lemma 4.2.

Let S be the Szegö map with parameters $(\lambda, n{+}1)$. If $f \in \mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ then

$$S(f)(x) = \int_K \tau(t)(i_n + i_{n+2})\, f(xt)\, dt$$

Proof of Lemma 4.2. Using the change of variable

$$\int_K h(k)\, dk = \int_K h(\kappa(x^{-1}t)) e^{-2\delta H(x^{-1}t)}\, dt$$

for $h(k) = \tau(k)(i_n + i_{n+2})\, f(xk)$ the following equality holds

$$\int_K \tau(k)(i_n + i_{n+2})\, f(xk) dk =$$

$$= \int_K \tau(\kappa(x^{-1}t)) e^{-2\delta H(x^{-1}t)} (i_n + i_{n+2})\, f(x\kappa(x^{-1}t))\, dt$$

As A normalize N,

$$\begin{aligned} x^{-1}t &= \kappa(x^{-1}t)e^{H(x^{-1}t)}n(x^{-1}t) \\ x\kappa(x^{-1}t) &= tn(x^{-1}t)^{-1}e^{-H(x^{-1}t)} \\ &= te^{-H(x^{-1}t)}n' \qquad \text{with } n' \in N \end{aligned}$$

So, $f\left(x\kappa(x^{-1}t)\right) = f\left(te^{-H(x^{-1}t)}n'\right) = e^{(\lambda+1)\delta H(x^{-1}t)}f(t)$. And

$$\begin{aligned} \int_K \tau(k)(i_n + i_{n+2})\, f(xk)dk &= \int_K \tau(\kappa(x^{-1}t))e^{(\lambda-1)\delta H(x^{-1}t)}(i_n + i_{n+2})\, f(t)\, dt \\ &= \int_K S(x,t)\, f(t)\, dt \qquad \square \end{aligned}$$

Proof of the Proposition 4.1. By the lemma 4.2 the Szegö map is G-equivariant for left regular actions. As $\tilde{\mathbf{D}}$ also commute with the action of G, it is enough to see that if $f \in \mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$

$$\tilde{\mathbf{D}}(Sf)(1) = \alpha Sf(1)$$

If $f \in \mathrm{I}_{MAN}^G\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$, the image of f is in $W_\epsilon = \mathbf{C}w$ with $\epsilon = (-1)^n$, then $f(t) = h(t)w$ with h a complex valued function. So,

$$\begin{aligned} Sf(x) &= \int_K S(x,t)wh(t)\, dt \\ \tilde{\mathbf{D}}Sf(1) &= \int_K \tilde{\mathbf{D}}(S(x,t)w)_{x=1} h(t)\, dt \end{aligned}$$

from which we only need prove that

$$\begin{aligned} D(S(x,t)w)_{x=1} &= \alpha S(1,t)w \\ &= \alpha\, \tau(t)(i_n w + i_{n+2}w) \end{aligned}$$

Let X_1, X_2 be an orthonormal base of p. Then,

$$\begin{aligned} &\tilde{\mathbf{D}}(S(x,t)w)_{x=1} = \\ &= (I \otimes c)\left(\sum_{i=1}^{2}(X_i S(x,t)w)_{x=1} \otimes \bar{X}_i\right) \\ &= (I \otimes c)\left(\sum_{i=1}^{2} \frac{d}{du}\bigg|_{u=0} e^{(\lambda-1)\delta H(\exp(-uX_i)t)}\tau(\kappa(\exp(-uX_i)t))\right)(i_n + i_{n+2})w \otimes \bar{X}_i \\ &= (I \otimes c)\left(\sum_{i=1}^{2} \frac{d}{du}\bigg|_{u=0} e^{(\lambda-1)\delta H(exp(-u\mathrm{Ad}(t^{-1})X_i)}\tau(\kappa(t\exp(-uAd(t^{-1})X_i))\right) \\ &\qquad\qquad (i_n + i_{n+2})w \otimes \bar{X}_i \\ &= (I \otimes c)\left(\tau(t) \otimes Ad(t)\sum_{i=1}^{2}(Ad(t^{-1})X_i)S(1,1)w \otimes \overline{Ad(t^{-1})X_i}\right) \end{aligned}$$

As $\{Ad(t^{-1})X_i\}_{i=1,2}$ is another orthonormal base of p, and

$$\tau(t)(I\otimes c)=(I\otimes c)(\tau(t)\otimes Ad(t))$$

then

$$\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\tau(t)\tilde{\mathbf{D}}(S(x,1)w)_{x=1}$$

So we must prove

$$\begin{aligned}\tilde{\mathbf{D}}(S(x,1)w)_{x=1}&=\alpha S(1,1)w\\&=\alpha(i_n+i_{n+2})w\end{aligned}$$

Let $\frac{1}{2}E_-, \frac{1}{2}E_+$ be the orthonormal base of p given in §1, then

$$\begin{aligned}&\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\\&=(I\otimes c)\left(\left.\frac{d}{du}\right|_{u=0}e^{(\lambda-1)\delta H(\exp(-u\frac{1}{2}E_-))}\tau(\kappa(\exp(-u\tfrac{1}{2}E_-)))(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+\right.\\&\quad\left.+\left.\frac{d}{du}\right|_{u=0}e^{(\lambda-1)\delta H(\exp(-u\frac{1}{2}E_+))}\tau(\kappa(\exp(-u\tfrac{1}{2}E_+)))(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+\right)\end{aligned}$$

By (1.7)

$$\begin{aligned}\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=(I\otimes c)\Bigg(&-(\lambda-1)\delta\tfrac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-(\lambda-1)\delta\tfrac{1}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-\tau\left(\tfrac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_+-\\&-\tau\left(-\tfrac{1}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)(i_n+i_{n+2})w\otimes\tfrac{1}{2}E_-\Bigg)\end{aligned}$$

By (4.2) and (4.3) applying $I\otimes c$, the following holds

$$c(\tfrac{1}{2}E_+)i_nw=c(\tfrac{1}{2}E_-)i_{n+2}w=0$$

and by (4.4)

$$\begin{aligned}c(\tfrac{1}{2}E_-)i_nw&=\tfrac{1}{2}a_n\,i_{n+2}w\\c(\tfrac{1}{2}E_+)i_{n+2}w&=-\tfrac{1}{2}\tfrac{1}{a_n}i_w\end{aligned}$$

So that

$$\begin{aligned}&\tilde{\mathbf{D}}(S(x,t)w)_{x=1}=\\&=-\tfrac{1}{8}(-\lambda+1)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}(-\lambda+1)a_ni_{n+2}w+\tfrac{1}{8}(n+2)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}na_ni_{n+2}w\\&=\tfrac{1}{8}(\lambda+n+1)\tfrac{1}{a_n}i_nw+\tfrac{1}{8}(-\lambda+n+1)a_ni_{n+2}w\end{aligned}$$

because

$$\delta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 1$$

$$\tau_j\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} v = jv \qquad \text{si } v \in V_{j\delta} \qquad j = n, n+2$$

The coefficients of $i_n w$ and $i_{n+2} w$ are

$$\begin{aligned}\tfrac{1}{8}(\lambda+n+1)\tfrac{1}{a_n} &= \tfrac{1}{8}(\lambda+n+1)\left(\frac{-\lambda+n+1}{\lambda+n+1}\right)^{\frac{1}{2}} sg\,\alpha\\ &= \tfrac{1}{8}\left(-\lambda^2+(n+1)^2\right)^{\frac{1}{2}} sg\,\alpha\\ &= \alpha\end{aligned}$$

$$\begin{aligned}\tfrac{1}{8}(-\lambda+n+1)a_n &= \tfrac{1}{8}\left(-\lambda^2+(n+1)^2\right)^{\frac{1}{2}} sg\,\alpha\\ &= \alpha\end{aligned}$$

That is,

$$\tilde{\mathbf{D}}(S(x,1)w)_{x=1} = \alpha\, S(1,1)w$$

Now, we will prove that the Sezgö map of parameters $(\lambda, n+1)$ for negative λ maps onto $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. We know by proposition 3.1 that $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is irreducible. As S is non zero, if $\mathrm{Im}(S)$ is square integrable, then $\mathrm{Im}(S) = \mathrm{W}_\alpha(\tilde{\mathbf{D}})$. $\mathrm{Im}(S)$ is a subset of the eigenspace $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ of the Dirac operator $\tilde{\mathbf{D}}$. But $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$ is a subset of $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$. According with the notation of §2, as $\tilde{\mathbf{D}}^2$ differ with the Casimir operator Ω by a constant, $\mathrm{W}_{\alpha^2}(\tilde{\mathbf{D}}^2)$ is isomorphic to $A^n_\lambda \oplus A^{n+2}_\lambda$. But the only quotient of $\mathrm{I}^G_{MAN}\left((-1)^n \otimes e^{\lambda\delta} \otimes 1\right)$ isomorphic to a subspace of $A^n_\lambda \oplus A^{n+2}_\lambda$ is infinitesimally equivalent to a discrete serie. Let $\phi \in \mathrm{Im}(S)$ in a non zero K-type, as the action of this K-type is one and the set of K-finite elements of the square integrable function space is a subset of the K-finite elements of the C^∞, then ϕ is square integrable. So $\mathrm{Im}(S)$ is a subset of $\mathrm{W}_\alpha(\tilde{\mathbf{D}})$. The irreducibility concludes the proof. $\square$

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COMPARISON OF TWO WEAK VERSIONS OF THE ORLICZ SPACES

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Abstract: In this work two versions of weak Orlicz spaces that appear in the literature, $\mathcal{M}_A$ and M_A, are analyzed. One of those include the weak Lebesgue spaces for $1 \leq p < \infty$, while the other version gives these spaces only for $p > 1$, resulting the stronger space L^1 in the extrem case $p = 1$. Necessary and sufficient conditions about the growth function A in order that both spaces coincide are given. Moreover we prove that these same conditions characterize the normability of the $\mathcal{M}_A$ space.

1.INTRODUCTION.

We shall denote by M_A the weak Orlicz space associated to A, defined as in the work of O'Neil, [O], where he makes use of this kind of functions to obtain a generalization of the Hardy-Littlewood-Sobolev's theorem on fractional integration into the context of Orlicz spaces. This version of weak Orlicz spaces generalizes the weak L^p spaces, L^p_*, but only for $p > 1$. In fact the class M_A for A the identity function gives a proper subspace of L^1_*.

Our aim in this work is to present an alternative definition of a weak Orlicz space associated to the function A, denoted by $\mathcal{M}_A$, in order to include all L^p_* for $1 \leq p < \infty$. In this way our spaces $\mathcal{M}_A$ give L^1_* for A the identity function and they coincide with M_A for $A(t) = t^p$, $p > 1$. Moreover we shall prove that both spaces are exactly the same as long as A keeps a "little bit away" from the identity. In fact we establish in theorem (4.8) the necessary and sufficient conditions on A to guarantee the equality $M_A = \mathcal{M}_A$.

We would like to point out that the spaces M_A are easier to handle since they are defined in terms of a norm while in turn, $\mathcal{M}_A$ is given by means of a quantity which is not necessarily a norm. It is well known that the weak L^p spaces are

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normable for $p > 1$ while L^1_* is not. Following this line we shall give in theorem (4.11) the necessary and sufficient conditions on A for $\mathcal{M}_A$ to be normable.
As a last remark we may say that the usefulness of one version or the other it would depend on the type of problem we are dealing with. On one side the spaces M_A seem to be the appropriate ones when generalizing the Hardy- Littlewood-Sobolev's theorem, while on the other side the spaces $\mathcal{M}_A$ would fit better for a theorem on interpolation of operators for example.

2.THE ORLICZ SPACES.

(2.1)Definition: *Along this work, for a Young function A we shall mean a non-negative, convex and non decreasing function defined on $[0,\infty]$ with $A(0) = 0$, $A(\infty) = \infty$ and such that it is neither identically zero nor identically infinity. We notice that A may have an jump at some $x_1 > 0$, but in this case $\lim_{x\to x_1^-} A(x) = \infty$ and $A(x) = \infty$ for $x \geq x_1$. Under these assumptions the inverse function A^{-1} is well defined and it is also increasing and continuous.*

We introduce now some notions related to the role of growth of non-negative functions as above.

(2.2)Definitions: *We shall say that two non-negative functions are equivalent if and only if their ratio is bounded above and bellow by two positive constants.*
A non negative function A defined on $\mathbb{R}^+$ is of lower type p (upper type p) if $A(st) \leq Cs^pA(t)$ for any $s \leq 1$ ($s \geq 1$).

We notice that lower and upper types are preserved by equivalence of functions and also for any function we may choose another for which the definition of type is satisfied with $C = 1$. In particular A is of lower type zero if and only if is equivalent to a non decreasing function.

(2.3)Definition: *For a Young function A we define the Orlicz space $L_A = L_A(X)$ as the linear space of those measurable functions acting on the measure space (X,μ) for which there is a finite number $K > 0$ such that*

$$\int_X A\left(\frac{|f(x)|}{K}\right) d\mu \leq 1 \quad .$$

The infimun of such K is a norm which will be denoted by $\|f\|_A$.

3.WEAK ORLICZ SPACES.

For a complex or real valued and measurable f, defined on a measure space (X,μ), we will denote by $\mu_f(t)$ the distribution function of f given by

$$\mu_f(t) = \mu(\{x : |f(x)| > t\}).$$

Then for $t \in [0,\infty)$, $\mu_f(t)$ is a non increasing function taking non-negative values. Therefore we may define its inverse f^* by

$$f^*(s) = \inf\{t : \mu_f(t) \leq s\}$$

where $s \geq 0$. This function f^* usually called the non-increasing rearrangement of f, has the of property being equimeasurable with f in the sense that they share the distribution function.

