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$\mathcal{T}$ is in general non compatible with the vector space structure, it has the remarkable advantage of a simpler description; for instance it is easy to characterize the $\mathcal{T}$-closed sets.

In each case we derive some properties of these topologies; in particular we prove that $\mathcal{V} = \mathcal{L}$ (theorem 5.1 below), and also that $\mathcal{G} = \mathcal{V}$ (in a more general setting, as theorem 4.1 shows); next, we shall be interested to find conditions that guarantee the remainder equality $\mathcal{T} = \mathcal{G}$ of topologies. Since it is clear that this equality is equivalent to the assertion that $\mathcal{T}$ is compatible with the group structure, it turns out that this question involves the continuity of the sum map $E \times E \to E$ and a full discussion of the conditions that relate in general the product topology $\mathcal{T} \times \mathcal{T}$ in $E \times E$ with the inductive limit of the sequence $E_n \times E_n$. This can be done, for instance, when all the maps $E_n \to E_{n+1}$ are compact mappings (theorem 2.9 and its consequence 5.2 part b) in the locally convex setting), a condition fulfilled in the quoted concrete applications.

1. PRELIMINARIES

We start with an ordered set I; a directed or inductive system $\mathbb{X} = (X_i, f_{ij})$ over I is a family X_i of objects of a category $\mathcal{C}$, together with maps f_{ij} of $\mathcal{C}$, defined for $i \leq j$, and such that:

(L1) $f_{ik} = f_{jk} \circ f_{ij}$, when $i \leq j \leq k$, and

(L2) $f_{ii} =$ identity of X_i, for each $i \in I$.

A morphism $\Phi\colon \mathbb{X} = (X_i, f_{ij}) \longrightarrow (Y_i, g_{ij}) = \mathbb{Y}$ is a collection of $\mathcal{C}$-morphisms $u_i\colon X_i \to Y_i$ such that $u_j \circ f_{ij} = g_{ij} \circ u_i$ for all $i \leq j$ in I; the directed systems over I of objects and morphisms of $\mathcal{C}$, together with this notion of morphisms conform a new category denoted by $\mathrm{Dir}(\mathcal{C}, I)$. The directed or inductive limit $\mathrm{L}(\mathbb{X})$ of a directed system $\mathbb{X} = (X_i, f_{ij})$ is an object X of $\mathcal{C}$, and a family of $\mathcal{C}$-morphisms $f_i\colon X_i \to X$, such that $f_j \circ f_{ij} = f_i$ holds for each $i \leq j$, and such that for any object Y of $\mathcal{C}$ and any family of $\mathcal{C}$-morphisms $u_i\colon X_i \to Y$ which verifies $u_j \circ f_{ij} = u_i$ for every $i \leq j$, there exists a unique morphism $u\colon X \to Y$ such that $u \circ f_i = u_i$ for all $i \in I$.

Clearly this X is unique, up to $\mathcal{C}$-isomorphisms; however the existence of directed limit for objects of $\mathrm{Dir}(\mathcal{C}, I)$ depends on some particular properties of $\mathcal{C}$ and I. We say that $\mathcal{C}$ admits directed limits over I when every object of $\mathrm{Dir}(\mathcal{C}, I)$ has directed limit. This fact can be stated as follows: there is an obvious inclusion functor $\xi = \xi_I\colon \mathcal{C} \to \mathrm{Dir}(\mathcal{C}, I)$, defined by the rule $\xi(A) = (A,\ \text{identity of } A)$, and $\xi(f)_i = f$ for $f\colon X \to Y$. Then the previous definition can be formulated as

$$\mathcal{C} - \mathrm{Hom}(L(\mathbb{X}), Y) = \mathrm{Dir}(\mathcal{C}, I) - \mathrm{Hom}(\mathbb{X}, \xi(Y))$$

for every pair of objects $\mathbb{X}$ and Y.
Hence $\mathcal{C}$ admits directed limits over I if and only if ξ has L as a left adjoint functor; this is the so called "universal property" of the direct limit functor.

Remarks

1.1. The notation $\mathrm{L}(\mathbb{X}) = \varinjlim X_i$ is usual.

1.2. Assume that I has a greatest element, say ω. Then for every object $\mathbb{X}$ we have $\varinjlim X_i = X_\omega$, together with the morphisms $f_i = f_{i\omega}$, for $i \in I$.

1.3. When I is discrete, that is $i = j \iff i \leq j$, the directed limit is called "direct sum", with the traditional notation $\mathrm{L}(\mathbb{X}) = \sum_{i\in I} X_i$. We say that $\mathcal{C}$ admits direct sums over a set I when there exists the direct limit functor $\mathrm{Dir}(\mathcal{C}, I(\text{discrete})) \to \mathcal{C}$.

From now on we assume familiarity with the standard formal facts about inductive limits in categories (see [6] for instance); we only state here the following facts to be used sistematically in the sequel: suppose $F \colon \mathcal{C} \longrightarrow \mathcal{C}'$ is a covariant functor; if we denote by F_I the obvious extension of F as a functor $\mathrm{Dir}(\mathcal{C}, I) \longrightarrow \mathrm{Dir}(\mathcal{C}', I)$, then we have

Lemma 1.4. *If both $\mathcal{C}$, $\mathcal{C}'$ admit inductive limits over I, then there exists a natural morphism $\varinjlim F(X_i) \to F(\varinjlim X_i)$ for all $\mathbb{X}$. If F has a right adjoint, then this morphism is a natural isomorphim.*

Proof. The first assertion is obvious; for the second one we consider the right adjoint G of F, and observe that G_I is a right adjoint of F_I. Then

$$\begin{aligned}
\mathcal{C}' - \mathrm{Hom}(L'(F_I(\mathbb{X})), Y') &= \mathrm{Dir}(\mathcal{C}', I) - \mathrm{Hom}(F_I(\mathbb{X}), \xi(Y')) \\
&= \mathrm{Dir}(\mathcal{C}, I) - \mathrm{Hom}(\mathbb{X}, \xi(G(Y')) \\
&= \mathcal{C} - \mathrm{Hom}(L(\mathbb{X}), G(Y')) \\
&= \mathcal{C} - \mathrm{Hom}(F(L(\mathbb{X})), Y')
\end{aligned}$$

holds for every pair of objects $\mathbb{X}$ and Y'. Hence $L' \circ F_I = F \circ L$, as desired □

Example 1.5. a) In the category of sets the inductive limit of any directed system $\mathcal{X} = (X_i, f_{ij})$ can be obtained by a standard process (see [2]).
b) If A is a commutative ring with identity 1, and $\mathcal{A}$ denotes the category of A-modules and homomorphisms, we can construct the inductive limit of any object $\mathbb{E} = (E_i, f_{ij})$ in $\mathrm{Dir}(\mathcal{A}, I)$ as follows: first we construct the direct sum $\bigoplus_{i\in I} E_i = S(\mathbb{E})$ as the submodule of the product module $\prod_{i\in I} E_i$ which consists of all families $x = (x_i)_{i\in I}$ such that $\mathrm{supp}(x) = \{i \in I : x_i \neq 0\}$ is finite, together with the monomorphisms $h_i \colon E_i \to S(\mathbb{E})$ defined by $(h_i(x))_k = x$ if $i = k$, and 0 otherwise.

Now, for $i \leq j$ we set $d_{ij}\colon E_i \to S(\mathbb{E})$ as $d_{ij}(x) = h_i(x) - h_j(f_{ij}(x))$; let N_{ij} be the image of d_{ij}, and let N be the submodule generated by all the N_{ij} with $i \leq j$ in I. Finally the quotient module $q\colon S(\mathbb{E}) \to E = S(\mathbb{E})/N$ and the homomorphisms $f_i = q \circ h_i\colon E_i \to E$ give the inductive limit of $\mathbb{E}$. Note that E is generated by the submodules $f_i(E_i)$, so that $E = \bigcup_{i \in I} f_i(E_i)$ when I is a directed set. This shows also that the set E coincides with the inductive limit of the inductive system $\mathbb{X}$ of sets, when I is directed.

2. INDUCTIVE LIMITS OF TOPOLOGICAL SPACES

We consider here inductive limits in the category $\mathcal{T}$ of topological spaces and continuous maps. Let $\mathbb{X} = (X_i, f_{ij})$ be a directed system in $\mathcal{T}$ and let X be its inductive limit in the category of sets. We obtain the inductive limit of $\mathbb{X}$ in $\mathcal{T}$ if the finest topology for which all the maps $f_i\colon X_i \to X$ are continuous is given to X (the so-called final topology on X for the family f_i).

In particular, if I is discrete, then we obtain the "topological sum", that is the set $\sum_{i \in I} X_i$ with the obvious topology: a set U is open $\Leftrightarrow$ $U \cap X_i$ is open in X_i for every $i \in I$.

It is clear from the definitions that the topology of X is also the quotient topology of this topological sum via the canonical map $p\colon \sum_{i \in I} X_i \to X$, since $A \subset X$ is open (closed) $\Leftrightarrow$ $f_i^{-1}(A)$ is open (closed) in X_i for every $i \in I$.

Lemma 2.1. *Let $(u_i)\colon \mathbb{X} \to \mathbb{Y}$ be a map of inductive systems of topological spaces, and suppose each map $u_i\colon X_i \to Y_i$ is a quotient map. Then the induced map $u\colon \varinjlim X_i \to \varinjlim Y_i$ is also a quotient map (here "$A \to B$ is a quotient" means that B has the finest topology which makes the map continuous).*

Proof. This follows from the definition of the open sets in the inductive limit □

Now suppose X is the limit of an inductive system $\mathbb{X} = (X_i, f_{ij})$ of topological spaces and continuous maps, together with the maps $f_i\colon X_i \to X$ $(i \in I)$. If A is a subspace of X, then we have an inductive subsystem $\mathbb{A} = (A_i, f_{ij}|A_i)$, where $A_i = f_i^{-1}(A)$. Hence the identity map $\varinjlim A_i \to A$ is a bijective continuous map; in other words, the induced topology of A is surely weaker than the limit topology derived from $\mathbb{A}$. We note that in general these two topologies in A are distinct. However:

Lemma 2.2. *In the previous situation, assume that A is locally closed in X. Then $\varinjlim A_i = A$ as topological spaces.*

Proof. This is very easy if A is open, or if A is closed. The general case follows from this since a locally closed set is closed in an open set □

These kind of complications mean that the analogous of 2.1 is false when we replace "quotient" by "subspace"; in fact, if X is any T_2 (Hausdorff) space and we consider the directed system of the finite subspaces F of X, then $\varinjlim F = (X, \text{discrete})$.
On the other hand, now suppose for each $i \in I$ we have a subspace S_i, in such a way that $f_{ij}(S_i) \subset S_j$ when $i \leq j$ in I, and consider the inductive system $\mathbb{S} = (S_i, f_{ij}|S_i)$.

Lemma 2.3. *In the previous situation, assume that I is a directed set and that all S_i are open. Then the subspace $S = \bigcup_{i\in I} f_i(S_i)$ of X is open and it identifies with the topological space $\varinjlim S_i$.*

Proof. First note that $A_i = \bigcup_{j\geq i} f_{ij}^{-1}(S_j)$ is open in X_i, $S_i \subset A_i$ if $i \in I$, and that $f_{ik}^{-1}(A_k) = A_i$ for $i \leq k$ in I. Hence $S = \bigcup_{i\in I} f_i(A_i)$ and $f_i^{-1}(S) = A_i$ for each $i \in I$ and the remainder follows from 2.2 □

Corollary 2.4. *If I is a directed set, and $\Phi \colon \mathbb{X} \to \mathbb{Y}$ is a morphism in $\mathcal{T}$ such that all the maps $u_i \colon X_i \to Y_i$ are open, then the map $u \colon \varinjlim \mathbb{X} \to \varinjlim \mathbb{Y}$ is also open.*

Remark 2.5. By the construction above we can define a basis for the neighborhood system of a point $a \in X$ when I is directed, as follows: pick an $i \in I$ and a point $a_i \in X_i$ such that $f_i(a_i) = a$. Then for any $j \geq i$ we set $a_j = f_{ij}(a_i)$, and select for each $j \geq i$ an open set $U_j \subset X_j$ in a basis of the neighborhood system of a_j, such that $f_{jk}(U_j) \subset U_k$ when $i \leq j \leq k$, and finally set $U = \bigcup_{j\geq i} f_j(U_j)$. Then U is a neighborhood of a, and any neighborhood of a in X contains a set of this kind.
The difficulties encountering when questions of subspaces appear in inductive limits occur also when we consider cartesian products. In fact, suppose $\mathbb{X}$ and $\mathbb{Y}$ are inductive systems of topological spaces, and set $X = \varinjlim X_i$, $Y = \varinjlim Y_j$. Then we have another inductive system $\mathbb{X} \times \mathbb{Y}$, and hence a natural continuous bijection

$$\xi \colon \varinjlim (X_i \times Y_j) \longrightarrow X \times Y$$

induced by the obvious map of $\sum_{ij}(X_i \times Y_j)$ onto $(\sum_i X_i) \times (\sum_j Y_j)$, which is compatible with the equivalence relations in both spaces.
In the sequel we analyze this map under suitable conditions.