By f^{**} we shall denote the average of f^* over the interval $[0, x]$, that is

$$f^{**}(x) = \begin{cases} \frac{1}{x}\int_0^x f^*(t)dt & x > 0 \\ f^*(0) & x = 0. \end{cases}$$

Given a Young function A, it is possible to define a class of functions M_A in terms of the size of the f^{**}, wider than the Orlicz space L_A. The following definition of a version of weak Orlicz spaces is taken from the work of O'Neil [O], where the author used this class in connection with the boundedness of convolution operators on strong Orlicz spaces.

(3.1)Definition: *For a Young function A we will say that f defined on (X, μ) belongs to M_A if and only if there exists a real number λ large enough so that for $x > 0$*

$$f^{*}(x) \leq \lambda A^{-1}\left(\frac{1}{x}\right).$$

We define $\|f\|_{M_A}$ as the infimum of such λ. Therefore

$$\|f\|_{M_A} = \sup_{s>0} \frac{f^{**}(s)}{A^{-1}(1/s)} \quad .$$

In [O], O'Neil shows that the quantity $\|f\|_{M_A}$ is indeed a norm wich makes M_A a Banach space.

For $A(t) = t^p$ with $p > 1$, it is well known that M_A agrees with the space L^p_* or weak L^p, defined as those functions satisfying

$$\|f\|_p^* = \sup_{t>0} t^{1/p} f^*(t) < \infty$$

since for $1 < p \leq \infty$ both quantities $\|f\|_p^*$ and $\|f\|_{M_{t^p}}$, are in fact equivalent. Moreover it is known that for $p \geq 1$ the Lebesgue spaces $L^p(I\!R^n)$ are proper subspaces of $L^p_*(I\!R^n)$ (see for example [SW]. However the situation changes for $A(t) = t$, that is for $p = 1$. In this case the O'Neil version of weak L^1 is no longer the same that L^1_*; it rather coincides with the strong L^1 space. In fact if $A(t) = t$, $f \in M_A$ if and only if for some λ

$$f^{**}(x) \leq \lambda \frac{1}{x}$$

which means

$$\int_0^x f^*(t)dt \leq \lambda.$$

This is equivalent to f^* being integrable, that is, f in L^1.

At this point it appears in a natural way another version of weak Orlicz spaces as to include all the L^p_* spaces for $p \geq 1$.

(3.2)Definition: *We will say that a μ-measurable function f defined on X belongs to the weak Orlicz space $\mathcal{M}_A$ if and only if there is a constant C so that for $t > 0$*

$$A(t)\mu(\{x : |f(x)| > t\}) \leq C.$$

This definition implies that the quantity

$$\|f\|_{\mathcal{M}_A} = \inf\left\{\lambda > 0/ \sup_{t>0} \mu_f(\lambda t)A(t) \leq 1\right\}$$

is finite. Moreover the following properties hold

a) $\|cf\|_{\mathcal{M}_A} = |c| \quad \|f\|_{\mathcal{M}_A}$

b) $\|f+g\|_{\mathcal{M}_A} \leq 2(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})$

We notice that the factor 2 in b) does not allow to say that $\| \ \|_{\mathcal{M}_A}$ is a norm.

The proof of a) is immediate. On the other hand we observe that b) will follow if we are able to prove the inequality

$$\mu\left(\left\{\frac{|f(x)+g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t) \leq 1$$

for all $t > 0$. But

$$\mu\left(\left\{\frac{|f(x)+g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t) \leq \mu\left(\left\{\frac{|f(x)|+|g(x)|}{c(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t)$$

$$= \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}} \frac{\|f\|_{\mathcal{M}_A}}{(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} + \frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}} \frac{\|g\|_{\mathcal{M}_A}}{(\|f\|_{\mathcal{M}_A} + \|g\|_{\mathcal{M}_A})} > t\right\}\right) A(t)$$

$$= \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}}\theta_1 + \frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}}\theta_2 > t\right\}\right) A(t)$$

$$\leq \mu\left(\left\{\frac{|f(x)|}{c\|f\|_{\mathcal{M}_A}} > t\right\}\right) A(t) + \mu\left(\left\{\frac{|g(x)|}{c\|g\|_{\mathcal{M}_A}} > t\right\}\right) A(t)$$

since $\theta_1 + \theta_2 = 1$. The convexity of A implies $A(st) \leq sA(t)$ for $0 \leq s \leq 1$. Then, if $c \geq 1$, we can bound the above sum by

$$\mu\left(\left\{\frac{|f(x)|}{\|f\|_{\mathcal{M}_A}} > ct\right\}\right) \frac{A(ct)}{c} + \mu\left(\left\{\frac{|g(x)|}{\|g\|_{\mathcal{M}_A}} > ct\right\}\right) \frac{A(ct)}{c}$$

$$\leq \frac{1}{c} + \frac{1}{c}$$

which in turn is bounded by one as long as we take $c \geq 2$.■

4.RELATIONSHIP BETWEEN THE TWO DEFINITIONS.

As we already apointed out $L^1(\mathbb{R}^n)$ is a proper subespace of $L^1_*(\mathbb{R}^n)$. Consequently the spaces M_A and $\mathcal{M}_A$ are not always the same. Indeed when A is the identity function there are functions on $\mathbb{R}^n$ for which

$$\mu(\{x : |f(x)| > t\}) \leq \frac{C}{t}$$

for some finite constant C, even though they are not integrable. Such is the case of for example $f(x) = \frac{1}{|x|^n}$. However M_A is always a subspace of $\mathcal{M}_A$. In fact we have the following result.

(4.1)Lemma: *For any Young function A, we have*

$$M_A \subset \mathcal{M}_A \quad .$$

Moreover we have the inequality

$$\|f\|_{\mathcal{M}_A} \leq \|f\|_{M_A} \quad .$$

First we will find an expression for $\|f\|_{\mathcal{M}_A}$ in terms of the non increasing rearrengement of f. From this lemma (4.1) will be an obvious consequence.

(4.2)Lemma: *If f is a measurable function and by $\mu_f(t)$ and $f^*(s)$ we denote its distribution and rearrengement function, then the following identity holds*

$$\sup_{t>0} \mu_f(\lambda t) A(t) = \sup_{s>0} sA\left(\frac{f^*(s)}{\lambda}\right),$$

and hence

$$\|f\|_{\mathcal{M}_A} = \sup_{s>0} \frac{f^*(s)}{A^{-1}(1/s)}.$$

Proof:

First, let us assume that f is a non-negative simple function. Then it may be written as

$$f = \sum_{j=1}^{n} c_j \chi_{E_j},$$

where $\mu(E_j) > 0, E_j \cap E_k = \emptyset$ if $j \neq k$ and $c_1 > c_2 > ... > c_n > 0$. Set $d_j = \mu(E_1) + ... + \mu(E_j)$, $1 \leq j \leq n$, and let us define $d_0 = 0, c_{n+1} = 0$. Then, if we set $\mu_f(t) = |\{x : |f(x)| > t\}|$, this function and its inverse f^* are given by

$$\mu_f(\lambda t) = \begin{cases} d_j & \frac{c_{j+1}}{\lambda} \leq t < \frac{c_j}{\lambda} \\ 0 & t \geq c_1 \end{cases}$$

$$f^*(s) = \begin{cases} c_j & d_{j-1} \leq s < d_j \\ 0 & s \geq d_n. \end{cases}$$

Therefore, using that A is non-decreasing we have

$$\sup_{t>0} A(t)\mu_f(\lambda t) = \sup_{j>0} A\left(\frac{c_j}{\lambda}\right) d_j = \sup_{s>0} A\left(\frac{f^*(s)}{\lambda}\right) s.$$

Now, for a general measurable function f, we can find a non-decreasing sequence of non-negative simple functions f_n such that $\lim_{n\to\infty} f_n(x) = |f(x)|$, for each x in the domain of f. Therefore, for each $t > 0$, the sequence $\{\mu_n(t)\}$ is non-decreasing and $\lim_{n\to\infty} \mu_n(t) = \mu(t)$, where μ_n and μ denote the distribution functions of f_n and f respectively. Likewise, for each $s > 0$ we also have that $f_n^*(s)$ increases to $f^*(s)$ and the first claim of the lemma follows immediately.

As for the second equality we just notice that

$$\|f\|_{\mathcal{M}_A} = \inf\left\{\lambda > 0/ \sup_{t>0} \mu_f(\lambda t)A(t) \leq 1\right\}$$

$$(4.3) \qquad = \inf\left\{\lambda > 0/ \sup_{s>0} sA\left(\frac{f^*(s)}{\lambda}\right) \leq 1\right\}$$

$$= \sup_{s>0} \frac{f^*(s)}{A^{-1}(1/s)} \quad .$$

where in the last equality we have used that $sA(\frac{f^*(s)}{\lambda}) \leq 1$ is equivalent to $f^*(s) \leq \lambda A^{-1}(\frac{1}{s})$.■

Proof of lemma (4.1):

From of definition of f^{**} it follows that for any $s > 0$ we have $f^*(s) \leq f^{**}(s)$. This observation together with lemma (4.2) give the desired conclusion.■

As we shall see the difference between the spaces M_A and $\mathcal{M}_A$ may appear in other cases besides $A(t) = t$. In fact if for $x > 0$ we denote by $\log^+ x$ the maximum between $\log t$ and zero and for $x \in I\!R^n$ we take the function

$$f(x) = \frac{9e^2}{\omega_n |x|^n (3 + \log^+(\frac{1}{\omega_n |x|^n}))^2}$$

then f belongs to the space $\mathcal{M}_A$ for $A(t)$ such that $A^{-1}(t) = 9e^2 t(3 + \log^+ t)^{-2}$. First, $A(t)$ is a Young function because we have chosen the constants in such a way that A^{-1} is increasing, continuous and concave on $[0, \infty]$. Also,it is not hard to check that $A(t)$ behaves at infinity like $t(\log^+ t)^2$. Second, for any increasing function A^{-1}, the function defined on $\mathbb{R}^n$ by $f(x) = A^{-1}\left(\frac{1}{\omega_n |x|^n}\right)$ is such that $f^*(s) = A^{-1}(1/s)$ proving our assertion that $f \in \mathcal{M}_A$. Finally let us see that f is not in M_A. If it were, there would be a constant $\lambda > 0$ such that

$$\frac{1}{s}\int_0^s A^{-1}(1/t)dt \leq \lambda A^{-1}(1/s).$$

But then, for any $s < 1$ we have

$$\int_0^s \frac{9e^2}{t(3+\log(1/t))^2}dt = -9e^2\int_{\infty}^{-\log s} \frac{1}{(3+u)^2}du$$

$$= 9e^2(3-\log s)^{-1}.$$

This together with our assumption would lead to

$$9e^2(3+\log(1/s))^{-1} < \lambda 9e^2(3+\log(1/s))^{-2}$$

for some $\lambda > 0$. But this impossible because it would imply that $-\log s$ is a bounded function on (0,1).
This example shows that when $X = \mathbb{R}^n$ and μ is the Lebesgue measure there are other Young functions different from $A(t) = t$ for which the space M_A is strictly contained in $\mathcal{M}_A$. In our next step we will characterize all the Young functions for which both spaces are exactly the same. In what follows we shall restrict ourselves to the case of $X = \mathbb{R}^n$ with μ the Lebesgue measure. Nevertheless the main results contained in theorems (4.8) and (4.11) could also be derived working in more general measure spaces.

We start by giving two real functions lemmas; the first can be found in [M], and the second is an stronger version of a result proved by Viviani in [V]. This last result will be an essential tool in looking for necessary and sufficient conditions on A to ensure that $M_A = \mathcal{M}_A$.

(4.4)Lemma: *Let $h(t)$ be a non negative and non decreasing function on $[0,j]$ for which there exists a constant D such that for $0 \leq s \leq j/20$, $\int_0^s h(t)dt \leq Dsh(s)$. Then if $1 \leq r < D/(D-1)$,*

$$\int_0^j [h(t)]^r dt \leq \frac{(20)^r j^{1-r} D}{D - r(D-1)}\left[\int_0^j h(t)dt\right]^r. \tag{4.5}$$

(4.6)Lemma: *Let η be a non negative function such that $\frac{\eta(t)}{t}$ is non increasing. Then $\eta(t)$ is equivalent to $\tilde{\eta}(t) = \int_0^t \frac{\eta(s)}{s} ds$ if and only if η has a positive lower type.*

Proof:

Since $\frac{\eta(t)}{t}$ is non increasing the inequality $\eta(t) \leq \int_0^t \frac{\eta(s)}{s} ds$ is always true no matter what the lower type of η is. Also, the fact that the inequality $\int_0^t \frac{\eta(s)}{s} ds \leq C\eta(t)$ holds whenever η is of positive lower type is proved in [V]. Conversely the equivalence between η and $\tilde{\eta}$ implies that $\int_0^t h(s)ds \leq Cth(t)$, for $h(t) = \frac{\eta(t)}{t}$ and $\forall t > 0$. This allows us to use (4.5) from Muckenhoupt lemma for any finite interval in order to obtain that η is of positive lower type. In fact, if $r > 1$, as in the conclusion of the previous lemma, $0 < u \leq 1$ and $s > 0$ we have

$$ush^r(us) \leq \int_0^{us} h^r(t)dt \leq \int_0^s h^r(t)dt \leq Cs^{1-r}\left[\int_0^s h(t)dt\right]^r \leq Csh^r(s)$$

Therefore

$$h(us) \leq C\left(\frac{1}{u}\right)^{\frac{1}{r}} h(s)$$

$$\frac{\eta(us)}{us} \leq C\left(\frac{1}{u}\right)^{\frac{1}{r}} \frac{\eta(s)}{s}.$$

Since $r > 1$ we arrive to the desired conclusion.∎

Now we make an useful remark on the relationship between the types of a Young function and its inverse.

(4.7)Lemma: *Let A be a Young function. Then A has a lower type m if and only if A^{-1} has an upper type $1/m$.*

Proof:

The Young function A has a lower type m if and only if there is a constant $C > 0$ such that

$$A(st) \leq Ct^m A(s) \quad \text{for any} \quad 0 < t \leq 1 \ .$$

Now taking a pair $t \leq s$ the latter inequality can be written

$$A(t) = A(s\frac{t}{s}) \leq C\left(\frac{t}{s}\right)^m A(s)$$

which is equivalent to say

$$\frac{A(t)}{t^m} \leq C\frac{A(s)}{s^m}$$

for any $t \leq s$. Setting $\alpha = A(t)$ and $\beta = A(s)$, by the continuity of A the above inequality can be written

$$\frac{\alpha}{[A^{-1}(\alpha)]^m} \leq C\frac{\beta}{[A^{-1}(\beta)]^m} \quad .$$

Since A is non decreasing we get that the inequality

$$\frac{A^{-1}(\beta)}{\beta^{\frac{1}{m}}} \leq C\frac{A^{-1}(\alpha)}{\alpha^{\frac{1}{m}}}$$

holds for any $\alpha \leq \beta$, but this is to say that A^{-1} has an upper type $\frac{1}{m}$.■

Now we are in position to state and prove the anounced characterization.

(4.8)Theorem: *Let A be a Young function. Then the following statements are equivalent*

i) $M_A = \mathcal{M}_A$,

ii) $\frac{1}{s}\int_0^s A^{-1}(1/t)dt$ *is equivalent with* $A^{-1}(1/s)$,

iii) A has a lower type greater than one 1.

Proof:

Let us assume i) is true. Since by (4.1) $M_A \subset \mathcal{M}_A$ always holds, we must obtain ii) from $\mathcal{M}_A \subset M_A$. Take the function $f(x) = A^{-1}\left(\frac{1}{\omega_n|x|^n}\right)$; since it is radial and non increasing it is easy to check that its rearrengement is $f^*(s) = A^{-1}(\frac{1}{s})$ and hence $f \in \mathcal{M}_A$. Now, our hypothesis implies that f belongs also to M_A which means that for some $\lambda > 0$ the inequality

$$\frac{1}{s}\int_0^s A^{-1}\left(\frac{1}{t}\right) = f^{**}(s) \leq \lambda A^{-1}\left(\frac{1}{s}\right)$$

holds for any $s > 0$ giving one of the inequalities in ii). Finally, the other inequality follows using that $A^{-1}(1/t)$ is a non increasing function.

To check that ii) $\Rightarrow$ iii) we set $\eta(t) = tA^{-1}(1/t)$ and we make use of lemma (4.6) to conclude that η has a positive lower type, say a. Therefore we have

$$\eta(ut) = utA^{-1}\left(\frac{1}{ut}\right) \leq Cu^a tA^{-1}\left(\frac{1}{t}\right) \quad (0 < u \leq 1, t > 0 \text{ y } y\ a > 0)$$

which implies

$$A^{-1}\left(\frac{1}{ut}\right) \leq Cu^{a-1}A^{-1}\left(\frac{1}{t}\right) \quad (0 < u \leq 1, t > 0 \text{ and } a > 0)$$

setting $\sigma = \frac{1}{u}$ and $z = \frac{1}{t}$ the above expresion is equivalent to

$$A^{-1}(\sigma z) \leq C\sigma^{1-a}A^{-1}(z) \quad (\sigma \geq 1, z > 0 \text{ y } a > 0)$$

which means that A^{-1} has an upper type less than one. By using now Lemma 4.7 we may conclude that A has lower type greater than one.
In order to prove iii) $\Rightarrow$ ii) we use again lemma (4.7) to conclude that A^{-1} has an upper type, say b, less than one and that, in consequence, the function $\eta(t) = tA^{-1}(1/t)$ has a positive lower type. In fact, if $0 < u \leq 1$ and $t > 0$ we have

$$\eta(ut) = utA^{-1}\left(\frac{1}{ut}\right) \leq Cut\left(\frac{1}{u}\right)^{b} A^{-1}\left(\frac{1}{t}\right) = Cu^{1-b}\eta(t).$$

Since $1 - b > 0$ we may apply lemma (4.6) to get ii).
It remains to prove that ii)$\Rightarrow$i). First we observe that by lemma 4.1 it is enough to check $\mathcal{M}_A \subset M_A$. Let us assume $f \in \mathcal{M}_A$, that is $f^*(s) \leq \lambda A^{-1}\left(\frac{1}{s}\right)$. Then we have

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt \leq \frac{\lambda}{x}\int_0^x A^{-1}\left(\frac{1}{s}\right) ds.$$

But, using ii) we get

$$f^{**}(x) \leq KA^{-1}\left(\frac{1}{x}\right)$$

and hence $f \in M_A$.■

(4.9)Corollary: *If A has a lower type greater than one, then there exists a constant C such that*

$$\|f\|_{M_A} \leq C\|f\|_{\mathcal{M}_A}$$

holds for any $f \in \mathcal{M}_A$ and moreover $\mathcal{M}_A$ is normable.

Proof:

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt$$

$$= \frac{1}{x}\int_0^x \frac{f^*(t)}{A^{-1}(1/t)} A^{-1}(1/t)dt$$

$$\leq C\|f\|_{\mathcal{M}_A}\frac{1}{x}\int_0^x A^{-1}(1/t)dt$$

$$\leq C\|f\|_{\mathcal{M}_A} A^{-1}(1/x),$$

where we have used iii) $\Rightarrow$ ii) from theorem (4.8). Taking supremum over all x, we get

$$\|f\|_{M_A} \leq C\|f\|_{\mathcal{M}_A}.$$

Finally, since by lemma (4.1) the reverse inequality between $\|f\|_{M_A}$ and $\|f\|_{\mathcal{M}_A}$ always holds, our space $\mathcal{M}_A$ is normable so that the proof of the corollary is complete.■

(4.10) Remark: As we have just seen the space $\mathcal{M}_A$ is normable, with the norm $\|.\|_{M_A}$, whenever A has a lower type greater than one. For A a Young function without this property (i.e. A has lower type one and no greater than) we already know that our space $\mathcal{M}_A$ is much bigger than M_A and consequently the quantity $\|.\|_{\mathcal{M}_A}$ is not longer equivalent to the norm $\|.\|_{M_A}$. A natural question then arises: is there a norm on the space $\mathcal{M}_A$ equivalent to the quantity $\|.\|_{\mathcal{M}_A}$?. In other words we would like to know whether or not this spaces $\mathcal{M}_A$ are normable for Young functions A without a lower type greater than one. It is known that the space L^1_* is not normable. Our next result shows that this situation extends to all $\mathcal{M}_A$ with A having a lower type at most one.

(4.11) Theorem: *Let A be a Young function. Then the weak Orlicz space $\mathcal{M}_A$ is normable, with a norm equivalent to $\|.\|_{\mathcal{M}_A}$ if and only if A has a lower type greater than one.*

Proof:

By corollary (4.9) we only have to show that $\mathcal{M}_A$ normable implies that A must have a lower type greater than one. For simplicity we will work out the proof only in the one dimensional case. For higher dimensions it follows the same lines. For given $s > 0$ and $N \in I\!N$ we define the function

$$f(x) = \sum_{k=1}^{N} A^{-1}\left(\frac{1}{2|x - \frac{ks}{N}|}\right) .$$

If we call $f_{k,s}(x) = A^{-1}\left(\frac{1}{2|x-\frac{ks}{N}|}\right)$ it is easy to check that they all belong to $\mathcal{M}_A$ for any $1 \le k \le N$ and $s > 0$ and moreover we have $\|f_{k,s}\|_{\mathcal{M}_A} \le 1$ since all of these functions sheare the same distribution $\frac{1}{A(t)}$. Therefore, if by $\|.\|$ we denote a norm equivalent to the quantity $\|.\|_{\mathcal{M}_A}$, we get

$$\|f\| \le \sum_{k=1}^{N} \|f_{k,s}\| \le C_1 \sum_{k=1}^{N} \|f_{k,s}\|_{\mathcal{M}_A} \le C_1 N .$$

However, elementary computations show that the derivative of f is negative on $[0, s + \frac{s}{N}]$ which implies that $f(x) \le f(0)$ for $x \in [0, s + \frac{s}{N}]$, . Then if we set

$$H_{N,s} = f(0) = A^{-1}(\frac{1}{2}\frac{N}{s}) + A^{-1}(\frac{1}{2}\frac{N}{2s}) + ... + A^{-1}(\frac{1}{2}\frac{1}{s})$$

we obtain

$$1 < \mu_f(H_{N,s})\frac{1}{s} = \mu_f(\lambda_{N,s}t_s)A(t_s)$$

where $t_s = A^{-1}(\frac{1}{s})$ y $\lambda_{N,s} = \frac{H_{N,s}}{A^{-1}(\frac{1}{s})}$. Then

$$\lambda_{N,s} \leq \|f\|_{\mathcal{M}_A} \leq C_2 N \quad .$$

Thus

$$H_{N,s} \leq C_2 N A^{-1}(\frac{1}{s})$$

and

$$H_{N,s} = \sum_{k=1}^{N} A^{-1}(\frac{1}{2}\frac{N}{ks}) \geq \frac{N}{s}\sum_{k=1}^{N}\int_{\frac{s}{N}k}^{\frac{s}{N}(k+1)} A^{-1}(\frac{1}{2u})du.$$

Since A^{-1} is non decreasing we obtain

$$\frac{N}{s}\int_{\frac{s}{N}}^{\frac{s}{N}(N+1)} A^{-1}(1/2u)du \leq C_2 N A^{-1}(1/s) \quad .$$

Letting N go to infinity we get that for any fixed $s > 0$

$$\int_0^s A^{-1}(1/2u)du \leq C_2 s A^{-1}(1/s).$$

Changing variables $v = 2u$ we get

$$\int_0^{2s} A^{-1}(1/v)dv \leq C s A^{-1}(1/s).$$

Finally since A^{-1} is non negative we arrived to the inequality

$$\int_0^s A^{-1}(1/v)dv \leq C s A^{-1}(1/s)$$

which by theorem (4.8), implies that A has a lower type greater than one.∎

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MOLECULAR CHARACTERIZATION OF HARDY-ORLICZ SPACES

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Abstract: We give a molecular characterization of the Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$ (Theorem 2.18), which generalizes similar results for the Hardy spaces $H^p(\mathbb{R}^n)$ for $p \leq 1$. This result is applied to provide a proof of the boundedness of singular integral operators on $H_w(\mathbb{R}^n)$. (Theorem 3.10).

INTRODUCTION. The purpose of this work is to study the Hardy-Orlicz spaces H_w. The usual Hardy spaces H^p can be obtained as particular cases taking $w(t) = t^p$. In [V] Viviani gives an atomic decomposition of H_w. The molecular theory can be found in [GC-RF]. Several authors have used this technique to deal with operators defined on Hardy spaces, see for instance [C], [C-W], [M], [M-S], [T-W].

In this paper we obtain a molecular characterization for H_w with a general w, see section 2, Theorem (2.18). Then, in section 3, we apply this result to study the boundedness of singular integral operators on $H_w(\mathbb{R}^n)$. One of the main difficulties is to define a suitable gauge, that is a notion of molecular "norm", in the context of Orlicz spaces. The one we introduce in (1.41) it is not the same as that considered in the papers above when $w(t) = t^p$. However, in view of Theorem (2.18), they turn out to be equivalent. In the first section we give the notation, definitions and some properties that we shall use in the sequel. We introduce the maximal spaces H_w, the atomic spaces $H^{\rho,q}$, $1 < q \leq \infty$ and the molecular spaces $\mathcal{M}_{(\rho,q,\varepsilon)}, 1 < q \leq \infty,\ \varepsilon > 0$.

1. NOTATION AND DEFINITIONS

Let w be a positive function defined on $\mathbb{R}^+ = \{x \in \mathbb{R}; x > 0\}$. We shall say that w is of lower type l (respectively, upper type l), if there exists a positive constant C such that

$$w(st) \leq Ct^l w(s)$$

for every $0 < t \leq 1$ (respectively, $t \geq 1$). It is easy to see that if w is of positive lower type l, then $\lim_{t \to 0^+} w(t) = 0$, therefore we define $w(0) = 0$.

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We shall say that a positive function w defined on $I\!R^+$ is quasi-increasing (respectively, quasi-decreasing) if there exists a constant C such that

$$w(s) \leq Cw(t)$$

for every $s \leq t$ (respectively $s \geq t$).

We shall understand that two positive functions are equivalent if their quotient is bounded above and below by two positive constants.