Theorem 2.6. *If Y is a locally compact space, then for any inductive system $\mathbb{X}$ of topological spaces the natural map*

$$\varinjlim (X_i \times Y) \to (\varinjlim X_i) \times Y$$

is a homeomorphism.

Proof. A direct argument is possible, but we prefer a more structural one, following the ideas of lemma 1.4: let $F\colon \mathcal{T} \to \mathcal{T}$ be the functor given by $F(X) = X \times Y$ on spaces and $F(u) = u \times id_Y$ on maps. Now F has a left adjoint functor, which associates to each space Z the space $C_c(Y, Z)$ of continuous maps with the compact-open topology (see [1]) □

It should be noted that any attempt to develop further results of this type require some assumptions on the spaces X_i or else on the maps f_{ij} involved. In this sense, a very useful notion is the following:

Definition 2.7. *A continuous map $f\colon X \to Y$ between T_2 (or Hausdorff) spaces is compact if for every $x \in X$ there exists a basis $\mathcal{B}(x)$ of neighborhoods of x such that $f(U)$ has compact closure for every $U \in \mathcal{B}(x)$.*

Remarks 2.8. a) If f is a compact mapping, $K \subset X$ is compact, and Ω is a neighborhood of K, then there exists a neighborhood V of K such that $K \subset V \subset \Omega$ and $f(V)^-$ is compact in Y.

Proof. We have an open covering V_x $(x \in K)$ such that $V_x \subset \Omega$ and $f(V_x)^-$ is compact for each $x \in K$. Then we have a finite subcovering V_i $(i = 1, \ldots, n)$ and $V = \bigcup_{i=1}^{n} V_i$ solves the problem.

b) If $u_i\colon X_i \to Y_i$ $(i = 1, 2)$ are compact mappings and $K_i \subseteq X_i$ are compact sets, then for any neighborhood Ω of $K_1 \times K_2$ in $Y_1 \times Y_2$ there exist neighborhoods V_1 of K_1, V_2 of K_2 such that $K_1 \times K_2 \subset V_1 \times V_2 \subset \Omega$, and also each $u_i(V_i)^-$ $(i = 1, 2)$ is compact.

Proof. Since each K_i $(i = 1, 2)$ is compact, we have open sets U_i $(i = 1, 2)$ such that $K_1 \times K_2 \subset U_1 \times U_2 \subset \Omega$. Then a) gives the desired $V_i \subset U_i$ for $i = 1, 2$.

Theorem 2.9. *Let (X_i, f_{ij}), (Y_i, g_{ij}) be inductive systems of T_2 topological spaces, and assume that*

(1) I is a denumerable directed set, and

(2) All maps f_{ij}, g_{ij} are compact mappings.

Then the natural continuous bijection

$$\varinjlim (X_i \times Y_j) \to (\varinjlim X_i) \times (\varinjlim Y_j)$$

is a homeomorphism.

Proof. Since any countable directed set has a cofinal sequence, we can assume without loss of generality that $I = \mathbb{N}$. Let $a \in X = \varinjlim X_n$, $b \in Y = \varinjlim Y_n$ and

let W be a neighborhood of (a,b) in $\varinjlim(X_n \times Y_n)$; we must show that there exists a neighborhood U (resp. V) of a (resp. b) such that $U \times V \subset W$.
Now we can assume that $a \in f_0(X_0)$, $b \in g_0(Y_0)$, hence $(f_n \times g_n)^{-1}(W) = W_n$ is open in $X_n \times Y_n$ for each n. In particular $(a_0, b_0) \in W_0$, where $f_0(a_0) = a$, $g_0(b_0) = b$; then we have an open neighborhood U_0 (resp. V_0) of a_0 (resp. b_0) such that $U_0 \times V_0 \subset W_0$, and $\overline{f_{01}(U_0)} \times \overline{g_{01}(V_0)} \subset W_1$ is compact in $X_1 \times Y_1$.
If we assume that there are open neighborhoods U_i, V_i of $f_{0i}(a_0)$, $g_{0i}(b_0)$, defined for $0 \leq i \leq n$ in such form that

(1_n) $U_i \times V_i \subset W_i$ $(0 \leq i \leq n)$,
(2_n) $f_{ij}(U_i) \subset U_j$, $g_{ij}(V_i) \subset V_j$ $(0 \leq i \leq j \leq n)$,
(3_n) $\overline{f_{n,n+1}(U_n)} \times \overline{g_{n,n+1}(V_n)} \subset W_{n+1}$ is compact in $X_{n+1} \times Y_{n+1}$.

By using 2.8 we can define U_{n+1}, V_{n+1} in such a way that 1_{n+1}, 2_{n+1}, 3_{n+1} hold. Finally we set

$$U = \bigcup_{i=0}^{\infty} f_i(U_i), \qquad V = \bigcup_{i=0}^{\infty} g_i(V_i)$$

in order to complete the proof (see 2.5) □

Corollary 2.10. *Let $\mathbb{X} = (X_n, f_{nm})$ be a directed system over $\mathbb{N}$ of Hausdorff topological spaces, such that every $f_{n,n+1}$ is a compact mapping. If we denote by R_{nm} the equivalence relation*

$$(x,y) \in R_{nm} \Longleftrightarrow f_{nm}(x) = f_{nm}(y)$$

well defined for $n \leq m$, then the following assertions are equivalent:

(i) $X = \varinjlim X_i$ is Hausdorff,
(ii) $\bigcup_{m=n}^{\infty} R_{nm}$ is closed in $X_n \times X_n$ for each $n \geq 0$.

In particular, $\varinjlim X_n$ is Hausdorff if all the maps $f_{n,n+1}$ are injective.

Proof. According with 2.8 the diagonal $\Delta \subset X \times X$ is closed if and only if it is closed as a subspace of $\varinjlim(X_n \times X_n)$, and this occurs if and only if $(f_n \times f_n)^{-1}(\Delta)$ is closed in $X_n \times X_n$ for each $n \geq 0$. Now observe that this set coincides with $\bigcup_{m=n}^{\infty} R_{nm}$ □

3. INDUCTIVE LIMITS OF TOPOLOGICAL ABELIAN GROUPS

Here we consider the category TAG of abelian topological groups, with morphisms the continuous homomorphisms of groups. As we have seen in 1.10, with $A = \mathbb{Z}$, if (G_i, f_{ij}) is an inductive system in TAG, then we can form the abelian group

$G = \varinjlim G_i$ and consider in G the finest topology compatible with the group structure that makes continuous all the homomorphisms $f_i \colon G_i \to G$. This shows that the category TAG has inductive limits over any ordered set I.
Alternatively we can form first the "TAG-direct sum" $S = \bigoplus_{i \in I} G_i$, that is, the algebraic direct sum of 1.11 with the finest topology compatible with the group structure which makes all the homomorphisms $h_i \colon G_i \to S$ continuous. Secondly, recalling that the quotient topology is compatible with the group structure, we see that the quotient topological group S/N coincides with the inductive limit G.

Remarks

3.1. From general considerations it follows that if (G_i, f_{ij}) is an inductive system in TAG, and for each $i \in I$ we have a subgroup $H_i \subset G_i$ such that $f_{ij}(H_i) \subset H_j$ when $i \leq j$ in I, then the homomorphisms $p_i \colon G_i \to G_i/H_i$ induce a quotient homomorphism $p \colon \varinjlim G_i \to \varinjlim G_i/H_i$; the kernel of p is the -algebraic- subgroup $\varinjlim H_i$ of $\varinjlim G_i$.
We note that, in general the topology of this subgroup is weaker than the topology of $\varinjlim H_i$ in TAG (see 5.2 c)).

3.2. If (G_i, f_{ij}), (H_i, g_{ij}) are inductive systems in TAG, then we have a natural isomorphism $\varinjlim(G_i \times H_i) \simeq (\varinjlim G_i) \times (\varinjlim H_i)$; this is clear, since in the category TAG the finite sums and finite products coincide.

3.3. If (G_i, f_{ij}) is an inductive system of abelian topological groups, the topology of the topological group $G = \varinjlim G_i$ can be defined by a set of invariant seudometrics: the set of all translation invariant seudometrics d in G such that $x \mapsto d(x,0) = N_d(x)$ is continuous in 0. Equivalently, the set of all invariant seudometrics d in G such that $N_d \circ f_i \colon G_i \to \mathbb{R}$ is continuous in 0 for each $i \in I$.

3.4. If $(G_i)_{i \in I}$ is a family of topological abelian groups, then their direct sum G in TAG is simply the algebraic direct sum, with the finest compatible topology such that all inclusion maps $G_i \to G$ are continuous. Clearly G is also the inductive limit TAG-$\varinjlim S_F$, where $S_F = \bigoplus_{k \in F} G_k$ for every finite set $F \subset I$.
Now we can also consider the algebraic direct sum as a subgroup (with the induced topology $\mathcal{T}$) of the topological abelian group P, the cartesian product of the family G_i with the so called "box-topology", a neighborhood basis of 0 in P is the family of all products $\prod_{i \in I} U_i$ ([4]), where U_i is a neighborhood of 0 in G_i for each $i \in I$.
In general this topology $\mathcal{T}$ is weaker than the topology of the TAG-direct sum.
Nevertheless we have:

Proposition 3.5. *If I is a denumerable direct set, then the topology $\mathcal{T}$ induced by the box-topology in the algebraic direct sum S is equal to the direct sum topology in TAG.*

Proof. Again we can restrict ourselves to the case $I = \mathbb{N}$, and of course it suffices to show that every 0-neighborhood U in the direct sum topology G is a 0-neighborhood in $\mathcal{T}$. There exists a sequence U_n $(n \geq 0)$ of 0-neighborhoods in the direct sum topology such that $U_0 = U$ and $U_{n+1} + U_{n+1} \subset U_n$ for each $n \geq 0$. Now let $S_n = \prod_{i=1}^{n} G_i$ $(n \geq 1)$, and also let $i_n \colon S_n \to G$ be the obvious map for each $n \geq 1$. Clearly all i_n are continuous, hence for each $n > 0$ there exist 0-neighborhoods V_i^n $(1 \leq i \leq n)$ in G_i such that $V_1^n \times \cdots \times V_n^n \subset i_n^{-1}(U_n)$. If we set $V = S \cap \prod_{k=1}^{\infty} V_k^k$, then clearly we have $V \subset U$, since $U_1 + \cdots + U_m \subset U$ for every $m > 0$ □

Now let (G_i, f_{ij}) be an inductive system of abelian topological groups, and let G be the topological group $\varinjlim G_i$; we denote by G' the inductive limit of the groups G_i as topological spaces. Clearly the identity map $G' \to G$ is continuous, since this means that the finest group topology defined by the f_{ij} is weaker than the finest topology defined by the same maps.
Note that in G' the maps $x \mapsto -x$ and $x \mapsto x + a$ $(a \in G')$ are both continuous. However G' is not a topological group; in general, more precisely: G' is a topological group $\Leftrightarrow$ these two topologies are the same.

Theorem 3.6. *Let (G_i, f_{ij}) be an inductive system of topological abelian groups, and assume that*

(1) I is a denumerable directed set, and

(2) The homomorphims f_{ij} are compact mappings.

Then G' is a topological group, and $G = G'$.

Proof. This follows from theorem 2.9, since under these hypothesis the continuity of the sum map $G' \times G' \to G$, $(x, y) \mapsto x + y$ is a consequence of the continuity of all the sum maps $G_i \times G_i \to G_i$ □

Remark 3.7. With minor modifications all the previous considerations are also true for general, non necessarily commutative, topological groups.

4. INDUCTIVE LIMITS OF TOPOLOGICAL VECTOR SPACES

We denote by TVS_k the category of topological vector spaces over a valuated non discrete field k, with continuous linear maps as morphisms; for E, F topological vector spaces over k, we denote by $\mathrm{L}_k(E, F)$ the vector space of all continuous linear maps $u \colon E \to F$.
The category TVS_k has inductive limits over any ordered set I; for (E_i, f_{ij}) an inductive system in TVS_k, we can form first as in 1.5 b) the algebraic limit E

of the vector spaces and then we consider in E the finest topology of topological vector space which makes continuous all the linear maps $f_i\colon E_i \to E$. It is easy to see that this topology can be described by the set of all quasi-norms q in E such that $q \circ f_i$ is continuous for each $i \in I$ ([8]).

We precede the proof of the following result by a formal argumentation. First, we define a functor $v\colon \mathrm{TAG} \to \mathrm{TVS}_k$ by the rule $v(G) = \mathrm{TAG}-\mathrm{Hom}(\Box k, G)$, where $\Box$ denotes the forgetful functor $\mathrm{TVS}_k \to \mathrm{TAG}$; here $v(G)$ is a k-vector space when we define $(\alpha h)(x) = h(\alpha x)$ for $\alpha \in k$, $x \in k$, $h \in v(G)$. The topology of $v(G)$ has the sets $W(\epsilon, U)$ as a basis of neighborhoods of 0, where $W(\epsilon, U)$ is the set of all $h \in v(G)$ such that $h(\Delta_\epsilon) \subset U$; here we set $\Delta_\epsilon = \{\lambda \in k : |\lambda| < \epsilon\}$. This topology is compatible with the vector space structure, since:

(i) $W(\epsilon, U) + W(\epsilon, U) \subset W(\epsilon, U + U)$,
(ii) $W(\epsilon, U_1 \cap U_2) \subset W(\epsilon, U_1) \cap W(\epsilon, U_2)$,
(iii) $W(\epsilon, U)$ is balanced (or "equilibrée"),
(iv) $W(\epsilon, U)$ is absorbing;

in fact, the continuity of $h \in v(G)$ implies that for a given neighborhood U of 0 in G, there exits $r > 0$ such that $h(\Delta_r) \subset U$. Since k is non discrete we can take $\alpha \in k$, $\alpha \neq 0$ such that $|\alpha|\epsilon < r$, hence $\Delta_t h \in W(\epsilon, U)$ if $0 < t < |\alpha|$.

Of course we define $v(u)$ for a homomorphism $u\colon G_1 \to G_2$ by the rule $v(u)(h) = u \circ h$, for every $h \in v(G_1)$.

Note that the natural transformation in the TAG category

$$\Gamma_G\colon \Box(v(G)) \to G$$

is well defined by $\Gamma_G(h) = h(1)$, for every $h \in G$. Clearly $\mathrm{Ker}\,\Gamma_G$ is a closed subgroup of $v(G)$, and if S is a subspace of $v(G)$, then $S \subset \mathrm{Ker}\,\Gamma_G$ implies $S = 0$.

Note also the natural transformation in the TVS_k category

$$j_E\colon E \to v(\Box(E))$$

defined by the rule $j_E(x)(\lambda) = \lambda x$ ($x \in E$, $\lambda \in k$).

Now v is a right adjoint of $\Box$. In fact we can define two natural group homomorphisms

$$f\colon \mathrm{TAG}-\mathrm{Hom}(\Box(E), G) \to \mathrm{L}_k(E, v(G)),$$
$$g\colon \mathrm{L}_k(E, v(G)) \to \mathrm{TAG}-\mathrm{Hom}(\Box(E), G),$$

according with the following formulas:

$$f(\phi) = v(\phi) \circ j_E, \qquad g(u)(x) = u(x)(1) = \Gamma_G \circ u(x) \ \ (x \in E).$$

Clearly $g \circ f$ is the identity, and it is easy to see that $f \circ g$ is injective (since $f(g(u)) = 0$ implies that $u(E)$ is a subspace of $\operatorname{Ker} \Gamma_G$). Hence f, g are reciprocal isomorphisms.

Theorem 4.1. *Let (E_i, f_{ij}) be an inductive system of topological k-vector spaces and continuous linear maps. Then $TAG\text{-}\varinjlim \square E_i = TVS_k - \varinjlim E_i$, in other words in the algebraic inductive limit E, the finest topology compatible with the group structure which makes continuous all the f_i is also compatible with the k-vector space structure.*

Proof. It is a consequence of the preceeding construction and lemma 1.4 □

Remarks

4.2. A direct and more elementary method of proving 4.1 is to show -in a previous lemma- that the balanced hull V of a neighborhood of 0 in the topological group $E_0 = \text{TAG} - \varinjlim E_i$ is also a neighborhood of 0, since the maps $\lambda \to \lambda x$, $x \to \lambda x$ are continuous in this topology. After this lemma, it follows that E_0 is a topological k-vector space, hence the result 4.1.

4.3. When k is a locally compact field, 4.1 is a direct consequence of 2.6, which implies the automatic continuity of the map $k \times E_0 \to E_0$.

4.4. For $(E_i)_{i \in I}$ an arbitrary family of topological vector spaces, the box-topology in $\prod_{i \in I} E_i$ in general is not compatible with the vector space structure (the sets $\prod_{i \in I} U_i$ fails to be absorbent); but it is easy to see that the restriction $\mathcal{B}$ of this topology to the algebraic direct sum S makes this subspace a topological vector space; in general $(S, \mathcal{B})$ is weaker than the TVS$_k$-direct sum topology. But when I is a denumerable directed set it follows from 3.5 and 4.1 that $(S, \mathcal{B})$ is the direct sum in the TVS$_k$-category.

5. INDUCTIVE LIMITS OF LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

Here $k = \mathbb{R}$ or $\mathbb{C}$, and we look the category LCS of locally convex topological vector spaces as a full subcategory of TVS$_k$. For (E_i, f_{ij}) an inductive system in LCS, we consider in the algebraic inductive limit E_L the finest locally convex topology which makes continuous all those maps f_i; this construction shows that LCS has inductive limits over any ordered set.

A basis of 0-neighborhoods in the inductive limit E can be found by taking the convex hull of the sets $\bigcup_{i \in I} U_i$, where each U_i is an absolutely convex neighborhood of 0 in E_i for each $i \in I$.

Alternatively we can describe the topology of the inductive limit as the topology defined by all the seminorms p in E_L, such that $p \circ f_i \colon E_i \to \mathbb{R}$ is continuous for each $i \in I$.

For instance, in any real or complex vector space V we can consider the topology of $\varinjlim E_i$, where the E_i's run over all the finite dimensional subspaces of V (provided with the unique compatible Hausdorff topology); this topology is the finest locally convex topology in V, and it is defined by all seminorms in V. In special, the LCS-direct sum $\mathbb{C}^{(I)}$ equals $\varinjlim \mathbb{C}^F$ (here F denotes all finite subsets of I), and it is defined by all seminorms $p \colon \mathbb{C}^{(I)} \to \mathbb{R}$.

For (E_i, f_{ij}) an inductive system in LCS, in general the identity map $\mathrm{TVS}_k - \varinjlim E_i \to \mathrm{LCS} - \varinjlim E_i$ is only continuous; the two topologies in the algebraic limit E_L are in general different. The best result in this direction is the following:

Theorem 5.1. *In the previous situation, assume that I is a denumerable directed set. Then the two topologies in E_L are equal; in other words the inductive limit is the same in TVS_k and in LCS.*

Proof. Again we can restrict ourselves to the case $I = \mathbb{N}$, and we must prove under this hypothesis that $\mathrm{TVS}_k - \varinjlim E_i$ is locally convex. We give the steps of the proof:

a) First case: here we assume that the order of I is discrete, in other words we must prove that the TVS_k-direct sum E is locally convex. But this follows quickly from 4.4 since the sets $E \cap \prod_{i \in I} U_i$ are convex when all the U_i are convex.

b) General case: Note that $L = \mathrm{TVS}_k - \varinjlim E_i$ can be identify with E/N, where N is the kernel of the natural map $(x_i)_{i \in I} \mapsto \sum_{i \in I} f_i(x_i)$, and $f_i \colon E_i \to L$ are the canonical maps. Since the quotient of a locally convex space is also locally convex the general case follows from a) above $\square$

Remarks 5.2. a) Let E_n $(n \geq 0)$ be a sequence of locally convex spaces, and let $u_n \colon E_n \to E_{n+1}$ $(n \geq 0)$ be a continuous linear map. We obtain an inductive system $\mathbb{E}$, by setting $u_{nm} = u_{m-1} \circ \cdots \circ u_n$ for $n \leq m$. If E denotes the algebraic inductive limit, with the maps $f_n \colon E_n \to E$, then 4.1 and 5.1 show that the three limit topologies (defined when we consider $\mathbb{E}$ alternatively in the categories TAG, TVS_k or LCS) are the same.

b) If we assume that all maps u_n are compact, then 3.6 implies that these topologies are also equal to the topology of $\mathcal{T} - \varinjlim E_n$. In particular a set $A \subset E$ is closed $\Leftrightarrow$ $f_n^{-1}(A)$ is closed for every $n \geq 0$. This is a very strong form of theorem 6 of [5].

c) In the situation of b), if $S \subset E$ is a closed vector subspace, and we consider the inductive system defined by the subspaces $f_n^{-1}(S) = S_n$, then lemma 2.2 and the previous remark imply that the topology of LCS-$\varinjlim S_n$ coincides with the induced topology of S. This fact is in general false, even in the case of strict inductive limits, if we drop some hypothesis about the maps u_n (see [7] for an example).

Remark 5.3. In especial, in the vector space $\mathbb{R}^{(\mathbb{N})}$ (or in $\mathbb{C}^{(\mathbb{N})}$) the topology defined by saying that a set A is closed (resp. open) when $A \cap V$ is closed (resp. open) for each finite dimensional subspace V is a locally convex topology. This fact fails when the algebraic dimension is not countable (see below).

6. SOME EXAMPLES

6.1. Theorem 5.1 is in general false when I is a non-denumerable set. For instance, if I is non-denumerable, then we can consider in $E = \mathbb{R}^{(I)}$ the two topologies T_1, T_2 defined as the inductive limit over the finite dimensional subspaces in the category TVS and LCS respectively. Then T_1 is defined by all the invariant seudometrics continuous on finite dimensional subspaces (or else by all the quasi-norms in E, [8]), and T_2 is defined by all the seminorms in E; they define the direct sum of I-copies of the scalar field in the two categories and we can see that they are different topologies. In fact, T_1 is not locally convex.

Proof. Let $q(x) = \sum_{i\in I} |x_i|^{1/2}$, a continuous invariant metric in E; then the set $\{x \in E : q(x) \leq 1\}$ is not a 0-neighborhood in T_2. On the contrary, if p is a seminorm in E with $q \leq p$, then we must have $1 \leq p(e_i)$ for each vector e_i of the canonical basis of E. Also for some n the set $G = \{i \in I : n \leq p(e_i) \leq n+1\}$ is infinite, let n_0 be the first n for which this fact is true. Hence for every finite set $F \subset G$ the vector $x_F = \sum_{i\in F} \left(\frac{1}{n_F}\right) e_i$, where $n_F = \operatorname{card} F$, gives $q(x_F) = \sqrt{n_F} \leq p(x_F) \leq \frac{n_0+1}{n_F}$ or $n_F^{3/2} \leq n_0 + 1$ for every finite $F \subset G$, which is clearly absurd.