Let w be a function of positive lower type l such that $w(s)/s$ is non-increasing. Then the following functions are well defined

$$w^{-1}(s) = \sup\{t : w(t) \leq s\} \quad , \tag{1.1}$$

$$\rho(t) = \frac{t^{-1}}{w^{-1}(t^{-1})} \quad , \tag{1.2}$$

$$\tilde{w}(t) = \int_0^t \frac{w(s)}{s}\, ds \quad , \tag{1.3}$$

$$\tilde{w}^{-1}(s) = \sup\{t : \tilde{w}(t) \leq s\} \quad \text{and} \tag{1.4}$$

$$\bar{\rho}(t) = \frac{t^{-1}}{\tilde{w}^{-1}(t^{-1})} \quad . \tag{1.5}$$

We state the basic properties of these functions, the proofs can be found in [V].
(1.6) The lower type l is less than or equal to one.
(1.7) w is of upper type 1 with constant $C = 1$.
(1.8) w^{-1} is of lower type 1 and of upper type $1/l$.
(1.9) $\tilde{w}$ is a continuous function equivalent to w.
(1.10) $\tilde{w}$ is strictly increasing.
(1.11) $\tilde{w}$ is subadditive.
(1.12) $\tilde{w}(s)/s$ is non-increasing.
(1.13) $\tilde{w}$ is of lower type l and of upper type 1 with constant $C = 1$.
(1.14) $\tilde{w}^{-1}$ coincides with the ordinary inverse function of $\tilde{w}$ and is equivalent to w^{-1}.

(1.15) ρ is a function of upper type $1/l - 1$ equivalent to the non decreasing function
$\bar{\rho}$.
(1.16) $\rho(t)/t^p$ is quasi-decreasing for $p \geq 1/l - 1$.

In order to introduce the atomic spaces $H^{\rho,q}$ and the molecular spaces $\mathcal{M}_{(\rho,q,\varepsilon)}$, $1 < q \leq \infty$, $\varepsilon > 0$, we need the following definition.

(1.17) DEFINITION. *Let w be a function of positive lower type l. Assume that* $\mathbf{b}= \{b_j\}$ *is a sequence of functions in* $L^q(I\!R^n), 1 \leq q \leq \infty$, *and* $\mathbf{c}= \{c_j\}$ *is a sequence of positive constants such that*

$$\sum_j c_j w(||b_j||_q c_j^{-1/q}) = A < \infty. \tag{1.18}$$

We define

$$\Lambda_q(\mathbf{b},\mathbf{c}) = \inf\left\{\lambda > 0 : \sum_j c_j w\left(\frac{||b_j||_q c_j^{-1/q}}{\lambda^{1/l}}\right) \leq 1\right\}. \tag{1.19}$$

We observe that

$$\Lambda_q(\mathbf{b},\mathbf{c}) = 0 \text{ if and only if } b_j \equiv 0 \text{ for every } j. \tag{1.20}$$

If L is the lower type constant of w, then

$$0 \leq \Lambda_q(\mathbf{b},\mathbf{c}) \leq max(LA, 1)\ . \tag{1.21}$$

If we also assume that $w(s)/s$ is non-increasing, we have

$$0 \leq \Lambda_q(\mathbf{b},\mathbf{c}) \leq max(LA, A^l) \tag{1.22}$$

and

$$\sum_j c_j w\left(\frac{||b_j||_q c_j^{-1/q}}{\Lambda_q(\mathbf{b},\mathbf{c})^{1/l}}\right) = 1. \tag{1.23}$$

Moreover, arguing in the same way as in the proof of Lemma (4.7) in [V], we can show that if $\alpha_j = ||b_j||_q c_j^{-1/q}/w^{-1}(c_j^{-1})$, then

$$\sum_j \alpha_j \leq C\ (\Lambda_q(\mathbf{b},\mathbf{c}) + 1)^{1/l^2}\ , \tag{1.24}$$

with C independent of $\mathbf{b}$ and $\mathbf{c}$. If $\Lambda_q(\mathbf{b},\mathbf{c}) \geq \beta > 0$, we get

$$\sum_j \alpha_j \leq C_\beta(\Lambda_q(\mathbf{b},\mathbf{c}))^{1/l^2}\ , \tag{1.25}$$

where C_β depends on β but not on $\mathbf{b}$ and $\mathbf{c}$.

REMARK. *In the following we shall assume that*

(1.26) *w is a function of positive lower type l such that $w(s)/s$ is non increasing and*

$\rho(t)$ is defined by (1.2).

Given $G \in I\!N$, we define the G-maximal function of a distribution f on $\mathcal{S}$ by

$$f_G^*(x) = \sup |f(\psi)|,$$

where the supremum is taken over all functions ψ belonging to $C_c^\infty(I\!R^n)$ satisfying $dist(x, supp(\psi)) < |supp(\psi)|$ and

$$\int |\psi(x)|\, dx + |supp(\psi)|^{G+1} \sum_{|\alpha|=G+1} \int |D^\alpha \psi(x)|\, dx = 1.$$

(1.27) DEFINITION. *Let $G \in I\!N$ such that $Gl > 1$.*
We define

$$H_w = H_w(I\!R^n) = \left\{ f \in \mathcal{S}' : \int w(f_G^*(x))\, dx = A < \infty \right\}$$

and we denote

$$||f||_{H_w} = \inf \left\{ \lambda > 0 : \int w\left(\frac{f_G^*(x)}{\lambda^{1/l}}\right) dx \le 1 \right\}.$$

It is easy to verify that if $f \in H_w$, then

(1.28) $\quad 0 \le ||f||_{H_w} \le max(LA, A^l)$,

(1.29) $\quad ||f||_{H_w} = 0$ if and only if $f \equiv 0$ and

(1.30) $\quad \int w\left(\frac{f_G^*(x)}{||f||_{H_w}^{1/l}}\right) dx = 1.$

It is easy to see that H_w is a complete topological vector space with respect to the quasi-distance induced by $||\ ||_{H_w}$. Moreover H_w is continuosly included in $\mathcal{S}'$. Clearly, when $w(t) = t^p$, $0 < p \le 1, w$ satisfies (1.26) with $l = p$ and $H_w(I\!R^n) = H^p(I\!R^n)$.

In this work we shall denote $N = [n(1/l - 1)]$, where $[x]$ stands for the biggest integer less than or equal to x.

(1.31) DEFINITION. *A (ρ, q) atom, $1 < q \le \infty$ is a real valued function a on $I\!R^n$ satisfying:*

(1.32)
$$\int a(x) x^\beta\, dx = 0,$$

for every multi-index $\beta = (\beta_1, \ldots, \beta_n)$ such that $|\beta| = \beta_1 + \ldots + \beta_n \le N$, where $x^\beta = x_1^{\beta_1} \cdot x_2^{\beta_2} \cdot \ldots \cdot x_n^{\beta_n}$, *(1.33) the support of a is contained in a ball B and*

$$(1.34)\qquad \begin{cases} \|a\|_q|B|^{-1/q} \leq [|B|\rho(|B|)]^{-1} & \text{if } q < \infty, \text{ or} \\ \|a\|_\infty \qquad\qquad \leq [|B|\rho(|B|)]^{-1} & \text{if } q = \infty. \end{cases}$$

Clearly, when $w(t) = t^p$, $p \in (0,1]$, we have that $\rho(t) = t^{\frac{1}{p}-1}$ and a (ρ, q) atom is a (p, q) atom in the usual sense.

Let us observe that, in view of (1.24), if $\{b_j\}$ is a sequence of multiples of (ρ, q) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $supp(b_j) \subset B_j$ and (1.18) with $c_j = |B_j|$, then the series $\sum_j b_j$ converges in $\mathcal{S}'$.

(1.35) DEFINITION. *We define $H^{\rho,q} = H^{\rho,q}(I\!R^n)$, $1 < q \leq \infty$, as the linear space of all distributions f on $\mathcal{S}$ which can be represented by*

$$(1.36)\qquad f = \sum_j b_j \qquad in\ \mathcal{S}',$$

where $\{b_j\}$ is a sequence of multiples of (ρ, q) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $supp(b_j) \subset B_j = B(x_j, r_j)$ and (1.18) with $c_j = |B_j|$. We denote $\mathbf{b} = \{b_j\}$, $\mathbf{B} = \{|B_j|\}$ and let

$$\|f\|_{H^{\rho,q}} = \inf \Lambda_q(\mathbf{b}, \mathbf{B}),$$

where $\Lambda_q(\cdot,\cdot)$ is as in (1.19) and the infimum is taken over all possible representations of f of the form (1.36).

(1.37) REMARK. *It can be proved that $H_w(I\!R^n) \equiv H^{\rho,q}(I\!R^n)$, $1 < q \leq \infty$. Moreover, if we define $H^{\rho,q,k}$, $k \geq N$, as in (1.35) but taking atoms satisfying (1.32) for all $|\beta| \leq k$, we also have $H_w \equiv H^{\rho,q,k}$, $1 < q \leq \infty$. In particular, this implies that definition (1.27) does not depend on G. The atomic descomposition of H_w and the density of L^2 in H_w will be important tools in this work.*

The Remark can be proved following the lines of [V]. However, in our case, since the space of homogeneous type involved is $I\!R^n$, it is possible to consider Hardy-Orlicz spaces for a larger range of ρ, q, by using atoms with vanishing moments as in (1.32). The necessary modifications can be carry out.

We are now in conditions to introduce the main object of study of this work, the (ρ, q, ε) molecules and the molecular Hardy-Orlicz spaces.

(1.38) DEFINITION. *Assume that $\varepsilon > 0, x_0 \in I\!R^n$ and $1 < q \leq \infty$. A (ρ, q, ε) molecule centered at x_0 is a real valued function M on $I\!R^n$ satisfying*

$$(1.39)\qquad \|M\|_q \|M\rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\varepsilon + \frac{1}{q'})}\|_q \leq C,$$

where $q' = q(q-1)^{-1}$, *and*

$$\int M(x)x^{\beta}\,dx = 0 \tag{1.40}$$

for every multi-index β *such that* $|\beta| \le N$.

Given M, a (ρ, q, ε) molecule centered at x_0, and B, a ball with the same center, we denote

$$M^B = M\mathcal{X}_B \quad \text{and}$$

$$M^{CB} = \frac{M\mathcal{X}_{CB}\,\rho(|\cdot - x_0|^n)(|\cdot - x_0|^{n(\varepsilon + \frac{1}{q'})}}{\rho(|B|)|B|^{\varepsilon + \frac{1}{q'}}} \ .$$

(1.41) DEFINITION. *Assume* $1 < q \le \infty$ *and* $0 < \varepsilon$. *We define* $\mathcal{M}_{(\rho,q,\varepsilon)} = \mathcal{M}_{(\rho,q,\varepsilon)}(I\!R^n)$, *as the class of distributions* f *on* $\mathcal{S}$ *which can be represented by*

$$f = \sum_j M_j \qquad in\ \mathcal{S}', \tag{1.42}$$

where $\{M_j\}$ *is a sequence of* (ρ, q, ε) *molecules centered in* $\{x_j\}$, *such that there exists a sequence of balls* $\{B_j\} = \{B(x_j, r_j)\}$ *satisfying*

$$\sum_j |B_j| w(||M_j^{B_j}||_q |B_j|^{-1/q}) + \sum_j |B_j| w(||M_j^{CB_j}||_q |B_j|^{-1/q}) < \infty.$$

Let $\mathbf{M^B} = \{M_j^{B_j}\}$, $\mathbf{M^{CB}} = \{M_j^{CB_j}\}$ *and* $\mathbf{B} = \{|B_j|\}$. *We define*

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} = \inf(\Lambda_q(\mathbf{M^B}, \mathbf{B}) + \Lambda_q(\mathbf{M^{CB}}, \mathbf{B}),)$$

where $\Lambda_q(\cdot,\cdot)$ *is as (1.19) and the infimum is taken over all possible representations of* f *of the form (1.42).*

2. MOLECULAR CHARACTERIZATION OF H_w

In order to prove the molecular characterization of H_w (Theorem 2.18), we need some previous lemmas. Let us observe that, in view of the equivalences stated in (1.9) and (1.14), we can assume, without lost of generality, that w satisfies (1.9) through (1.13).

(2.1) LEMMA. *Assume that μ is a Borel measure on $\mathbb{R}^n$ and E is a bounded set such that $\mu(E) = 1$. Suppose that $\{x^\alpha\}_{|\alpha|\le m}$ is linearly independent on E and V is the linear space generated by $\{x^\alpha \mathcal{X}_E(x)\}_{|\alpha|\le m}$. If $u \in L^q(E), 1 \le q \le \infty$, then there exists a unique $v \in V$ such that*

$$\int (u(x)\mathcal{X}_E(x) - v(x))x^\beta \, d\mu(x) = 0, \qquad \text{for every } \beta, \ |\beta| \le m. \tag{2.2}$$

In addition

$$v(x) = \sum_{|\alpha|\le m} \int u(y)\mathcal{X}_E(y) y^\alpha \, d\mu(y) \cdot \ v_\alpha(x),$$

where v_α is the unique element of V which satisfies

$$\int v_\alpha(x)x^\beta \, d\mu(x) = \delta_{\alpha,\beta} \qquad \text{for every } \beta, \ |\beta| \le m. \tag{2.3}$$

PROOF. Let $v(x) = \sum_{|\alpha|\le m} c_\alpha x^\alpha \mathcal{X}_E(x), \ c_\alpha \in \mathbb{R}$. Clearly, v satisfies (2.2) if and only if

$$\sum_{|\alpha|\le m} c_\alpha \int_E x^\alpha x^\beta \, d\mu(x) = \int_E u(x)x^\beta \, d\mu(x), \qquad \text{for every } \beta, \ |\beta| \le m.$$

Then, since $\{x_\alpha\}_{|\alpha|\le m}$ is linearly independent on the bounded set E, there exists a unique $v \in V$ which satisfies (2.2). On the other hand, arguing as before, we have that for each α, $|\alpha| \le m$, there exists a unique $v_\alpha \in V$ which satisfies (2.3). Thus, if $\sum_{|\alpha|\le m} d_\alpha v_\alpha = 0, \ d_\alpha \in \mathbb{R}$, we have

$$d_\beta = \sum_{|\alpha|\le m} d_\alpha \int v_\alpha(x)x^\beta \, d\mu(x) = 0, \qquad \text{for every } \beta, \ |\beta| \le m.$$

Therefore, $\{v_\alpha\}_{|\alpha|\le m}$ is a basis of V and we can write $v = \sum_{|\alpha|\le m} a_\alpha v_\alpha, \ a_\alpha \in \mathbb{R}$. Finally, in view of (2.3) and (2.2), it follows that

$$a_\beta = \sum_{|\alpha|\le m} a_\alpha \int v_\alpha(x)x^\beta \, d\mu(x) = \int v(x)x^\beta \, d\mu(x) = \int u(x)\mathcal{X}_E(x)x^\beta \, d\mu(x)$$

for every $\beta, |\beta| \le m$.

(2.4) LEMMA. *Suppose that M is a (ρ, q, ε) molecule centered at x_0, with $1 < q \le \infty$ and $\varepsilon > \frac{1}{l} - 1$. Let σ be a positive constant and $B_k = B(x_0, 2^k\sigma)$, with k a* *non-negative integer. Then there exists a sequence of multiples of (ρ, q) atoms $\{b_k\}$, $supp(b_k) \subset B_k$, such that*

$$M = \sum_{k \geq 0} b_k \qquad \text{in } \mathcal{S}', \tag{2.5}$$

$$\|b_0\|_q \leq C \|M^{B_0}\|_q \qquad \text{if } k = 0, \text{ or} \tag{2.6}$$

$$\|b_k\|_q \leq C \|M^{CB_0}\|_q \quad 2^{-n(\varepsilon + \frac{1}{q'})k} \qquad \text{if } k \geq 1, \tag{2.7}$$

where C is a constant independent of M and σ. When $w(t) = t^p, p \in (0, 1]$, we have, without restriction for $\varepsilon > 0$, (2.5), (2.6) and

$$\|b_k\|_q \leq C \|M^{CB_0}\|_q 2^{-n(\varepsilon + \frac{1}{p} - \frac{1}{q})k}, \qquad \text{for } k \geq 1. \tag{2.8}$$

PROOF. Clearly, we can suppose that M is a (ρ, q, ε) molecule centered at 0. Let $E_0 = B_0, E_k = B_k - B_{k-1}, k \geq 1$, and $M_k = M\mathcal{X}_{E_k}$. Let V_k be the linear space generated by $\{x^\alpha \mathcal{X}_{E_k}\}_{|\alpha| \leq N}$. From Lemma (2.1), with $E = E_k, d\mu = \frac{1}{|E_k|}\, dx, m = N$ and $u = M_k$, there exists a unique $P_k \in V_k$ which verifies

$$\int (M_k(x) - P_k(x)) x^\beta \, dx = 0 \tag{2.9}$$

for every $\beta, |\beta| \leq N$. Moreover,

$$P_k = \sum_{|\alpha| \leq N} \frac{1}{|E_k|} \int M_k(x) \; x^\alpha \, dx. \; Q_{\alpha k} \;, \tag{2.10}$$

where $Q_{\alpha k}$ is the unique element of V_k such that

$$\int Q_{\alpha k}(x) \; x^\beta \, dx = |E_k| \delta_{\alpha\beta} \;\; \text{for every } \beta, |\beta| \leq N. \tag{2.11}$$

If we denote $m_{\alpha k} = \frac{1}{|E_k|} \int M_k(x) x^\alpha \, dx$, then we can write

$$M(x) = \sum_{k \geq 0} M_k(x) = \sum_{k \geq 0} (M_k(x) - P_k(x)) + \sum_{k \geq 0} \sum_{|\alpha| \leq N} m_{\alpha k} Q_{\alpha k}(x).$$

Since $\sum_{r \geq 0} |E_r| m_{\alpha r} = \int M(x) x^\alpha \, dx = 0$, applying summation by parts, we obtain

$$\begin{aligned}
\sum_{k \geq 0} \sum_{|\alpha| \leq N} m_{\alpha k} Q_{\alpha k}(x) &= \sum_{|\alpha| \leq N} \sum_{k \geq 0} (m_{\alpha k} |E_k|)(|E_k|^{-1} Q_{\alpha k}(x)) \\
&= \sum_{|\alpha| \leq N} \sum_{k \geq 0} \Big(\sum_{r \geq k} m_{\alpha r} |E_r| - \sum_{r \geq k+1} m_{\alpha r} |E_r|\Big)(|E_k|^{-1} Q_{\alpha k}(x)) \\
&= \sum_{|\alpha| \leq N} \sum_{k \geq 0} (|E_{k+1}|^{-1} Q_{\alpha k+1}(x) - |E_k|^{-1} Q_{\alpha k}(x)) \sum_{r \geq k+1} m_{\alpha r} |E_r| \\
&= \sum_{|\alpha| \leq N} \sum_{k \geq 0} \eta_{\alpha k} R_{\alpha k}(x) \;,
\end{aligned}$$

where $\eta_{\alpha k} = \sum_{r\geq k+1} m_{\alpha r}|E_r|$ and $R_{\alpha k}(x) = |E_{k+1}|^{-1}Q_{\alpha k+1}(x) - |E_k|^{-1}Q_{\alpha k}(x)$. Then, since $supp(M_k - P_k) \subset E_k$ and $supp(\eta_{\alpha k}R_{\alpha k}) \subset E_k \cup E_{k+1}$, it follows that

$$M = \sum_{k\geq 0}(M_k - P_k) + \sum_{|\alpha|\leq N}\sum_{k\geq 0}\eta_{\alpha k}R_{\alpha k}, \qquad \text{locally in } L^q. \tag{2.12}$$

Clearly, by (2.9) and (2.11), $M_k - P_k$ and $\eta_{\alpha k}R_{\alpha k}$ are multiples of (ρ, q) atoms. Furthermore, by (2.11), we get

$$|Q_{\alpha k}(x)| \leq C(2^k\sigma)^{-|\alpha|}. \tag{2.13}$$

Thus, by using (2.10) and Hölder's inequality, we have

$$|P_k(x)| \leq C\int \frac{|M_k(x)|}{|E_k|}\,dx \leq C\left(\int |M_k(x)|^q \frac{dx}{|E_k|}\right)^{\frac{1}{q}},$$

which inmediately yields

$$||M_k - P_k||_q \leq C||M_k||_q, \qquad \text{for every } k\geq 0. \tag{2.14}$$

Then, for $k \geq 1$, since ρ is increasing and of upper type $\frac{1}{l} - 1$, we obtain

$$\begin{aligned} ||M_k - P_k||_q \leq C||M_k||_q &\leq C\frac{||M\mathcal{X}_{CB_0}\rho(|\cdot|^n)|\cdot|^{n(\varepsilon+\frac{1}{q'})}||_q}{\rho((2^{k-1}\sigma)^n)(2^{k-1}\sigma)^{n(\varepsilon+\frac{1}{q'})}} \\ &\leq C||M^{CB_0}||_q 2^{-n(\varepsilon+\frac{1}{q'})k} \quad . \end{aligned} \tag{2.15}$$

On the other hand, applying Hölder's inequality, (2.15) and the restriction on ε, we have

$$\begin{aligned} |\eta_{\alpha k}| &\leq \sum_{r\geq k+1}\int |M_r(x)|x|^{|\alpha|}\,dx \\ &\leq \sum_{r\geq k+1} ||M_r||_q(2^r\sigma)^{|\alpha|}|E_r|^{\frac{1}{q'}} \\ &\leq C\sigma^{|\alpha|}||M^{CB_0}||_q|B_0|^{\frac{1}{q'}}\sum_{r\geq k+1}2^{r(|\alpha|-n\varepsilon)} \\ &\leq C\sigma^{|\alpha|}||M^{CB_0}||_q|B_0|^{\frac{1}{q'}}2^{(|\alpha|-n\varepsilon)k}. \end{aligned} \tag{2.16}$$

From (2.13), we get

$$|R_{\alpha k}(x)| \leq C(2^k\sigma)^{-|\alpha|-n}$$

and, applying (2.16), we obtain

$$||\eta_{\alpha k}\ R_{\alpha k}||_\infty \leq C||M^{CB_0}||_q|B_0|^{\frac{-1}{q}}2^{-n(\varepsilon+1)k}.$$

Hence, since $supp\ (\eta_{\alpha k}\ R_{\alpha k}) \subset B_{k+1}$, it follows that

$$(2.17) \qquad ||\eta_{\alpha k}\ R_{\alpha k}||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{q'})k}.$$

Finally, if we define $b_0 = M_0 - P_0$, $b_1 = \sum_{|\alpha|\leq N} \eta_{\alpha 0} R_{\alpha 0}$, $b_k = M_{k-1} - P_{k-1} + \sum_{|\alpha|\leq N} \eta_{\alpha k-1} R_{\alpha k-1}$, $k \geq 2$, by (2.12), (2.14), (2.15) and (2.17), we get (2.5), (2.6) and (2.7). When $w(t) = t^p, p \in (0,1]$, we can improve (2.15) and get

$$||M_k - P_k||_q \leq C||M_k||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{p}-\frac{1}{q})k}, \quad \text{for } k \geq 1.$$

Thus, arguing as before, but without restriction on ε, we have

$$||\eta_{\alpha k} R_{\alpha k}||_q \leq C||M^{CB_0}||_q\ 2^{-n(\varepsilon+\frac{1}{p}-\frac{1}{q})k}, \quad \text{for } k \geq 0$$

which proves (2.8).

(2.18) THEOREM. *Assume that w is a function of positive lower type l such that $w(s)/s$ is non increasing. Let $\rho(t)$ be the function defined by $\rho(t) = t^{-1}/w^{-1}(t^{-1})$. Then $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ with $1 < q \leq \infty$ and $\varepsilon > \frac{1}{l} - 1$. When $w(t) = t^p, p \in (0,1]$, we have $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ for $1 < q \leq \infty$ and every $\varepsilon > 0$.*

PROOF. By (1.37) is sufficient to prove that $H^{\rho,q} \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$.
First inclusion: $H^{\rho,q} \subset \mathcal{M}_{(\rho,q,\varepsilon)}$. Let f be a distribution in $H^{\rho,q}$. Assume that $\mathbf{b} = \{b_j\}$ is a sequence of multiples of (ρ, q) atoms such that $f = \sum_j b_j$ is a representation of f as in (1.35). Clearly, b_j is a (ρ, q, ε) molecule centered at x_j. Moreover, if we denote $M_j = b_j$, in view of (1.35), (1.38) and (1.41), we have

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq \Lambda_q(\mathbf{M^B}, \mathbf{B}) + \Lambda_q(\mathbf{M^{CB}}, \mathbf{B}) \leq \Lambda_q(\mathbf{b}, \mathbf{B}).$$

Thus, we have that $f \in \mathcal{M}_{(\rho,q,\varepsilon)}$ and

$$||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq ||f||_{H^{\rho,q}}.$$

Second inclusion: $\mathcal{M}_{(\rho,q,\varepsilon)} \subset H^{\rho,q}$. Let f be a distribution in $\mathcal{M}_{(\rho,q,\varepsilon)}$. According to definition (1.41) suppose that $\{M_j\}$ is a sequence of (ρ, q, ε) molecules centered at $\{x_j\}$ and $\{B_j\}$ is a sequence of balls, $B_j = B(x_j, r_j)$, such that

$$(2.19) \qquad f = \sum_j M_j \ \text{ in } \mathcal{S}' \text{ and } 0 < \Lambda_q(\mathbf{M^B}, \mathbf{B}) + \Lambda_q(\mathbf{M^{CB}}, \mathbf{B}) < \infty.$$

In view of (1.20), we can assume that $\Lambda_q(\mathbf{M^B}, \mathbf{B}) > 0$ and $\Lambda_q(\mathbf{M^{CB}}, \mathbf{B}) > 0$. Applying lemma (2.4) to each M_j with $\sigma = r_j$, from (2.19), we have

$$(2.20) \qquad f = \sum_j \sum_{k\geq 0} b_k^j \qquad \text{in } \mathcal{S}',$$

where b_k^j is a multiple of a (ρ, q) atom, $supp\ (b_k^j) \subset B_k^j = B(x_j, 2^k r_j)$, and $||b_0^j||_q \leq C||M_j^{B_j}||_q$ if $k = 0$ or $||b_k^j||_q \leq C||M_j^{CB_j}||_q 2^{-n(\varepsilon+\frac{1}{q'})k}$ if $k \geq 1$. Let $\eta \geq 1$ be a constant to be determinated later. Since w is an increasing function of lower type l and of upper type 1 we have

$$\sum_j \sum_{k\geq 0} |B_k^j| w \left(\frac{||b_k^j||_q \, |B_k^j|^{-1/q}}{[\eta(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}))]^{1/l}} \right)$$
$$\leq \frac{C}{\eta} \left[\sum_j |B_j| w \left(\frac{||M_j^{B_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^B},\mathbf{B})^{1/l}} \right) \right.$$
$$\left. + \sum_j \sum_{k\geq 1} 2^{kn(1-(\varepsilon+1)l)} |B_j| w \left(\frac{||M_j^{CB_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^{CB}},\mathbf{B})^{1/l}} \right) \right],$$

which, by the restriction on ε, is less than or equal to

$$\frac{C}{\eta} \left(\sum_j |B_j| \; w \left(\frac{||M_j^{B_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^B},\mathbf{B})^{1/l}} \right) + \sum_j |B_j| \; w \left(\frac{||M_j^{CB_j}||_q \, |B_j|^{-1/q}}{\Lambda_q(\mathbf{M^{CB}},\mathbf{B})^{1/l}} \right) \right) \leq \frac{2C}{\eta}.$$

Choosing $\eta = 2C$, we get

$$\sum_j \sum_{k\geq 0} |B_k^j| \; w \left(\frac{||b_k^j||_q |B_k^j|^{-1/q}}{[2C(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B}))]^{1/l}} \right) \leq 1. \tag{2.21}$$

From (2.20), (2.21) and the observation above (1.35), we obtain

$$||f||_{H^{\rho,q}} \leq C(\Lambda_q(\mathbf{M^B},\mathbf{B}) + \Lambda_q(\mathbf{M^{CB}},\mathbf{B})). \tag{2.22}$$

Then, since we have (2.22) for every possible representation of f in the form (2.19), we get

$$||f||_{H^{\rho,q}} \leq C||f||_{\mathcal{M}_{(\rho,q,\varepsilon)}}.$$

Note that the restriction $\varepsilon > \frac{1}{l} - 1$ was only used in the proof of the inclusion $\mathcal{M}_{(\rho,q,\varepsilon)} \subset H^{\rho,q}$. When $w(t) = t^p, p \in (0,1]$, we can apply (2.8) and, following the same lines as above, we get $H_w \equiv \mathcal{M}_{(\rho,q,\varepsilon)}$ with $\varepsilon > 0$ and $1 < q \leq \infty$.