6.2. The following example shows that the considerations of 5.2 are false if we drop some hypothesis about the maps of an inductive system.

Let $j_n \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, $j_n(\lambda_1, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_n, 0)$ be the canonical inclusion and let $E_n = \ell^1 \times \mathbb{R}^n$. Now we have a denumerable direct system of Banach spaces defined by the maps $id \times j_n \colon E_n \to E_{n+1}$. Algebraically the limit space is $E = \ell^1 \times \mathbb{R}^{(\mathbb{N})}$, and 5.2 implies the equality of all limit topologies in $\mathbb{R}^{(\mathbb{N})}$; hence the identity map $\varinjlim(\ell^1 \times \mathbb{R}^n) \to \ell^1 \times \mathbb{R}^{(\mathbb{N})}$ is an isomorphism in the categories TAG, TVS and LCS. We shall see that this fact is false for the category $\mathcal{T}$; in fact we construct a map $u \colon E \to \ell^\infty$ continuous in $\varinjlim E_n$ and discontinuous in $\ell^1 \times \mathbb{R}^{(\mathbb{N})}$.

For this, let $F\colon \ell^1 \to \mathbb{R}$ be the continuous map defined by $F(x) = \|x\|_\infty$ when $\|x\|_\infty + \|x\|_1 \geq 1$, and $F(x) = 1 - \|x\|_1$ when $\|x\|_\infty + \|x\|_1 \leq 1$; then for every positive α in $\mathbb{R}$ set $F_\alpha(x) = \alpha F(\alpha^{-1}x)$, and finally for each $n \geq 1$, let $f_n = F_{1/n}$. Note that $\inf\{f_n(x) : \|x\|_1 = 1/n\} = 0$, since $\inf\{F(x) : \|x\|_1 = 1\} = 0$.

Now define u as $u(x,\lambda) = (\lambda_n/f_n(x))_{n\geq 1}$, which is clearly continuous on $\varinjlim E_n$, but $U = \{(x,\lambda) : \|u(x,\lambda)\|_\infty \leq 1\}$ is not a 0-neighborhood in $\ell^1 \times \mathbb{R}^{(\mathbb{N})}$; suppose the contrary, then for some seminorm p in $\mathbb{R}^{(\mathbb{N})}$ and some positive r we must have: $f_n(x) \geq \lambda_n$ for every n, when $\|x\|_1 + p(\lambda) \leq r$. But this is absurd, since we can take $k \in \mathbb{N}$ such that $1/k \leq 1/2r$ and then select some positive α such that the vector $b \in \mathbb{R}^{(\mathbb{N})}$ given by $b = \sum_{i=1}^{k} \alpha e_i$ verifies $p(b) \leq 1/2r$. This in turn means that for every $x \in \ell^1$ such that $\|x\|_1 = 1/k$ we must have $f_k(x) \geq \alpha$, a contradiction.

Note that the above argument implies that the set U is a 0-neighborhood in the space $\mathcal{T}-\varinjlim(\ell^1 \times \mathbb{R}^n)$, but U is not a 0-neighborhood in the space $\ell^1 \times (\mathcal{T}-\varinjlim \mathbb{R}^n)$; this shows that the conditions about the spaces or the maps in theorems 2.6 and 2.9 are non superfluous.

References

1. R. Arens, *A topology for spaces of transformations*, Ann. of Math(2) (1946), 480-495.
2. N. Bourbaki, *Elements de Mathématique, Livre I, Theorie des ensembles*, Chap. III, §7, No 5, Hermann Paris, 1970.
3. J. Horvath, *Topological Vector Spaces and Distributions*, Addison-Wesley, 1966.
4. J.Kelley, *General topology*, Van Nostrand, Inc, 1959.
5. H. Komatsu, *Proyective and Injective limits of weakly compact sequences of locally convex spaces*, J. Math. Soc. Japan **19** No 3 (1967), 366-383.
6. S. Mac Lane, *Categories for the Working Mathematician*, GTM 5, Springer-Verlag, 1971.
7. V. Retah, *Subspaces of a countable inductive limit*, Soviet Math. Dokl. **11** No 5 (1970), 1384-1386.
6. C. Swartz, *An Introduction to Functional Analysis*, Pure and Appl. Math. Series, M. Dekker Inc., 1992.

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SOLUTIONS TO THE MEAN CURVATURE EQUATION FOR NONPARAMETRIC SURFACES BY FIXED POINT METHODS

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ABSTRACT

We study the existence of solutions for the equation of prescribed mean curvature when the surface is the graph of $u : \overline{\Omega} \longrightarrow R$, with mean curvature $H(x, y, u(x, y))$. We give conditions on the boundary data in order to obtain at least one solution for the quasilinear Dirichlet problem (1) below, with H a given continuous function.

INTRODUCTION

We consider the quasilinear Dirichlet problem in a bounded domain $\Omega \subset R^2$ with $\partial\Omega \in C^2$

$$(1)\begin{cases} (1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} - 2u_x u_y u_{xy} = 2H(x,y,u)\left(1+|\nabla u|^2\right)^{\frac{3}{2}} & \text{in} \quad \Omega \\ u(x,y) = \varphi(x,y) \quad \text{on } \partial\Omega \end{cases}$$

where $H : \overline{\Omega} \times [\epsilon, \epsilon] \longrightarrow R$ is continuous for some $\epsilon > 0$ and $\varphi \in W^{2,p}(\Omega)$ is the boundary data.

The problem above is the mean curvature equation for nonparametric surfaces which has been studied in general for hypersurfaces in R^{n+1} by Gilbarg, Trudinger, Simon, Serrin, Díaz, Saa and Thiel among other authors. For H independent of u it has been proved [GT] that there exists a solution for any smooth boundary data if the mean curvature H' of $\partial\Omega$ satisfies:

$$H'(x_1, ..., x_n) \geq \frac{n}{n-1}|H(x_1, ..., x_n)|$$

for any $(x_1, ..., x_n) \in \partial\Omega$, and $H \in C^1(\overline{\Omega}, R)$ satisfying the inequality:

$$|\int_\Omega H\varphi| \leq \frac{1-\epsilon}{n}\int_\Omega |D\varphi|$$

for any $\varphi \in C_0^1(\Omega, R)$ and some $\epsilon > 0$. The sharpness of the geometric condition on the curvature of $\partial\Omega$ is shown by a non-existence result ([GT], corollary 14.13): if $H'(x_1, ..., x_n) < \frac{n}{n-1}|H(x_1, ..., x_n)|$ for some $(x_1, ..., x_n)$ and the sign of H is constant, then for any $\epsilon > 0$ there exists $g \in C^\infty(\overline{\Omega})$ such that $\|g\|_\infty \leq \epsilon$ for which the Dirichlet's problem is not solvable.

On the other hand, Díaz, Saa and Thiel [DST] studied the general quasilinear elliptic equation $div(Q(|\nabla u|)\nabla u) + f(u) = g(x_1, ..., x_n)$ in R^n under Dirichlet and Neumann conditions. They studied existence and uniqueness of the problem for nonincreasing f by finding apriori bounds for ∇u. The case $Q(r) = (1 + r^2)^{-1/2}$ corresponds to the mean curvature equation (1), and the condition on f becomes: $h'(u) \geq 0$.

In the present paper we study the problem by topological methods, obtaining a solution under some restrictions on $\|H\|_\infty$ and $\|\varphi\|_{2,p}$ but avoiding the conditions on the curvature of $\partial\Omega$. The condition $\frac{\partial h}{\partial u} \geq 0$ will not be necessary either.

The general Plateau problem and the Dirichlet asociated problem, have been studied in [AMR],[BC],[H],[LD-M],[MR] [S1],[S2],[WG], etc.

The quasilinear operator associated to problem (1) is strictly elliptic since its eigenvalues are $\lambda = 1$ and $\Lambda = 1 + |p|^2$, where $p = (u_x, u_y)$ (see [GT] chapter 10).

The main result is the following theorem

THEOREM 1

Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough with respect to $|\Omega|$, the Sobolev's constant and the apriori bounds for Δ in Ω. Then there exists at least one solution $u \in W^{2,p}(\Omega)$ of (1).

SOLUTIONS BY FIXED POINT METHODS

First we note that u is a solution of (1), if and only if $w = u - \varphi$ is a solution of the following equation:

$$(2)\begin{cases} (1+(w_y+\varphi_y)^2)w_{xx} + (1+(w_x+\varphi_x)^2)w_{yy} - 2(w_x+\varphi_x)(w_y+\varphi_y)w_{xy} \\ \qquad = 2H(x,y,w+\varphi)\left(1+|\nabla(w+\varphi)|^2\right)^{\frac{3}{2}} - (1+(w_y+\varphi_y)^2)\varphi_{xx} \\ \qquad -(1+(w_x+\varphi_x)^2)\varphi_{yy} + 2(w_x+\varphi_x)(w_y+\varphi_y)\varphi_{xy} \qquad \text{in } \Omega \\ w = 0 \qquad \text{on } \partial\Omega \end{cases}$$

For each $\overline{v} \in C^1(\overline{\Omega})$ such that $\|\overline{v} + \varphi\|_\infty \leq \epsilon$ we consider the elliptic linear Dirichlet problem associated to equation (2)

$$(3)\begin{cases} L_{\overline{v}}(v) = F(\overline{v}) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where

$$L_{\overline{v}}(v) = (1 + (\overline{v}_y + \varphi_y)^2)v_{xx} + (1 + (\overline{v}_x + \varphi_x)^2)v_{yy} - 2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)v_{xy}$$

and

$$F(\overline{v}) = 2H(x, y, \overline{v} + \varphi)\left(1 + |\nabla(\overline{v} + \varphi)|^2\right)^{\frac{3}{2}} - (1 + (\overline{v}_y + \varphi_y)^2)\varphi_{xx} - (1 + (\overline{v}_x + \varphi_x)^2)\varphi_{yy}$$
$$+2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)\varphi_{xy}$$

The linear equation (3) has a unique solution $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [GT], theorem 9.15). Thus, if we consider the Sobolev imbedding $W^{2,p} \hookrightarrow C^1$ with imbedding constant k (i.e. $\|u\|_{1,\infty} \leq k\,\|u\|_{2,p}$), we may define an operator $T : \overline{B_\epsilon(-\varphi)} \subset C^1(\overline{\Omega}) \longrightarrow C^1(\overline{\Omega})$ given by $T(\overline{v}) = v$ if v is the solution of (3) for $\overline{v}$.

We'll see that the operator T has at least one fixed point in C^1, and this will give a solution of the original problem (1).

Our main tool will be the Schauder fixed point theorem (see [GT] theorem 11.1 and corollary 11.2). We prove first the following lemma and proposition.

LEMMA 2

There exists a constant C (depending only on $|\Omega|$, p) and $R > 0$ such that if

$$\|\overline{v} + \varphi\|_{1,\infty} \leq R$$

then for every $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

$$\|w\|_{2,p} \leq C\,\|L_{\overline{v}}(w)\|_p$$

Proof

We can write

$$L_{\overline{v}}(w) = \Delta w + S_{\overline{v}}(w)$$

where $S_{\overline{v}}(w) = (\overline{v}_y + \varphi_y)^2 w_{xx} + (\overline{v}_x + \varphi_x)^2 w_{yy} - 2(\overline{v}_x + \varphi_x)(\overline{v}_y + \varphi_y)w_{xy}$.

The operator Δ satisfies the hypotheses of [GT], lemma 9.17, then there exists a constant C_1 (independent of w) such that

$$\|w\|_{2,p} \leq C_1 \|\Delta w\|_p$$

for all $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then

$$\|L_{\overline{v}}(w)\|_p \geq \|\Delta w\|_p - \|S_{\overline{v}}(w)\|_p \geq \frac{1}{C_1}\|w\|_{2,p} - \|S_{\overline{v}}(w)\|_p$$

and being

$$\|S_{\overline{v}}(w)\|_p \leq 4\|\overline{v}+\varphi\|_{1,\infty}^2 \|w\|_{2,p}$$

we obtain

$$\|L_{\overline{v}}(w)\|_p \geq \left(\frac{1}{C_1} - 4\|\overline{v}+\varphi\|_{1,\infty}^2\right)\|w\|_{2,p}$$

The second member of the last inequality is positive if $\|\overline{v}+\varphi\|_{1,\infty} \leq R < \frac{1}{2\sqrt{C_1}}$, and setting $C = \dfrac{C_1}{1-4C_1R^2}$ the lemma holds.