3. APPLICATION OF THE MOLECULAR CHARACTERIZATION OF H_w

In this section we shall assume that T is a singular integral operator in $\mathbb{R}^n$ with a kernel K of class C^{k+1} outside the origin with k a non-negative integer, satisfying

$$\left| \int_{r<|x|<R} K(x)\,dx \right| \leq C, \qquad 0 < r < R, \tag{3.1}$$

$$\lim_{r\to 0} \int_{r<|x|<1} K(x)\,dx \qquad \text{exists, and} \tag{3.2}$$

$$(3.3) \qquad |D^\beta K(x)| \le C|x|^{-n-|\beta|} ,$$

for every multi-index β such that $|\beta| \le k+1$, and every $x \neq 0$. It is well known that, under these conditions, T is a bounded operator on L^q, $1 < q < \infty$. Moreover, if we define the maximal operator

$$T^* f(x) = \sup_{\delta>0} |T_\delta f(x)| ,$$

where

$$T_\delta f(x) = \int_{\delta<|y|} K(y) f(x-y)\, dy ,$$

we have that T^* is bounded on L^q, $1 < q < \infty$ and

$$(3.4) \qquad Tf(x) = \lim_{\delta\to 0} T_\delta f(x) \quad \text{a.e. } x$$

The purpose of this section is to show the boundedness of T on H_w. The main tool will be the molecular characterization obtained in section 2. In [H-V], Harboure and Viviani, using another technique, proved a similar result in the context of the spaces of homogeneous type. In that work, the cancellation property of the kernel K is stronger than (3.1). Moreover, since in our case the space involved is $\mathbb{R}^n$, we can impose more regularity to the Kernel and by using atoms with vanishing moments as in (1.37), it is possible to consider Hardy Orlicz spaces for a larger range of w.

(3.5) LEMMA. *Let w and ρ be as in theorem (2.18). Let T be a singular integral operator with a kernel K satisfying (3.1), (3.2) and (3.3) with $k+1 > n(\frac{1}{l} - 1)$. Assume that b is a function belonging to L^q, $1 < q < \infty$, with vanishing moments up to the order k and $supp(b) \subset B = B(x_0, r)$. Let $0 < \varepsilon < 1 - \frac{1}{l} + \frac{k+1}{n}$, then Tb is a (ρ, q, ε) molecule centered at x_0 and*

$$(3.6) \qquad ||Tb||_q \le C||b||_q,$$

$$(3.7) \qquad ||Tb\ \rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\varepsilon+\frac{1}{q'})}||_q \le C\rho(|B|)\ |B|^{\varepsilon+\frac{1}{q'}}\ ||b||_q ,$$

where C is a constant independent of b.

PROOF. Since T conmutes with translations we may assume that b is supported in a ball $B = B(0, r)$. Clearly Tb satisfies (3.6). Let $\tilde{B} = B(0, 2r)$, then

$$||Tb\ \rho(|\cdot|^n)(|\cdot|^n)^{\varepsilon+\frac{1}{q'}}||_q^q = \left(\int_{\tilde{B}} + \int_{C\tilde{B}}\right) |Tb(x)\ \rho(|x|^n)|x|^{n(\varepsilon+\frac{1}{q'})}|^q\, dx = I_1 + I_2.$$

Since ρ is increasing and of upper type, applying (3.6), we have that I_1 is bounded by

$$C[\rho(|B|)\ |B|^{\varepsilon+\frac{1}{q'}}\ ||b||_q]^q.$$

On the other hand, if $x \in C\tilde{B}$ we have

$$Tb(x) = \int_B (K(x-y) - P(x-y))b(y)\,dy,$$

where P is the Taylor polynomial of K at x of degree k. The typical estimate for the remainder in Taylor's formula for this function, (3.3) and Hölder's inequality yield

$$(3.8) \qquad |Tb(x)| \leq C\int_B |b(y)|\frac{|y|^{k+1}}{|x|^{n+k+1}}\,dy \leq C\frac{||b||_q |B|^{\frac{k+1}{n}+\frac{1}{q'}}}{|x|^{n+k+1}} \qquad , x \in C\tilde{B}.$$

From this estimate and (1.16) we have that I_2 is bounded by

$$C[\rho(|B|)|B|^{\frac{k+1}{n}+\frac{1}{q'}+1-\frac{1}{l}}||b||_q]^q \int_{C\tilde{B}} |x|^{n(\varepsilon-\frac{1}{q}-\frac{k+1}{n}+\frac{1}{l}-1)q}\,dx.$$

Then, since $\varepsilon < 1 - \frac{1}{l} + \frac{k+1}{n}$, I_2 is less than or equal to

$$C[\rho(|B|)\ |B|^{\varepsilon+\frac{1}{q'}}\ ||b||_q]^q \ ,$$

which completes the proof of (3.7). In order to prove that Tb has vanishing moments up to the order k we shall use the following partition of unity. Take functions $\phi_j(t)$, $j = 0, 1, 2, \ldots, C^\infty$ in $(0, \infty)$ satisfying $\phi_j \geq 0$, $\sum_{j=0}^{\infty} \phi_j(t) = 1$ for every t in $(0,\infty)$. Moreover, we can assume that $supp(\phi_0) \subset [0, 2r]$, $supp(\phi_j) \subset [2^{j-1}r, 2^{j+1}r]$ for $j \geq 1$ and $|\phi_j^{(k)}(t)| \leq C_k t^{-k}$ for every $t > 0$, every $k = 0, 1, 2, \ldots$ and every j, with C_k depending only on k. Now, we define for each j, $K_j(x) = K(x)\phi_j(|x|)$, and observe that all the K_j's satisfy the same estimates as K with a uniform constant. Moreover, we have

$$\sum_{j\geq 0} \mathcal{X}_{supp(K_j * b)}(x) \leq 4, \text{ at each } x \in I\!R^n.$$

Then we can write

$$(3.9) \qquad \int Tb(x)x^\beta\,dx = \int \sum_{j\geq 0} K_j * b(x)x^\beta\,dx, \quad \text{for every } \beta, |\beta| \leq k.$$

Clearly,

$$|\sum_{j=0}^{n} K_j{*}b(x)x^\beta| \leq \sum_{j=0}^{n} |K_j{*}b(x)||x|^{|\beta|}\chi_{\tilde{B}}(x) + \sum_{j=0}^{n} |K_j{*}b(x)||x|^{|\beta|}\chi_{C\tilde{B}}(x) = A_1 + A_2$$

For $j \geq 1$, by Hölder's inequality, there exists a constant C, independent of j, such that

$$||K_j * b||_\infty \leq C|B|^{-1/q}||b||_q \ .$$

On the other hand, arguing as in (3.8), for $x \in C\tilde{B}$, we get

$$|K_j * b(x)| \leq C\frac{||b||_q|B|^{\frac{k+1}{n}+1/q'}}{|x|^{n+k+1}} \qquad \text{for } j \geq 0,$$

where C is again independent fo j. Thus, since the overlap of the supports of $K_j * b$ is uniformly bounded we have that

$$A_1 \leq C(|K_0 * b(x)| + ||b||_q|B|^{-1/q})|x|^{|\beta|}\chi_{\tilde{B}}(x)$$

and

$$A_2 \leq C\frac{||b||_q|B|^{\frac{k+1}{n}+1/q'}}{|x|^{n+k+1-|\beta|}}\chi_{C\tilde{B}}(x).$$

Then, by (3.9) and the dominated convergence theorem, we obtain

$$\int Tb(x)x^\beta\, dx = 0, \qquad |\beta| \leq k,$$

since $K_j * b$ has vanishing moments up to order k.

(3.10) THEOREM. *Let w and ρ be as in theorem (2.18). Let T be a singular integral operator with a kernel K satisfying (3.1), (3.2) and (3.3) with $k+1 > 2n(\frac{1}{l}-1)$. Then there exists a constant C such that*

$$||Tf||_{H_w} \leq C||f||_{H_w}.$$

PROOF. By (1.37) and (2.18) it is enough to show that

$$||Tf||_{\mathcal{M}_{(\rho,q,\varepsilon)}} \leq C||f||_{H^{\rho,q,k}}\ , \tag{3.11}$$

for every $f \in L^2 \cap H^{\rho,q,k}$, where $\frac{1}{l}-1 < \varepsilon < 1-\frac{1}{l}+\frac{k+1}{n}$ and $1 < q < \infty$. Let $f \in L^2 \cap H^{\rho,q,k}$ and $\mathbf{b} = \{b_j\}$ be a sequence of multiples of (ρ,q) atoms with vanishing moments up to the order k, $sup(b_j) \subset B_j = B(x_j, r_j)$, such that

$$f = \sum_j b_j \text{ in } \mathcal{S}'. \tag{3.12}$$

From the previous lemma we have that Tb_j is a (ρ,q,ε) molecule centered at x_j satisfying (3.6) and (3.7). Let $M_j = Tb_j$. Arguing in a similar way as it was done in the proof of Theorem (2.20) in [H-V], it can be shown that

$$Tf = \sum_j M_j \qquad \text{in } S'. \tag{3.13}$$

Let η a positive constant to be determinated. In view of (1.35), (1.38) and (1.41), applying (3.6), we have

$$\sum_j |B_j|w\left(\frac{||M_j^{B_j}||_q|B_j|^{-1/q}}{(\eta\Lambda_q(\mathbf{b},\mathbf{B}))^{1/l}}\right) \leq \sum_j |B_j|w\left(\frac{C||b_j||_q|B_j|^{-1/q}}{(\eta\Lambda_q(\mathbf{b},\mathbf{B}))^{1/l}}\right).$$

Then taking $\eta = C^l$ we obtain

$$\Lambda_q((\mathbf{M})^{\mathbf{B}}, \mathbf{B}) \leq C^l \Lambda_q(\mathbf{b}, \mathbf{B}).$$

In a similar way, from (3.7), we get

$$\Lambda_q((\mathbf{M})^{\mathbf{CB}}, \mathbf{B}) \leq C^l \Lambda_q(\mathbf{b}, \mathbf{B}).$$

Then, by (3.13), we have

$$||Tf||_{(\rho,q,\varepsilon)} \leq C \Lambda_q(\mathbf{b}, \mathbf{B}),$$

which completes the proof of the Theorem.

(3.14) REMARK. *When* $w(t) = t^p$, $p \in (0,1]$, *since from (2.18),* $H_w \equiv \mathcal{M}(\rho, q, \varepsilon)$, *with* $\varepsilon > 0$, *we have that* T *is a bounded operator in* H_w *with the only restriction* $k + 1 > n(\frac{1}{l} - 1)$.

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FOURIER VERSUS WAVELETS: A SIMPLE APPROACH TO LIPSCHITZ REGULARITY

Hugo Aimar and Ana Bernardis

Abstract: We give a very simple proof of the caracterization of Lipschitz regularity of a function by the size of its Haar coefficients.

It is well known that given a real function f periodic with period 2π satisfying a Lipschitz α condition for $0 < \alpha \leq 1$, its k^{th} Fourier coefficient is bounded by $|k|^{-\alpha}$. More precisely, the following result holds (see for example Chapter 12 of [9]).

(A) *Let f be a 2π periodic real function satisfying a Lipschitz α condition for $0 < \alpha \leq 1$, i.e., there exists a positive finite constant M such that, $|f(x+h)-f(x)| \leq M|h|^{\alpha}$, for every pair of real numbers h and x. Then, there exists a constant C such that, for every $k \in \mathbb{Z}$, $|C_k[f]| \leq C|k|^{-\alpha}$, where $C_k[f] = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$.*

The result is an easy consequence of the fact that $\int_0^{2\pi} e^{-ikx}dx = 0$, for $k \neq 0$. Nevertheles, it does not constitute a characterization of Lipschitz α. This fact can easily be observed by taking the Fourier coefficients of the characteristic function of a subinterval of $[0, 2\pi]$. Moreover there is no way to characterize the regularity of a function in terms of the size of its Fourier coefficients, this is a very deep fact implied by the results in the article "Sur les coefficients de Fourier des fonctions continues" by J.P. Kahane, Y. Katznelson and K. de Leeuw, see [4]. On the other hand, we can easily obtain an analogous of **(A)** for the Haar coefficients. We define the Haar coefficients of a locally integrable function f as $C_{a,b} = \int_{\mathbb{R}} f(x)H_{a,b}(x)dx$, where $H_{a,b}(x) = a^{-1/2}H(\frac{x-b}{a})$, $a > 0$, $b \in \mathbb{R}$ and H is the Haar function i.e., H is defined by 1, for $0 \leq x < 1/2$; by -1, for $1/2 \leq x < 1$ and 0 otherwise. More precisely we get the following result

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(B) *Let f be a Lipschitz α function for $0 < \alpha \leq 1$. Then, there exists a constant C such that $|C_{a,b}| \leq C a^{1/2+\alpha}$, for every $a > 0$ and $b \in I\!R$.*

Proof of (B):

$$\begin{aligned} |C_{a,b}| =& \Big| \int_{I\!R} H_{a,b}(x) f(x) dx \Big| \\ =& a^{1/2} \Big| \int_0^1 H(u)[f(au+b) - f(b)] du \Big| \\ \leq& C a^{1/2+\alpha}. \blacksquare \end{aligned}$$

Clearly we have the following general version of (B):

(B') *Let $\varphi : I\!R^+ \to I\!R^+$ be a non-decreasing function and let f be a function satisfying a Lipschitz (φ) condition, i.e., there exists a constant C such that $|f(x) - f(y)| \leq C\varphi(|x-y|)$, for every x, y in $I\!R$. Then, there exists a constant C such that*

(1) $$|C_{a,b}| \leq C a^{1/2} \varphi(a); \qquad a > 0, \quad b \in I\!R.$$

The aim of this note is to give a very simple proof of the converse of the preceding result, moreover, we shall prove the Lipschitz (ψ) regularity of a function whose Haar coefficients $C_{a,b}$ satisfy (1), with $\psi(t) = \int_0^t \varphi(s)/s\, ds$. Notice that if $\psi(t) \leq C\varphi(t)$, condition (1) is equivalent to Lipschitz (φ) regularity, which is certainly the case for $\varphi(t) = t^\alpha$, $0 < \alpha \leq 1$. The proof of this converse can be extended to get a characterization of Lipschitz spaces with some non-isotropic metrics in higher dimensions.
By using the inversion formula for the continuous wavelet transform, Holschneider and Tchamitchian prove in [3] that Lipschitz α regularity of a function is completely characterized by the size of the wavelet coefficient, see also [2]. The inversion formula itself relies on the Fourier transform. Nevertheless the notion of Lipschitz regularity can be naturally extended to metric spaces and generally, wavelet coefficients can be computed for functions defined on spaces of homogeneous type where Fourier transform is not available. Since the work by Campanato [1], Meyers [5], Spanne [6] among others it has become classical the integral characterization of pointwise regular functions such us Lipschitz α or more generally Lipschitz (φ). A simple proof of these facts for one dimension as can be found in the book [8], can be adapted to give a direct proof of the desired result. The adventage of this approach is that it can be used to get an analog of this result for some families of non-isotropic dilations in dimension higher than one whithout an explicit inversion formula.
Let us first observe that the inequality $|C_{a,b}| \leq C a^{1/2} \varphi(a)$ can be rewritten as

(2) $$|m(I^-) - m(I^+)| \leq C\varphi(|I|),$$

where $I = [b, b+a]$, I^- is the left half of I, I^+ is its right half, $m(J) = \frac{1}{|J|} \int_J f(x)\, dx$, and $|J|$ is the measure of the interval J.

(3) Theorem: *Let $\varphi : I\!R^+ \to I\!R^+$ be a non-decreasing function such that*

$$\int_0^1 \varphi(t)/t dt < \infty.$$

Let f be a locally integrable function. If the Haar coefficients of f satisfy (1), then f is Lipschitz (ψ), with $\psi(t) = \int_0^t \varphi(s)/s\, ds$.

Proof: Let x and y be two real numbers with $x < y$. Let us now construct two sequences of subintervals $\{I_k^-\}$ and $\{I_k^+\}$ of $I = [x, y]$ in the following way: I_1^- is the left half of I and I_1^+ its right half, I_2^- is the left half of I_1^-, I_2^+ the right half of I_1^+. And so, I_k^- is the left half of I_{k-1}^- and I_k^+ the right half of I_{k-1}^+. Notice now that

$$\begin{aligned}|f(x) - f(y)| \leq & |f(x) - m(I_k^-)| + \sum_{i=2}^{k} |m(I_i^-) - m(I_{i-1}^-)| + |m(I_1^-) - m(I_1^+)| \\ & + \sum_{i=1}^{k-1} |m(I_i^+) - m(I_{i+1}^+)| + |m(I_k^+) - f(y)|.\end{aligned}$$

By an application of (2) with $b = x$ and $a = y - x$ we get that the central term $|m(I_1^-) - m(I_1^+)|$ is bounded by $C\varphi(|I|)$. In order to estimate the general term of the first sum $|m(I_i^-) - m(I_{i-1}^-)|$ with $2 \leq i \leq k$, notice that

$$\begin{aligned}|m(I_i^-) - m(I_{i-1}^-)| &= |m(I_i^-) - 1/2m(I_i^-) - 1/2m(I_{i-1}^- \setminus I_i^-)| \\ &= 1/2|m(I_i^-) - m(I_{i-1}^- \setminus I_i^-)|.\end{aligned}$$

Since I_i^- and $I_{i-1}^- \setminus I_i^-$ are contiguous intervals with the same length, we apply (2) to the last term in the above equality to obtain $|m(I_i^-) - m(I_{i-1}^-)| \leq C\varphi(|I_{i-1}^-|)$. In a similar way, we can estimate the general term of the second sum by $\varphi(|I_i^+|)$. Therefore

$$\begin{aligned}|f(x) - f(y)| \leq & |f(x) - m(I_k^-)| + C\sum_{i=2}^{k} \varphi(|I_{i-1}^-|) + C\varphi(|I|) + C\sum_{i=1}^{k-1} \varphi(|I_i^+|) \\ & + |m(I_k^+) - f(y)| \\ \leq & |f(x) - m(I_k^-)| + 2C\sum_{i=0}^{\infty} \varphi(\frac{|I|}{2^i}) + |m(I_k^+) - f(y)|.\end{aligned}$$

Now by Lebesgue Differentiation Theorem, when k tends to infinity, and the properties on φ we get

$$\begin{aligned}|f(x) - f(y)| &\leq 2C\sum_{i=0}^{\infty} \varphi(\frac{|I|}{2^i}) \\ &\leq \frac{2C}{log2} \sum_{i=0}^{\infty} \int_{|I|/2^i}^{|I|/2^{i-1}} \varphi(t)/t\, dt \\ &\leq \frac{2C}{log2} \psi(|I|),\end{aligned}$$

for almost every x and y. So that, after redefining f on a null set, we have a Lipschitz (ψ) function. ■

To ilustrate the applicability of this method to the regularity problem in the parabolic setting, even when our method applies to more general situation, we shall restrict ourselves to the case of dilations $T_\lambda x = e^{A\log\lambda}x$ with $\lambda > 0$ and A the diagonal matrix with eigenvalues 1 and 2 in two dimensions. Actually $T_\lambda x = (\lambda x_1, \lambda^2 x_2)$, for $x = (x_1, x_2)$. The associated translation invariant metric $\rho(x)$ on $\mathbb{R}^2$ is the only solution of $|T_{1/\rho(x)}x| = 1$ (see for example [7]). Let us introduce the following two wavelets in $\mathbb{R}^2$

$$\eta_1(x,y) = \chi(x)H(y)$$

$$\eta_2(x,y) = H(x)\chi(y),$$

where χ is the characteristic function of the one dimensional interval [0,1). Performing the usual translations in $\mathbb{R}^2$ and the parabolic dilations induced by A we get an L^2-normalized family of functions $\eta_i^{a,b}(x) = a^{-3/2}\eta_i(T_{1/a}(x-b)) = a^{-3/2}\eta_i(\frac{x_1-b_1}{a}, \frac{x_2-b_2}{a^2})$, for $a > 0$ and $b \in \mathbb{R}^2$.

(4) Theorem: *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that*

$$\int_0^1 \varphi(t)/t dt < \infty.$$

Let f be a locally integrable function on $\mathbb{R}^2$. Assume that there is a constant C such that

$$(5) \qquad |< f, \eta_i^{a,b} >| \leq C a^{3/2}\varphi(a); \qquad a > 0, \quad b \in \mathbb{R}^2, \quad i = 1,2$$

then f satisfies the Lipschitz (ψ) condition with respect to ρ, i.e., $|f(x) - f(y)| \leq C\psi(\rho(x-y))$.

Proof: Let us first notice that the inequalities in (5) can be written as follows

$$(5.a) \qquad |m(I \times J^-) - m(I \times J^+)| \leq C\varphi(a),$$

$$(5.b) \qquad |m(I^- \times J) - m(I^+ \times J)| \leq C\varphi(a),$$

where I and J are two real intervals conforming a parabolic rectangle, i.e. $|I|^2 = |J|$, I^- is the left half of I, while I^+ is its right half. Similar notation applies to J. Given $x = (x_1, x_2)$ and $y = (y_1, y_2)$ two points in the plane, in order to estimate $|f(x) - f(y)|$, we introduce the point $z = (y_1, x_2)$ which satisfies both $\rho(x-z) \leq \rho(x-y)$ and $\rho(z-y) \leq \rho(x-y)$, so that we look for the following inequalities

$$(6) \quad |f(x) - f(z)| \leq C\psi(\rho(x-z)) \qquad \text{and} \qquad |f(z) - f(y)| \leq C\psi(\rho(z-y)).$$

We shall work out with some detail the proof of the first inequality in (6) and we shall only sketch the similar proof of the second. Let us assume that $x_1 < y_1$, call $I = [x_1, y_1]$. As in the proof of Theorem 3 we are lead to two subinterval sequences of I, $\{I_i^-\}$ and $\{I_i^+\}$ with $k_i = |I_i^-| = |I_i^+|$. Let $J_i = [x_2, x_2 + (2k_i)^2]$, $R_i^- = I_i^- \times J_i$ and $R_i^+ = I_i^+ \times J_i$ for $i \in I\!N$. For each $k \in I\!N$ we have

$$\begin{aligned}|f(x) - f(z)| \leq &|f(x) - m(R_k^-)| + \sum_{i=2}^{k} |m(R_i^-) - m(R_{i-1}^-)| + |m(R_1^-) - m(R_1^+)| \\ &+ \sum_{i=1}^{k-1} |m(R_i^+) - m(R_{i+1}^+)| + |m(R_k^+) - f(z)|.\end{aligned}$$

By $(5.b)$ the central term in the right hand side above satisfies the desired bound. For the general term in each of the sums, for example for $|m(R_i^-) - m(R_{i-1}^-)|$, we proceed in the following way: decompose R_{i-1}^- into eight equal parabolic rectangles $R_1, \ldots, R_8$ with $R_1 = R_i^-$, so that

$$8m(R_{i-1}^-) = m(R_1) + m(R_2) + \ldots + m(R_8).$$

Clearly we may assume that the R_j's are indexed in such a way that R_j shares one side with R_{j+1}. Now, since

$$|m(R_i^-) - m(R_{i-1}^-)| \leq \sum_{j=1}^{7} |m(R_j) - m(R_{j+1})|,$$

we only need to show that each of the terms $|m(R_j) - m(R_{j+1})|$ satisfies the desired inequality. Let us first observe that if R_j and R_{j+1} have a common vertical side we can apply again $(5.b)$. On the other hand, when R_j and R_{j+1} share a horizontal side the rectangle defined by the union $R_j \cup R_{j+1}$ of both is not a parabolic rectangle, so that we divide both of them in eight equal parabolic rectangle by dividing only the vertical sides of R_j and R_{j+1} in eight equal intervals. Let us write $R_1^*, R_2^*, \ldots, R_{16}^*$ to denote these new rectangles and assume that they are indexed from top to bottom. Hence

$$\begin{aligned}|m(R_j) - m(R_{j+1})| &= 1/8 |\sum_{k=1}^{8} m(R_k^*) - \sum_{k=9}^{16} m(R_k^*)| \\ &\leq 1/8 \sum_{i=0}^{7} |m(R_{8-i}^*) - m(R_{9+i}^*)| \\ &\leq 1/8 \sum_{i=0}^{7} \sum_{k=0}^{2i} |m(R_{8-i+k}^*) - m(R_{9-i+k}^*)|.\end{aligned}$$

Now, since each R_j^* is parabolic, the general term in the last sum can be bounded by $(5.a)$. Finally, by the Differentiation Theorem, which is still valid in the parabolic setting, we may obtain the first inequality in (6). The proof of the second follows the same lines provided we change the iteration of the diadic decomposition on the x-axis by the iteration of the procedure of dividing in four equal parts the vertical intervals containing x_2 and y_2. ∎

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XLV REUNION ANUAL DE COMUNICACIONES CIENTIFICAS DE LA UNION MATEMATICA ARGENTINA Y XVIII REUNION DE EDUCACION MATEMATICA

En la Universidad Nacional de Río Cuarto, desde el lunes 16 de octubre hasta el viernes 20 de octubre de 1995, se realizaron la XLV Reunión Anual de Comunicaciones Científicas y la XVIII Reunión de Educación Matemática, con el auspicio de las Universidad Nacional de la Plata, Universidad Nacional del Litoral, Universidad Nacional de la Patagonia "San Juan Bosco", Universidad Nacional de Rosario, la Municipalidad de la ciudad de Río Cuarto y del Consejo Nacional de Investigaciones Científicas y Técnicas.