In the following proposition we'll find $0 < R < \frac{1}{2\sqrt{C_1}}, \epsilon$ such that $T(\overline{B_R(-\varphi)}) \subset \overline{B_R(-\varphi)} \subset C^1(\overline{\Omega})$.

PROPOSITION 3

Let $p > 2$ and assume that $\|\varphi\|_{2,p}$ and $\|H\|_{L^\infty(\overline{\Omega}\times[\epsilon,\epsilon])}$ are small enough. Then there exists $R \leq \epsilon$ such that if

$$\|\overline{v}+\varphi\|_{1,\infty} \leq R$$

then

$$\|T(\overline{v})+\varphi\|_{1,\infty} \leq R$$

Furthermore, the operator T is continuous in the closed ball $\overline{B_R(-\varphi)}$, and its range is a precompact set.

Proof

Assume that $\|\overline{v}+\varphi\|_{1,\infty} \leq R < \frac{1}{2\sqrt{C_1}}$. Then

$$\|v+\varphi\|_{1,\infty} \leq k\|v\|_{2,p} + \|\varphi\|_{1,\infty} \leq Ck\|L_{\overline{v}}(v)\|_p + \|\varphi\|_{1,\infty} = \frac{C_1 k}{1-4C_1R^2}\|F(\overline{v})\|_p + \|\varphi\|_{1,\infty}$$

and

$$\|F(\overline{v})\|_p \leq 2(1+\|\overline{v}+\varphi\|_{1,\infty}^2)^{3/2}\|H(x,y,\overline{v}+\varphi)\|_p + 2(1+2\|\overline{v}+\varphi\|_{1,\infty}^2)\|\varphi\|_{2,p} \leq$$

$$\leq 2(1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + 2(1+2R^2)\,\|\varphi\|_{2,p}$$

We look for a number R such that

$$\frac{2C_1 k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+2R^2)\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} \leq R$$

or, equivalently, such that $f(R) \leq 0$, where

$$f(R) = \frac{2C_1 k}{1-4C_1R^2}((1+R^2)^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+2R^2)\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} - R$$

It is clear that $f \leq \frac{P}{1-4C_1R^2}$ in the interval $(0, \frac{1}{2\sqrt{C_1}})$, where

$$P(R) = 2C_1 k((1+\frac{1}{4C_1})^{3/2}|\Omega|^{1/p}\,\|H\|_\infty + (1+\frac{1}{2C_1})\,\|\varphi\|_{2,p}) + \|\varphi\|_{1,\infty} - R + 4C_1R^3$$

P achieves a minimum in $R_0 = \frac{1}{\sqrt{12C_1}}$ and $P(R_0) \leq 0$ if $\|\varphi\|_{2,p}$ and $\|H\|_\infty$ are small enough. Then $f(R_0) \leq 0$.

In order to complete the proof we must see that T is continuous and compact. Indeed, for $\overline{u}$, $\overline{v} \in \overline{B_R(-\varphi)}$:

$$\|u-v\|_{2,p} \leq C\,\|L_{\overline{u}}(u-v)\|_p \leq C\left(\|F(\overline{u})-F(\overline{v})\|_p + \|L_{\overline{v}}(v) - L_{\overline{u}}(v)\|_p\right).$$

But

$$\|F(\overline{u})-F(\overline{v})\|_p \leq \|2H(x,y,\overline{u}+\varphi) - 2H(x,y,\overline{v}+\varphi))\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}}\|_p$$

$$+\|2H(x,y,\overline{v}+\varphi)(\left(1+|\nabla(\overline{u}+\varphi)|^2\right)^{\frac{3}{2}} - \left(1+|\nabla(\overline{v}+\varphi)|^2\right)^{\frac{3}{2}})\|_p$$

$$+\|((\overline{u}_y+\varphi_y)^2-(\overline{v}_y+\varphi_y)^2)\varphi_{xx}\|_p+\|((\overline{u}_x+\varphi_x)^2-(\overline{v}_x+\varphi_x)^2)\varphi_{yy}\|_p+2\|((\overline{u}_x+\varphi_x)(\overline{u}_y+\varphi_y)-$$

$$(\overline{v}_x+\varphi_x)(\overline{v}_y+\varphi_y))\varphi_{xy}\|_p \leq 2\|H(x,y,\overline{u}+\varphi) - H(x,y,\overline{v}+\varphi)\|_p(1+R^2)^{3/2}$$

$$+6R(1+R^2)^{\frac{1}{2}}(R+\|\varphi\|_{1,\infty})\|\overline{u}-\overline{v}\|_{1,\infty}|\Omega|^{1/p}\|H\|_\infty + 8R\|\overline{u}-\overline{v}\|_{1,\infty}\|\varphi\|_{2,p}$$

and

$$\|L_{\overline{v}}(v) - L_{\overline{u}}(v)\|_p \leq 8R\|\overline{u}-\overline{v}\|_{1,\infty}\|v\|_{2,p}$$

Being H uniformely continuous and $\|u-v\|_{1\ \infty} \leq k\|u-v\|_{2,p}$, the continuity follows. Moreover, fixing any $\overline{v} \in \overline{B_R(-\varphi)}$ we see that $\overline{T(B_R(-\varphi))}$ is bounded in $W^{2,p}$, and the result follows from the compactness of the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$.

REMARK

In the situation of proposition 3, if we write $P(R) = 4C_1R^3 - R + a$, *the smallness of* H *and* φ *can be stated in the more precise condition* $a \leq \frac{2}{3\sqrt{12C_1}}$.

Proof of theorem 1: From proposition 3 we know that the operator T satisfies the assumptions of Schauder fixed point theorem (see [GT] corollary 11.2). Thus, we obtain a fixed point $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for the operator T, which corresponds to a solution of equation (2).

References

[A] Adams, R: Sobolev Spaces, Academic Press, (1975).

[AMR] Amster P., Mariani M.C., Rial, D.F: Existence and unicity of H-System's solutions with Dirichlet conditions. To appear in Nonlinear Analysis, Theory, Methods, and Applications.

[BC] Brezis H. and Coron J: Multiple solutions of H systems and Rellich's conjeture, Comm.Pure Appl. Math. 37 (1984), 149-187.

[DST] Díaz J., Saa J., Thiel U: Sobre la ecuación de curvatura media prescripta y otras ecuaciones cuasilineales elípticas con soluciones anulándose localmente, Revista de la Unión Matemática Argentina vol.35, 1989, 175-206.

[GT] Gilbarg D. and Trudinger N: Elliptic Partial Differential Equations of Second Order, Springer-Verlag. Second Edition

[H] Hildebrandt S: On the Plateau problem for surfaces of constant mean curvature. Comm. Pure Appl. Math. 23 (1970), 97-114.

[LD-M] Lami Dozo E. and Mariani M.C: A Dirichlet problem for an H system with variable H. Manuscripta Math. 81 (1993), 1-14.

[MR] Mariani M.C, Rial D: Solution to the mean curvature equation by fixed point methods. To appear in Bulletin of The Belgian Mathematical Society - Simon Stevin.

[S1] Struwe M: Pläteau's problem and the calculus of variations, Lecture Notes, Princeton Univ. Press (1988).

[S2] Struwe M: Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature, Preprint.

[WG] Wang Guofang, The Dirichlet problem for the equation of prescribed mean curvature, Preprint.

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ON SIMULTANEOUS REPEATED 2×2-GAMES

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ABSTRACT: We consider a game with two choices, two players and infinitely number of rounds (repeated game). A small error probability occur due to mistakes in implementation or in perception. We compute the payoff matrix by means of a perturbation approach using the isomorphisms between the transition matrices.

1. INTRODUCTION.

We consider a game with two players I and II and two choices (strategies) a_1,a_2 for I and b_1 ,b_2 for II .The process of choosing one option of each player is called one round. The game which contains more than one round is called repeated game. For each round of the game, we have four possible outcomes $(a_1,b_1),(a_1,b_2),(a_2,b_1)$ and (a_2,b_2). If player I options a_i (i=1,2) and player II options b_j (j=1,2), then player I gets payoff value a_{ij} and player II gets payoff value b_{ji} . Therefore in each round, player I has 2×2 matrix $A=(a_{ij})$ called the payoff matrix of player I and player II has 2×2 matrix $B=(b_{ij})$ called the payoff matrix of player II. The game in this case is described by two matrices A and B (asymmetric game or two population game). In the case when player II has the same possible strategies and payoff values as player I, i.e. $a_1 = b_1$, $a_2 = b_2$, the game is described by symmetric payoff matrix (symmetric game or one population game).

If there is a chance of moves in the game, then when player I plays a_i against player II who plays b_j, the payoff depends on the outcome of the chance move. So we multiply each payoff by the probability of the chance event that will give rise to it, and add all these products together. This gives the average or expected payoff.

There are two models of the game, simultaneous model and alternating model. In simultaneous model, both players choose their options at the same time without knowing the choice of the other, otherwise the game is alternating. In this work, we shall be interested in infinite repeated simultaneous one population games (simultaneous and symmetric).

Keywords. Game Dynamics, Perturbation, Binary Representation, Isomorphism, Transition Matrix.

2.SIMULTANEOUS 2×2-REPEATED GAME.

If we consider a symmetric 2×2-game with two pure strategies C and D, then the payoff matrix is given by

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array} \qquad (1)$$

The letters indicate that we have primarily in mind the Prisoner's Dilemma, where C stands for Cooperate, D for Defect see [6]. The payoff value R for Reward, S for the Sucker payoff, T for Temptation and P for Punishment. These payoff values corresponding to the four possible outcomes (C,C),(C,D),(D,C) or (D,D), where the first position denotes the option chosen by the player and the second that of the co-player. If we numbered the outcomes by i=1,2,3,4, then we have 2^4 transition rules that can be labeled by the quadruples (u_1,u_2,u_3,u_4) of zeros and ones. Here, u_i is 1 if the player plays C and 0 if he plays D after outcome i (i=1,2,3,4). Thus (0,0,0,0) is the rule that always defects (ALL D), while the rule (1,0,0,0) with initial state C plays Grim, see[5]: after a single D of the adversary, it never reverts to C again. The rule (1,1,1,1) plays always Cooperate (ALL C). There are 16 rule altogether, which we number from 0 to 15, using the integers given in binary notation, by u_1 u_2 u_3 u_4. The strategy corresponding to rule i will be denoted by S_i .Thus S_0 is ALL D, S_8 is Grim and S_{15} is ALL C. The S_i strategies are exactly the 16 corner points of the four dimensional strategy space formed by all (p_1,p_2,p_3,p_4) strategies where p_i is the probability to play C after the corresponding outcome of the previous round.

3. ISOMORPHISMS OF MARKOV MATRICES.

The Markov matrix or transition probability matrix is the matrix M denoted by M =<E,T> where E is a finite nonempty set of states and T is the transition matrix of probabilities. If E = $\{e_0,e_1,\ldots,e_r\}$ and T = (p_{ij}), then M is given by:

$$M = \begin{array}{c} \begin{array}{cccc} e_0 & e_1 & & e_r \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0r} \\ p_{10} & p_{11} & \cdots & p_{1r} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_{r0} & p_{r1} & \cdots & p_{rr} \end{pmatrix} \end{array} \qquad (2)$$

The transition probabilities p_{ij} satisfy the conditions

$$\sum_{j=0}^{r} p_{ij} = 1 \text{ and } p_{ij} \geq 0 \ (i,j = 0,1,\ldots,r).$$

If we have Markov matrices $M = <E,T>$, $M' = <E',T'>$ with $E = \{e_0,e_1,\ldots,e_r\}$, $E' = \{e'_0,e'_1,\ldots,e'_r\}$ and $T = (p_{ij})$, $T' = (p'_{ij})$ respectively, then for each one-one mapping

Ψ from the index set $I = \{0,1,\ldots,r\}$ onto I, we can define a one-one mapping Φ from E onto E' such that $\Psi(i) = k$ iff $\Phi(e_i) = e'_k$ $\forall$ i,j $\in$ I.
This leads to the following definition

Definition 1. A Markov matrix $M = \langle E, T \rangle$ is called isomorphic to $M' = \langle E', T' \rangle$ if there exist one-one mapping Φ from E onto E' such that $p_{ij} = p'_{\Psi(i)\Psi(j)}$ (i.e.the probability of going from state $e_i \in E$ to state $e_j \in E$ in one step equal to the probability of going from state $\Phi(e_i) \in E'$ to $\Phi(e_j) \in E'$ in one step). In this case we write $M \underset{\Phi}{\cong} M'$, which means M is isomorphic to M' by Φ.

Definition 2. The vector $\Pi = (\pi_0, \pi_1, \ldots, \pi_r)$, where π_i satisfy the following conditions $\pi_i \geq 0$, $\sum_{i=0}^{r} \pi_i = 1$ and $\pi_j = \sum_{i=0}^{r} \pi_i p_{ij}$ is called a stationary probability distribution of M.

Theorem (Isomorphism theory for Markov matrices).
Let Π and Π' be stationary distributions of $M = \langle E, T \rangle$ and $M' = \langle E', T' \rangle$ respectively. If Π is uniquely defined and $M \underset{\Phi}{\cong} M'$, then $\pi_i = \pi'_{\Psi(i)}$.

Proof. We have $\Pi = (\pi_0, \pi_1, \ldots, \pi_r)$ and $\Pi' = (\pi'_0, \pi'_1, \ldots, \pi'_r)$, then

$$\pi_i \geq 0, \quad \sum_{i=0}^{r} \pi_i = 1 \quad \text{and} \quad \pi_j = \sum_{i=0}^{r} \pi_i p_{ij} \tag{3}$$

$$\pi'_i \geq 0, \quad \sum_{i=0}^{r} \pi'_i = 1 \quad \text{and} \quad \pi'_j = \sum_{i=0}^{r} \pi'_i p'_{ij} \tag{4}$$

From (4) we obtain

$$\pi'_{\Psi(j)} = \sum_{\Psi(i)=0}^{r} \pi'_{\Psi(i)} p'_{\Psi(i)\Psi(j)} \tag{5}$$

By rearrangement of (5) we can write

$$\pi'_{\Psi(j)} = \sum_{i=0}^{r} \pi'_{\Psi(i)} p'_{\Psi(i)\Psi(j)}.$$

Since $p_{ij} = p'_{\Psi(i)\Psi(j)}$, then

$$\pi'_{\Psi(j)} = \sum_{i=0}^{r} \pi'_{\Psi(i)} p_{ij} \tag{6}$$

From (3) and (6), since stationary distribution of M is uniquely defined, we get $\pi_i = \pi'_{\Psi(i)}$. □

4.GROUP OF ISOMORPHISMS BETWEEN MARKOV MATRICES.

The game between the two players, $P_0 = (p_1,p_2,p_3,p_4)$ and $Q_0 = (q_1,q_2,q_3,q_4)$ where p_i and q_i are the probabilities to play C after outcome i (i=1,2,3,4), is a Markov process given by the transition probability matrix between the four states R,S,T and P from one round to the next

$$M_0 = \begin{pmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{pmatrix} \quad (7)$$

where for example $(1-p_2)(1-q_3)$ means that the transition probability from state S to state P . The stationary distribution $\Pi_0 = (\pi_1, \pi_2, \pi_3, \pi_4)$ is the left-hand eigenvector corresponding to the eigenvalue 1 of M_0 in (7). Therefore the payoff for strategy P_0 is then given by

$$R\pi_1 + S\pi_2 + T\pi_3 + P\pi_4 .$$

In (7), if we replace p_1 by p_2, then to get an isomorphism between M_0 and the resulting matrix (other transition matrix), we find that P_0 must become P_1 and Q_0 must become Q_1, where $P_1 = (p_2,p_1,p_4,p_3)$, $Q_1 = (1-q_3,1-q_4,1-q_1,1-q_2)$. The resulting matrix is given by

$$M_1 = \begin{pmatrix} p_2(1-q_3) & p_2q_3 & (1-p_2)(1-q_3) & (1-p_2)q_3 \\ p_1(1-q_1) & p_1q_1 & (1-p_1)(1-q_1) & (1-p_1)q_1 \\ p_4(1-q_4) & p_4q_4 & (1-p_4)(1-q_4) & (1-p_4)q_4 \\ p_3(1-p_2) & p_3q_2 & (1-p_3)(1-q_2) & (1-p_3)q_2 \end{pmatrix} \quad (8)$$

The matrix in (8) is the transition matrix of P_1 against Q_1. We note that $M_0 \underset{\Phi_1}{\cong} M_1$ where Φ_1 (R,S,T,P)= (S,R,P,T), to simplify we denoted this function by Φ_1 = (S,R,P,T), and for Ψ_1 ; Ψ_1 (1,2,3,4)= (2,1,4,3) by Ψ_1 = (2,1,4,3). If Π_0 is uniquely defined, then (by isomorphism theory for Markov matrices) the payoff for strategy P_1 against strategy Q_1 is

$$R\pi_2 + S\pi_1 + T\pi_4 + P\pi_3 .$$

Using the same way to get the all isomorphic matrices to M_0, we get

i. $M_0 \underset{\Phi_2}{\cong} M_2$; M_2 is the transition matrix of $P_2 = (1-p_3,1-p_4,1-p_1,1-p_2)$ against $Q_2 = (q_2,q_1,q_4,q_3)$ and $\Phi_2 = (T,P,R,S)$.

i. $M_0 \underset{\Phi_3}{\cong} M_3$; M_3 is the transition matrix of $P_3 = (1-p_4,1-p_3,1-p_2,1-p_1)$ against $Q_3 = (1-q_4,1-q_3,1-q_2,1-q_1)$ and $\Phi_3 = (P,T,S,R)$.

Also we can show that $M_0 \underset{\Phi_r}{\cong} M_r$; M_r is the transition matrix of Q_k against P_k ,where

r = k mod 4 ; r = 4,5,6,7 and $\Phi_4 = (R,T,S,P)$, $\Phi_5 = (T,R,P,S)$, $\Phi_6 = (S,P,R,T)$

$\Phi_7 = (P,S,T,R)$.

Remarks:

(i) If Π_0 is uniquely defined, then we can find the stationary distribution Π_n for M_n ; $n = 0,1,\ldots,7$ (by using isomorphism theory for Markov matrices).

(ii) We see that the set$\{\Phi_0,\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5,\Phi_6,\Phi_7\}$ is a group with respect to composition. We call this group the group of isomorphisms between Markov matrices.

5. EQUIVALENCE CLASSES OF MARKOV MATRICES.

From the group isomorphisms of the transition matrices, we have the following table

P_0	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	Q_0
P_1	S_0	S_2	S_1	S_3	S_8	S_{10}	S_9	S_{11}	S_4	S_6	S_5	S_7	S_{12}	S_{14}	S_{13}	S_{15}	Q_2
P_2	S_{15}	S_{11}	S_7	S_3	S_{14}	S_{10}	S_6	S_2	S_{13}	S_9	S_5	S_1	S_{12}	S_8	S_4	S_0	Q_1
P_3	S_{15}	S_7	S_{11}	S_3	S_{13}	S_5	S_9	S_1	S_{14}	S_6	S_{10}	S_2	S_{12}	S_4	S_8	S_0	Q_3

Table 1.

This table represents the strategies P_0,Q_0 and the corresponding strategies P_1,P_2,P_3,Q_1,Q_2 and Q_3 (which determine in section 4). For instance if $P_0=S_7,Q_0=S_{13}$, then $P_1=S_{11}$, $Q_1=S_8$, $P_2=S_2$, $Q_2=S_{14}$ and $P_3=S_1$, $Q_3=S_4$.

We denoted the Markov matrix corresponding to the game between the two players S_i and $S_j(i,j= 0,1,\ldots,15)$ by M^i_j. The set $\left[\begin{smallmatrix} r\\ m\end{smallmatrix}\right]=\{\, M^i_j \mid M^r_m \underset{\Phi_n}{\cong} M^i_j\}$ is called the equivalence class of M^r_m, where the isomrphism relation for the transition matrices make equivalence relation. By using Table 1 we get for example.

$$\left[\begin{smallmatrix} 1\\ 4\end{smallmatrix}\right]=\{M^1_4,M^2_{14},M^{11}_8,M^7_{13},M^4_1,M^{14}_2,M^8_{11},M^{13}_7\}$$

and

$$\left[\begin{smallmatrix} 4\\ 12\end{smallmatrix}\right]=\{M^4_{12},M^8_{12},M^{14}_{12},M^{13}_{12},M^{12}_4,M^{12}_8,M^{12}_{14},M^{12}_{13}\}.$$

In this case there are 43 equivalence classes, which are given by

$$(\left[\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 2\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 3\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 4\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 5\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 6\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 8\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 9\end{smallmatrix}\right],\left[\begin{smallmatrix} 0\\ 12\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 1\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 2\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 3\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 4\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 5\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 6\end{smallmatrix}\right],$$

$$\left[\begin{smallmatrix} 1\\ 7\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 8\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 9\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 10\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 12\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 13\end{smallmatrix}\right],\left[\begin{smallmatrix} 1\\ 14\end{smallmatrix}\right],\left[\begin{smallmatrix} 3\\ 3\end{smallmatrix}\right],\left[\begin{smallmatrix} 3\\ 4\end{smallmatrix}\right],\left[\begin{smallmatrix} 3\\ 5\end{smallmatrix}\right],\left[\begin{smallmatrix} 3\\ 6\end{smallmatrix}\right],\left[\begin{smallmatrix} 3\\ 12\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 4\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 5\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 6\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 8\end{smallmatrix}\right],$$

$$\left[\begin{smallmatrix} 4\\ 9\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 10\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 12\end{smallmatrix}\right],\left[\begin{smallmatrix} 4\\ 13\end{smallmatrix}\right],\left[\begin{smallmatrix} 5\\ 5\end{smallmatrix}\right],\left[\begin{smallmatrix} 5\\ 6\end{smallmatrix}\right],\left[\begin{smallmatrix} 5\\ 10\end{smallmatrix}\right],\left[\begin{smallmatrix} 5\\ 12\end{smallmatrix}\right],\left[\begin{smallmatrix} 6\\ 6\end{smallmatrix}\right],\left[\begin{smallmatrix} 6\\ 12\end{smallmatrix}\right] \text{ and } \left[\begin{smallmatrix} 12\\ 12\end{smallmatrix}\right])$$

6. THE PAYOFF MATRIX CORRESPONDING TO THE ERROR IN IMPLEMENTATION.

If we assume that our game has an error in implementation in each move, then there is a probability for a mistake in each strategy. If $\varepsilon>0$ denotes the probability of a mistake in implementing a strategy, then S_j becomes $S_j(\varepsilon)$, which is given by the quadruple obtained (u_1,u_2,u_3,u_4) by replacing 0 with ε and 1 with $1-\varepsilon$. For instance the transition rule of the Grim strategy $S_8=(1,0,0,0)$ becomes $S_8(\varepsilon)=(1-\varepsilon,\varepsilon,\varepsilon,\varepsilon)$.

Thus, according to the equivalence classes we can find the 256 entries in 16×16 payoff matrix corresponding to the errors in implementation by finding 43 entries.