En el marco de estas se efectuó además el VII Encuentro de Estudiantes de Matemática. Hubo un total de 672 participantes de los cuales 162 fueron estudiantes.

Las actividades de la XVIII Reunión de Educación Matemática comenzaron el lunes 16. Durante su transcurso se dictaron 10 cursillos sobre temas variados, dos talleres sobre los Contenidos Básicos Comunes del E.G.B. y la Educación Polimodal. Del 18 al 20 de octubre se expusieron 11 Posters sobre la Enseñanza de la Matemática y se presentaron 37 Comunicaciones.

La XLV Reunión Anual de Comunicaciones Científicas se inició el miércoles 18 de octubre con la inscripción de los participantes, efectuándose por la tarde el acto inaugural en el Aula Mayor, en la oportunidad hicieron uso de la palabra el Decano de la Facultad de Ciencias Exactas Físico-Químicas y Naturales, el Presidente de la Unión Matemática Argentina y el Rector de la Universidad Nacional de Río Cuarto
También se entregaron sendas plaquetas recordatorias por su trayectoria a los socios honorarios de la Unión Matemática Argentina Dr. Félix Herrera e Ing. Roque Scarfiello. Se realizó la entrega de los premios del concurso "Rodolfo Ricabarra" a las mejores monografías sobre el tema "Teorema del Punto Fijo."

A continuación actuó el Grupo Instrumental de la U.N.R.C. Después de un cuarto intermedio el Dr. Rafael Panzone pronunció la conferencia " Dr. Julio Rey Pastor" sobre el tema " Conjuntos y curvas notables del plano". Luego los participantes y autoridades fueron agasajados con un vino de honor.

Los días jueves 19 y viernes 20 se expusieron 106 Comunicaciones, distribuídas en los siguientes temas: Geometría Diferencial y Grupos de Lie, Ecuaciones Diferenciales y Modelos, Ecuaciones Diferenciales Parabólicas, Análisis Numérico, Control, Optimización, Teoría de Juegos y Convexidad, Lógica, Análisis Real y Armónico, Conjuntos Borrosos, Grafos y Topología, Algebra y Teoría de Números y Análisis Funcional. Además se dictaron 6 cursos para estudiantes de Matemática.
Se realizó una mesa redonda en la que se discutieron los planes de estudio de las licenciaturas en matemática.

El viernes 20 a las 16:30 hs. tuvo lugar la Asamblea Anual de socios de U.M.A., en cuyo transcurso se elegieron nuevas autoridades.

El congreso se clausuró el viernes 20 a las 19 hs. con la conferencia "Alberto González Dominguez" sobre el tema " Mejor Aproximación de funciones", a cargo del Dr. Felipe Zó
Para terminar, el presidente de la Unión Matemática Argentina Dr. J. Tirao hizo uso de la palabra y el Rector de la U.N.R.C. entregó al presidente de la Unión Matemática Argentina una medalla recordatoria.

Posteriormente se agasajó a los participantes.

NOMINA DE LAS COMUNICACIONES PRESENTADAS A LA XLV REUNION ANUAL DE LA UNION MATEMATICA ARGENTINA

NOTA: Las comunicaciones que van precedidas por un asterisco no fueron expuestas.

Geometría Diferencial - Grupos de Lie

Jorge Lauret. (FaMAF. UNC.) *" Grupos de isometrías de una nilvariedad homogénea. "*

Cristián Sánchez, Walter Dal Lago, Alicia García, Eduardo Hulett.(FaMAF. UNC.) *"Algunas propiedades que caracterizan a los R-espacios. "*

D. Alekseevky, I. Dotti. (FaMAF. UNC.)*"Variedades de Einstein homogéneas"*.

Alfredo O. Brega.(FaMAF. UNC.) *" Sobre el dual unitario de Spin (2n, C). "*

Carina Boyallian. (FaMAF. UNC.) *"D-módulos y Operadores diferenciales G-invariantes"*.

Jorge Vargas.(FaMAF. UNC.) *" Restricciones de representaciones. "*

Leandro Cagliero, Juan Tirao (FaMAF. UNC). *"Los residuos de los operadores de entrelazamiento de Kunze-Stein. "*

José I. Liberati. (Fa MAF. UNC.) *" Propiedad biespectral y la Grassmanniana* Gr^{rat}. *"*

Guillermo Keilhauer, M. del Carmen Calvo (FCEyN. UBA). *" Tensores del tipo (0,2) sobre fibrados tangentes (I) "*

Cristián Sánchez (FaMAF. UNC.) *" El I-número de un R-espacio. "*

Marcos Salvai (FaMAF. UNC). *"Geodésicas asintóticas en el cubrimiento universal de Sl (2, R)"*

Mirta S. Iriondo (FaMAF. UNC) *" Superficies de curvatura media constante en espacios Lorentzianos. "*

Ana Forte Cunto, María Piacquadio (FCEyN. UBA.) *" Continuidad de la función de visibilidad en* R^n. *"*

Alejandro Tiraboschi (FaMAF. UNC.) *" Algebras reales nilpotentes matabelianas regulares. "*

Javier Fernández, Marcela Zuccalli (FCEyN. UBA.- UNLP) *" Grupos de Loops y la Orbita Coadjunta del 0. "*

J. P. Rossetti, P. Tirao (FaMAF. UNC.) *" Variedades compactas planas con grupo de holonomía* $Z_2 \oplus Z_2$

Walter Dal Lago, Alicia García, Cristián Sánchez (FaMAF. UNC) *" Espacios proyectivos en la variedad de secciones normales.*

Graciela S. Birman (U.N. del Centro de la Pcia. Bs. As. - CONICET) *" Métrica para un modelo homegéneo, isotrópico, 5-dimensional."*

Liliana Gysin, M. Cristina López (FC EyN. UBA). *" Esperanzas de funciones definidas sobre el transporte paralelo."*

(*)Salvador Gigena (UNR - UNC) *" La curvatura escalar Riemanniana de hipersuperficies descomponibles."*

Sergio Console, Carlos Olmos (FaMAF. UNC) *" Subvariedades que admiten un campo normal paralelo isoparamétrico."*

Carlos Olmos, Adrián L. E. Will (FaMAF. UNC.) *" Subvariedades Homogéneas del Espacio Hiperbólico."*

María J. Druetta (FaMAF. UNC) *" Métricas invariantes en el ejemplo generalizado de Pyatetskii-Shapiro."*

Guillermo Keilhauer (FCEyN. UBA) *"Tensores del tipo (0,2) sobre fibrados tangentes (II)."*

Bernardo Molina, Carlos Olmos (FaMAF. UNC.) *" Rango y Simetría de Variedades Riemannianas."*

Ecuaciones Diferenciales y Modelos

E. Lami Dozo, M. C. Mariani (FCEyN. UBA. - IAM. CONICET) *" Soluciones al problema de Plateau para la ecuación de curvatura media prescripta vía el Lema del Paso de la Montaña."*

M. Mariani, D. F. Rial (FCEyN. UBA- IAM. CONICET) *" Soluciones de la ecuación de curvatura media prescripta mediante técnicas de punto fijo."*

T. Godoy, E. Lami Dozo, S. Paczka. (IAM. CONICET- UBA- FaMAF. UNC) *"El problema parabólico periódicos de autovalores con peso* L^{∞}*."*

Marcela C. Falsetti (U.N. Gral Sarmiento) *"Aplicación de un nuevo modelo funcional al análisis de imágenes."*

María E. Torres, Lucas Gamero, Carlos D'Attellis (U.N. Entre Ríos Fac de Ing. y Bioingeniería- UBA.- Fac. de Ing.) *"Detección de patrones en señales no lineales mediante entropía multirresolución."*

Mónica Bocco (FCA. UNC) *" Análisis y Medida de la Mortalidad a través de los Años de Vida Perdidos. Su relación con la Esperanza de Vida."*

Graciela A. Canziani (UN Centro de la Pcia. Bs. As.) *" Modelo matemático de dinámica poblacional para copepodos calanoides."*

(*)Nora E. Muler (FCEyN. UBA.) *" Cota Uniforme para una discretización de una ecuación parabólica forward-backward."*

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RESEÑA DEL LIBRO ALGEBRA DE MICHAEL ARTIN

NICOLÁS ANDRUSKIEWITSCH

Algebra, M. Artin. Birkhäuser, Basel, 1993, 720 páginas. ISBN 3-7643-2927-0

Important though the general concepts and propositions may be with which the modern and industrious passion for axiomatizing and generalizing has presented us, in algebra perhaps more than anywhere else, nevertheless I am convinced that the special problems in all their complexity constitute the stock and core of mathematics, and to master their difficulties requires on the whole the harder labor.

HERMANN WEYL

Este libro, una introducción al álgebra diseñada para los primeros cursos universitarios de esta materia, está basado en notas de clases dictadas por el autor a lo largo de veinte años. La fuente de esta reseña es la edición en alemán de Birkhäuser (1993), traducción de la versión original en inglés publicada por Prentice Hall en 1991.

El enfoque adoptado para la elección de los temas y su presentación se sustenta, como lo expresa el autor en el prefacio, en los siguientes principios:

1. Los ejemplos fundamentales deben preceder a las correspondientes definiciones.

2. El libro no es una obra de consulta, de modo que puntos técnicos son desarrollados únicamente si son necesarios.

3. Los temas tratados deben ser significativos para todo matemático.

En este espíritu, ilustrado por la cita de H. Weyl que sirve de epígrafe al Prefacio del libro- y a esta reseña-, se privilegia el estudio de temas particulares, como simetrías, grupos lineales y extensiones cuadráticas de $\mathbb{Q}$.

El libro consta de catorce capítulos y un apéndice, donde se presentan algunos resultados y nociones de uso en el texto principal. A continuación se describen someramente los contenidos del libro.

Los primeros cuatro capítulos ("Matrices", "Grupos", "Espacios vectoriales", "Transformaciones lineales") cubren definiciones y resultados básicos.

La segunda parte del libro atañe a los grupos y sus relaciones con la geometría. Así, en el capítulo quinto ("Simetrías") se estudian las acciones de los grupos ortogonales en dos y tres dimensiones, y sus subgrupos discretos. Por ejemplo, se clasifican los subgrupos discretos de SO_3. El capítulo sexto ("Más sobre grupos") incluye, entre otros tópicos, los teoremas de Sylow, la clasificación de los grupos de orden 12, los grupos simétricos y la presentación de un grupo por generadores y relaciones. El capítulo séptimo está dedicado a las formas bilineales: clasificación de las formas bilineales simétricas y antisimétricas, formas hermíticas, teorema espectral. En el capítulo octavo ("Grupos lineales"), se definen los grupos clásicos y se estudia en detalle la estructura geométrica del grupo especial unitario SU_2. Se discuten los subgrupos monoparamétricos y las álgebras de Lie de los grupos clásicos. El capítulo noveno contiene los elementos básicos de la teoría de representaciones de dimensión finita: caracteres, relaciones de ortogonalidad, lema de Schur. Se clasifican las representaciones irreducibles del grupo del icosaedro y de SU_2.

La tercera parte del libro concierne a la aritmética y al álgebra conmutativa. En el capítulo décimo se introducen nociones básicas de la teoría de anillos; se esboza la relación entre álgebra conmutativa y geometría algebraica. En el capítulo undécimo se considera la factorialidad, a través de ejemplos- anillos de enteros en extensiones cuadráticas de los racionales- y de condiciones axiomáticas- dominios de ideales principales, dominios euclídeos. Se discute la factorización en ideales primos y el grupo de clases. En el capítulo duodécimo ("Módulos") se parte de la definición y se llega a la clasificación de los grupos abelianos finitamente generados; la prueba de este resultado es adaptada para obtener las formas racional y de Jordan de un endomorfismo de un espacio vectorial. El capítulo decimotercero está consagrado a la teoría de cuerpos e incluye, por ejemplo, la clasificación de los cuerpos finitos, una prueba del teorema fundamental del álgebra y la determinación de los puntos del plano constructibles con regla y compás. El capítulo decimocuarto y último ("Teoría de Galois") aborda el teorema de Galois y aplicaciones: ecuaciones solubles por radicales, ecuaciones de quinto grado, extensiones de Kummer y ciclotómicas.

Cada capítulo concluye con una larga lista de ejercicios; aquí también, como en el texto principal, se enfatiza la consideración de ejemplos y casos particulares.

El estilo del autor es claro, ameno y abundante en motivaciones. Así, por ejemplo, la definición de grupo en el capítulo 3 ocupa las páginas 40 a 42. La exposición de algunos temas se complementa con enunciados de teoremas más avanzados, sin demostración.

Hay una permanente intención de desarrollar en el estudiante una adecuada intuición mediante ejemplos e interpretaciones geométricas, así como de relacionar al álgebra con otras ramas de la matemática; por caso, en la sección 7 del capítulo 4 (Transformaciones lineales) se explica la resolución de sistemas lineales (diagonalizables) de ecuaciones diferenciales. En contrapartida, se minimiza deliberadamente el empleo de métodos axiomáticos; así por ejemplo, el principio de inducción es presentado en la página 397, capítulo 10.

Indudablemente este libro es un aporte valioso a la enseñanza del álgebra en el inicio del ciclo universitario y su uso, de provecho para el docente como fuente de ejemplos, permite acceder a las definiciones fundamentales del álgebra moderna a través de importantes problemas particulares de enunciado sencillo. Sin embargo, el autor de esta reseña vacila en sugerir ceñirse estrictamente al punto de vista mantenido en esta obra; a su juicio, un matemático moderno precisa también manejar con soltura las técnicas axiomáticas y no es desdeñable la idea de familiarizar al estudiante con ellas desde su ingreso a la Universidad.

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