If $P=S_i(\varepsilon)$ and $Q=S_j(\varepsilon)$, then the transition probability matrix M becomes $M(\varepsilon)$. We can write this matrix in the form

$$M(\varepsilon) = M + \varepsilon B_1 + \varepsilon^2 B_2 \qquad (9)$$

where M is a stochastic matrix, B_1 and B_2 have row sum 0. We may view $M(\varepsilon)$ as a perturbation of the matrix M and treat the problem of finding the left eigenvector $\Pi(\varepsilon)$ of $M(\varepsilon)$ as a perturbation problem. Thus $\Pi(\varepsilon)$ can be written in the form

$$\Pi(\varepsilon) = \Pi + \varepsilon X + \varepsilon^2 Y + \ldots \qquad (10)$$

where the stochastic vector Π is a solution of the equation $\Pi M = \Pi$. The components of the vectors X and Y must sum up to 0. Writing $M = I + B_0$ (where I is the identity matrix) and using (9) and (10), the eigenvalue equation $\Pi(\varepsilon) M(\varepsilon) = \Pi(\varepsilon)$ implies, upon comparing powers in ε, the three equations

$$\Pi B_0 = 0 \qquad (11)$$

$$XB_0 + \Pi B_1 = 0 \qquad (12)$$

$$YB_0 + XB_1 + \Pi B_2 = 0 \qquad (13)$$

The payoff for the player using $S_i(\varepsilon)$ against an opponent using $S_j(\varepsilon)$ is given by (we shall neglect ε^2)

$$A(S_i(\varepsilon), S_j(\varepsilon)) = \Pi \begin{pmatrix} R \\ S \\ T \\ P \end{pmatrix} + \varepsilon X \begin{pmatrix} R \\ S \\ T \\ P \end{pmatrix} \qquad (14)$$

For $P = S_4(\varepsilon)$ against $Q = S_{12}(\varepsilon)$, by solving the three equations (11),(12) and (13) (subjected to the condition that the components of Π sum up to 1, while those of X and Y sum up to 0) we get

$\Pi = (0, 1/6, 3/6, 2/6)$ and $X = (6/9, 1/9, -6/9, -1/9)$.

Thus

$$A(S_4(\varepsilon), S_{12}(\varepsilon)) = \tfrac{1}{18}(12\varepsilon R + (3+2\varepsilon) S + (9-12\varepsilon) T + (6-2\varepsilon) P).$$

Since $M_{12}^4 \underset{\Phi_5}{\cong} M_8^{12}$, then (by Isomorphism theory) we get

$$A(S_{12}(\varepsilon), S_4(\varepsilon)) = \tfrac{1}{18}((3+2\varepsilon) R + (6-2\varepsilon) S + 12\varepsilon T + (9-12\varepsilon) P).$$

By these calculations, we obtain the vectors Π for all 16 strategies which are in Table 2. The vector Π for S_i against S_j is $(\pi_1, \pi_2, \pi_3, \pi_4)$, with $\pi_i = h_i(h_1 + h_2 + h_3 + h_4)^{-1}$ and (h_1, h_2, h_3, h_4) is given by the element in the i-th row and j-th column of Table 2. Table 3 represents the vectors X for all 16 strategies. By using Table 2, Table 3 and equation (14) we can obtain the payoff matrix corresponding to the errors in implementations.

7. THE PAYOFF MATRIX CORRESPONDING TO THE ERRORS IN PERCEPTION.

We have assumed that there are errors in implementing a move. We can also analyse the effect of errors in perception -in misunderstanding the other's C or a D,(see[1],[4]). This type of errors can sometimes lead to quit different results. Let us denote by ε the probability of mistaking the other player's previous move, and by $\lambda\varepsilon$ the probability of mistaking one's own previous move (usually $0<\lambda<1$). The perturbation of the tit for tat strategy S_{10} is $(1-\varepsilon, \varepsilon, 1-\varepsilon, \varepsilon)$, just as with mistakes in implementation, the perturbation of S_9 is $(1-(1+\lambda)\varepsilon, (1+\lambda)\varepsilon, (1+\lambda)\varepsilon, 1-(1+\lambda)\varepsilon)$; that of S_8 is $(1-(1+\lambda)\varepsilon, \varepsilon, \lambda\varepsilon, 0)$, while S_0 and S_{15} are not affected by the perturbation at all. In general, the strategy (u_1, u_2, u_3, u_4) turns into

$$(1-(1+\lambda)\varepsilon)(u_1,u_2,u_3,u_4) + \varepsilon(u_2,u_1,u_4,u_3) + \lambda\varepsilon(u_3,u_4,u_1,u_2) + \lambda\varepsilon^2(v,-v,-v,v),$$

where $v = u_1 + u_4 - u_2 - u_3$.

In this case, the table which represents $P_0, P_1, P_2, P_3, Q_0, Q_1, Q_2$ and Q_3 is Table 1 (as the same table in the previous case). Thus we have the same 43 equivalence classes, i.e. we can find the 256 entries in 16×16 payoff matrix corresponding to the errors in perception by finding 43 entries (clear by isomorphism theory). Again, one can use the same perturbation method as before to find the payoff values. For $P = S_4$ against $Q = S_{12}$, for instance, one obtains

$$\Pi = (0,\lambda,1+3\lambda,1+2\lambda)/(2(1+3\lambda)) \text{ and } X = \lambda^2(1+3\lambda,-\lambda,-1-3\lambda,\lambda)/(2(1+3\lambda)^2).$$

We can obtain the vectors Π for all 16 strategies from Table 4. The vector Π for S_i against S_j is $(\pi_1,\pi_2,\pi_3,\pi_4)$, with $\pi_i = n_i(n_1,n_2,n_3,n_4)^{-1}$ and (n_1,n_2,n_3,n_4) is given by the element in the i-th row and j-th column of Table 4. We can obtain the vectors X for all 16 strategies from Table 5. The vector X for S_i against S_j is (x_1,x_2,x_3,x_4), with $x_i = m_i(n_1,n_2,n_3,n_4)^{-2}$ where (m_1,m_2,m_3,m_4) and (n_1,n_2,n_3,n_4) are given by the element in the i-th row and j-th column of Table 5 and Table 4 respectively.

S_i is outcompeted by S_j if both $A(S_j,S_i) \geq A(S_i,S_i)$ and $A(S_j,S_j) \geq A(S_i,S_j)$, with at least one inequality being strict. Writing $S_i \ll S_j$ if S_i is outcompeted by S_j.

For Prisoner's Dilemma game by using the Axelrod's payoff values (T = 5, R = 3, P = 1 and S = 0) with $\varepsilon = 0.001$, $\lambda = 0.01$ we get that

$S_0 \ll$ —
$S_1 \ll S_0,S_4,S_8$
$S_2 \ll S_9,S_{10},S_{11}$
$S_3 \ll S_0,S_4,S_8,S_9,S_{11}$
$S_4 \ll S_0,S_8$
$S_5 \ll S_0,S_1,S_2,S_4,S_8,S_9$
$S_6 \ll S_0,S_1,S_2,S_3,S_4,S_5,S_8,S_9,S_{10},S_{11},S_{12}$
$S_7 \ll S_0,S_1,S_2,S_3,S_4,S_5,S_8,S_9,S_{11},S_{12}$
$S_8 \ll$ —
$S_9 \ll S_0,S_1,S_4,S_8$
$S_{10} \ll S_9,S_{11}$
$S_{11} \ll S_0,S_1,S_4,S_5,S_8,S_{12}$
$S_{12} \ll S_0,S_1,S_2,S_4,S_8,S_9$
$S_{13} \ll S_0,S_1,S_2,S_3,S_4,S_5,S_7,S_8,S_9,S_{12}$
$S_{14} \ll S_0,S_1,S_2,S_3,S_4,S_5,S_7,S_8,S_9,S_{12},S_{13}$
$S_{15} \ll S_0,S_1,S_2,S_3,S_4,S_5,S_7,S_8,S_9,S_{12},S_{13}$

We see that all strategies except S_0 and S_8 are outcompeted by at least one other strategy. The strategies S_0 and S_8 can outcompete the greatest number of rival strategies (exactly 12). On the other hand, every strategy except S_6, S_{14} and S_{15} can outcompete some other strategies. There are only two heteroclinic three-cycle which are S_2 S_{11} S_5 and S_2 S_{11} S_{12}: these are triples of strategies S_i, S_j and S_k where $S_i \ll S_j$, $S_j \ll S_k$ and $S_k \ll S_i$.

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
S_0	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,2)	(0,0,1,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,0)	(0,0,2,1)	(0,0,1,
S_1	(0,1,0,1)	(1,0,0,1)	(0,1,1,1)	(1,0,0,1)	(0,2,1,2)	(1,0,1,1)	(0,0,1,0)	(1,0,2,1)	(0,1,0,1)	(1,0,1,1)	(0,1,1,1)	(1,0,1,1)	(0,1,2,1)	(0,0,1,0)	(0,0,1,0)	(0,0,1,
S_2	(0,0,0,1)	(0,1,1,1)	(0,1,1,2)	(0,1,1,0)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(0,1,1,1)	(0,1,1,1)	(0,1,1,0)	(1,0,1,2)	(1,0,1,0)	(2,0,2,1)	(1,0,1,
S_3	(0,1,0,1)	(1,0,0,1)	(0,1,1,0)	(1,1,1,1)	(0,1,0,1)	(1,0,0,1)	(1,1,1,1)	(1,0,0,1)	(0,1,0,1)	(1,1,1,1)	(0,1,1,0)	(0,1,1,0)	(1,1,1,1)	(1,0,1,0)	(1,0,1,0)	(1,0,1,
S_4	(0,1,0,2)	(0,1,2,2)	(0,0,0,1)	(0,0,1,1)	(0,1,1,2)	(0,0,1,0)	(0,0,1,2)	(0,0,1,0)	(0,1,0,2)	(0,1,2,2)	(0,0,0,1)	(0,0,1,1)	(0,1,3,2)	(0,0,1,0)	(0,0,2,1)	(0,0,1,
S_5	(0,1,0,0)	(1,1,0,1)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,1)	(1,0,1,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,1)	(1,0,1,1)	(0,1,1,0)	(0,0,1,0)	(0,0,1,0)	(0,0,1,
S_6	(0,1,0,1)	(0,1,0,0)	(0,0,0,1)	(1,1,1,1)	(0,1,0,2)	(1,1,1,1)	(0,0,0,1)	(1,0,1,1)	(0,2,0,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,0)	(1,1,1,1)	(2,1,2,0)	(2,0,2,1)	(1,0,1,
S_7	(0,1,0,0)	(1,2,0,1)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(1,1,0,1)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(0,1,0,0)	(1,1,1,0)	(1,1,1,0)	(1,2,1,0)	(2,1,2,0)	(1,0,1,0)	(1,0,1,
S_8	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,2)	(0,0,1,0)	(0,0,2,1)	(0,0,1,0)	(0,0,0,1)	(1,0,2,2)	(0,0,0,1)	(1,0,2,2)	(1,0,2,3)	(1,0,2,0)	(1,0,2,1)	(1,0,2,
S_9	(0,1,0,1)	(1,1,0,1)	(0,1,1,1)	(1,1,1,1)	(0,2,1,2)	(1,1,1,1)	(0,0,1,0)	(0,0,1,0)	(1,2,0,2)	(1,0,0,0)	(1,1,1,1)	(1,0,0,0)	(1,1,1,1)	(2,0,1,0)	(1,0,2,0)	(1,0,1,
S_{10}	(0,0,0,1)	(0,1,1,1)	(0,1,1,1)	(0,1,1,0)	(0,0,0,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,0)	(0,0,0,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,0)	(1,0,0,1)	(1,0,0,0)	(1,0,0,0)	(1,0,0,
S_{11}	(0,1,0,1)	(1,1,0,1)	(0,1,1,0)	(0,1,1,0)	(0,1,0,1)	(1,1,0,1)	(1,1,1,0)	(1,1,1,0)	(1,2,0,2)	(1,0,0,0)	(1,1,1,0)	(2,1,1,0)	(2,1,0,1)	(1,0,0,0)	(1,0,0,0)	(1,0,0,
S_{12}	(0,1,0,1)	(0,2,1,1)	(1,1,0,2)	(1,1,1,1)	(0,3,1,2)	(0,1,1,0)	(1,1,1,1)	(1,1,2,0)	(1,2,0,3)	(1,1,1,1)	(1,0,0,1)	(2,0,1,1)	(1,1,1,1)	(2,1,3,0)	(3,0,2,1)	(1,0,1,
S_{13}	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(0,1,0,0)	(0,1,0,0)	(2,2,1,0)	(2,2,1,0,)	(1,2,0,0)	(2,1,0,0)	(1,0,0,0)	(1,0,0,0)	(2,3,1,0)	(2,1,1,0)	(2,0,1,0)	(2,0,1,
S_{14}	(0,2,0,1)	(0,1,0,0)	(2,2,0,1)	(1,1,0,0)	(0,2,0,1)	(0,1,0,0)	(2,2,0,1)	(1,1,0,0)	(1,2,0,1)	(1,2,0,0)	(1,0,0,0)	(1,0,0,0)	(3,2,0,1)	(2,1,0,0)	(1,0,0,0)	(1,0,0,
S_{15}	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(1,2,0,0)	(1,1,0,0)	(1,0,0,0)	(1,0,0,0)	(1,1,0,0)	(2,1,0,0)	(1,0,0,0)	(1,0,0,

Table.2

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
S_0	(0,1,1,−2)	(2,2,−3,−1)/4	(0,1,2,−3)	(1,1,−1,−1)/2	(3,6,−1,−8)/9	(1,0,−3,2)	(1,1,−1,−1)/2	(1,0,−3,2)	(0,1,1,−2)	(1,1,−1,−1)/2	(0,1,2,−3)	(2,2,−1,−3)/4	(1,1,−1,−1,)/2	(1,0,−2,1)	(6,3,−8,−1)/9	(1,0,−2
S_1	(2,−3,2,−1)/4	(−3,2,2,−1)/2	(6,−2,−5,1)/9	(−2,1,2,−1)	(15,−14,3,−4)/25	(−4,6,−1,−1)/9	(1,2,−5,2)	(0,1,−1,0)/2	(1,−2,2,−1)/2	(−1,2,−1,0)/3	(6,−4,−1,−1)/9	(−5,6,1,−2)/9	(6,−1,−6,1)/8	(2,0,−3,1)	(1,1,−3,1)	(2,0,−3
S_2	(0,2,1,−3)	(6,−5,−2,1)/9	(1,0,0,1)/2	(1,−2,−1,2)	(1,1,1,−3)	(−4,6,−1,−1)/9	(2,1,2,−5)	(−2,6,1,−5)/9	(0,2,1,−3)	(2,−1,0,−1)/3	(6,−4,−1,−1)/9	(2,−3,−1,2)/2	(−1,6,1,−6)/8	(−2,1,−1,2)/2	(−14,15,−4,3)/25	(−3,2,−1,
S_3	(1,−1,1,−1)/2	(−2,2,1,−1)	(1,−1,−2,2)	(1,−1,−1,1)/16	(2,−2,1,−1)/2	(−1,1,1,−1)	(0,0,0,0)	(−1,1,2,−2)	(1,−1,2,−2)/2	(0,0,0,0)	(1,−1,−1,1)	(2,−2,−1,1)	(1,−1,−1,1)/8	(−1,1,−2,2)/2	(−2,2,−1,1)/2	(−1,1,−1,
S_4	(3,−1,6,−8)/9	(15,3,−14,−4)/25	(1,1,1,−3)	(2,1,−2,−1)/2	(1,0,0,−1)/2	(1,1,−4,2)	(9,6,−1,−14)/9	(1,0,−3,2)	(3,−2,9,−10)/9	(3,0,−1,−2)/5	(1,1,2,−4)	(2,1,−1,−2)/2	(6,1,−6,−1)/9	(2,1,−5,2)/2	(9,3,−10,−2)/9	(1,0,−2
S_5	(1,−3,0,2)	(−4,−1,6,−1)/9	(−4,−1,6,−1)/9	(−1,1,1,−1)	(1,−4,1,2)	(0,0,0,0)	(0,0,0,0)	(−1,6,−1,−4)/9	(1,−4;1,2)	(0,0,0,0)	(0,0,0,0)	(−1,6,−1,−4)/9	(1,−1,−1,1)	(2,1,−4,1)	(2,1,−4,1)	(2,0,−3
S_6	(1,−1,1,−1)/2	(1,−5,2,2)	(2,2,1,−5)	(0,0,0,0)	(9,−1,6,−14)/9	(0,0,0,0)	(2,1,1,−4)	(0,2,−1,−1)/3	(6,−14,9,−1)/9	(1,−4,2,1)	(0,0,0,0)	(−1,−1,0,2)/3	(0,0,0,0)	(−2,0,−1,3)/5	(−1,3,−2,0)/5	(−1,1,−1,
S_7	(1,−3,0,2)	(0,−1,1,0)/2	(−2,1,6,−5)/9	(−1,2,1,-2)	(1,−3,0,2)	(−1,−1,6,−4)/9	(0,−1,2,−1)/3	(−1,2,2,−3)/2	(1,−3,1,1)	(2,−5,2,1)	(−1,−1,−4,6)/9	(1,−5,−2,6)/9	(1,−6,−1,6)/8	(−4,3,−14,15)/25	(−1,2,−2,1)/2	(−1,2,−3,
S_8	(0,1,1,−2)	(1,2,−2,−1)/2	(0,1,2,−3)	(1,2,−1,−2)/2	(3,9,−2,−10)/9	(1,1,−4,2)	(6,9,−14,−1)/9	(1,1,−3,1)	(1,2,2,−5)/2	(0,3,−2,−1)/5	(1,1,2,−4)	(3,15,−4,−14)/25	(1,6,−1,−6)/9	(−2,3,−10,9)/9	(0,1,−1,0)/2	(−1,3,−8,
S_9	(1,−1,1,−1)/2	(−1,−1,2,0)/3	(2,0,−1,−1)/3	(0,0,0,0)	(3,−1,0,−2)/5	(0,0,0,0)	(1,2,−4,1)	(2,2,−5,1)	(0,−2,3,−1)/5	(−4,1,1,2)	(0,0,0,0)	(−5,1,2,2)	(0,0,0,0)	(−14,6,−1,9)/9	(−1,9,−14,6)/9	(−1,1,−1,
S_{10}	(0,2,1,−3)	(6,−1,−4,−1)/9	(6,−1,−4,−1)/9	(1,−1,−1,1)	(1,2,1,−4)	(0,0,0,0)	(0,0,0,0)	(−1,−4,−1,6)/9	(1,2,1,−4)	(0,0,0,0)	(0,0,0,0)	(−1,−4,−1,6)/9	(−1,1,1,−1)	(−4,1,2,1)	(−4,1,2,1)	(−3,1,2
S_{11}	(2,−1,2,−3)/4	(−5,1,6,−2)/9	(2,−1,−3,2)/2	(2,−1,−2,1)	(2,−1,1,−2)/2	(−1,−1,6,−4)/9	(−1,0,−1,2)/3	(1,−2,−5,6)/9	(3,−4,15,−14)/25	(−5,2,1,2)	(−1,−1,−4,6)/9	(−1,0,0,1)/2	(−6;1,6,−1)/8	(−3,1,1,1)	(−3,1,2,0)	(−3,1,2
S_{12}	(1,−1,1,−1)/2	(6,−6,−1,1)/8	(−1,1,6,−6)/8	(1,−1,−1,1)/8	(6,−6,1,−1)/9	(1,−1,−1,1)	(0,0,0,0)	(1,−1,−6,6)/8	(1,−1,6,−6)/9	(0,0,0,0)	(−1,1,1,−1)	(−6,6,1,−1)/8	(0,0,0,0)	(−1,1,−6,6)/9	(−6,6,−1,1)/9	(−1,1,−1,
S_{13}	(1,−2,0,1)	(2,−3,0,1)	(−2,−1,1,2)/2	(−1,−2,1,2)/2	(2,−5,1,2)/2	(2,−4,1,1)	(−2,−1,0,3)/5	(−4,3,−14,15)/25	(−2,−10,3,9)/9	(−14,−1,6,9)/9	(−4,2,1,1)	(−3,1,1,1)	(−1,−6,1,6)/9	(−1,0,0,1)/2	(−10,9,−2,3)/9	(−8,6,−1,
S_{14}	(6,−8,3,−1)/9	(1,−3,1,1)	(−14,−4,15,3)/25	(−2,−1,2,1)/2	(9,−10,3,−2)/9	(2,−4,1,1)	(−1,−2,3,0)/5	(−1,−2,2,1)/2	(0,−1,1,0)/2	(−1,−14,9,6)/9	(−4,2,1,1)	(−3,2,1,0)	(−6,−1,6,1)/9	(−10,−2,9,3)/9	(−5,2,2,1)/2	(−2,1,1
S_{15}	(1,−2,0,1)	(2,−3,0,1)	(−3,−1,2,2)/4	(−1,−1,1,1)/2	(1,−2,0,1)	(2,−3,0,1)	(−1,−1,1,1)/2	(−1,−3,2,2)/4	(−1,−8,3,6)/9	(−1,−1,1,1)/2	(−3,2,1,0)	(−3,2,1,0)	(−1,−1,1,1)/2	(−8,−1,6,3)	(−2,1,1,0)	(−2,1,1

Table.3

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
S_0	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,λ,1+λ)	(0,0,1,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,1,1)	(0,0,1,0)	(0,0,1+λ,λ)	(0,0,1,0)
S_1	(0,1,0,1)	(1,0,0,1)	(0,1,1,1)	(1,0,0,1)	(0,1,0,1)	(2,0,2+λ,2)	(0,0,1,0)	(1,0,1+λ,1)	(0,1,0,1)	(1,0,1,1)	(0,1,1,1)	(1,0,1,1)	(0,1,2,1)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)
S_2	(0,0,0,1)	(0,1,1,1)	(0,1,1,1+λ)	(0,1,1,0)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(1,0,1,1)	(0,0,0,1)	(0,1,1,1)	(0,2,2,2+λ)	(0,1,1,0)	(1,0,1,2)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
S_3	(0,1,0,1)	(1,0,0,1)	(0,1,1,0)	(1,1,1,1)	(0,1,0,1)	(1,0,0,1)	(1,1,1,1)	(1,0,0,1)	(0,1,0,1)	(1,1,1,1)	(0,1,1,0)	(0,1,1,0)	(1,1,1,1)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
S_4	(0,λ,0,1+λ)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,λ,λ,1+λ)	(0,0,1,0)	(0,0,1,1)	(0,0,1,0)	(0,λ,0,1+2λ)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,λ,1+3λ,1+2λ)	(0,0,1,0)	(0,0,1+2λ,λ)	(0,0,1,0)
S_5	(0,1,0,0)	(2,2+λ,0,2)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,1)	(2,0,2+λ,2)	(0,1,0,0)	(1,1,1,1)	(1,1,1,1)	(1,0,1,1)	(0,1,1,0)	(0,0,1,0)	(0,0,1,0)	(0,0,1,0)
S_6	(0,1,0,1)	(0,1,0,0)	(0,0,0,1)	(1,1,1,1)	(0,1,0,1)	(1,1,1,1)	(0,0,0,1)	(1,0,1,1)	(0,1,0,1)	(0,1,0,0)	(1,1,1,1)	(1,1,1,0)	(1,1,1,1)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
S_7	(0,1,0,0)	(1,1+λ,0,1)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(2,2+λ,0,2)	(1,1,0,1)	(1,0,0,1)	(0,1,0,0)	(0,1,0,0)	(1,1,1,0)	(1,1,1,0)	(1,2,1,0)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
S_8	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(0,0,λ,1+2λ)	(0,0,1,0)	(0,0,1,1)	(0,0,1,0)	(0,0,0,1)	(0,0,1,1)	(0,0,0,1)	(0,0,1,1)	(λ,0,1+2λ,1+3λ)	(λ,0,1+2λ,0)	(λ,0,1+λ,λ)	(λ,0,1+λ,0)
S_9	(0,1,0,1)	(1,1,0,1)	(0,1,1,1)	(1,1,1,1)	(0,1,0,1)	(1,1,1,1)	(0,0,1,0)	(0,0,1,0)	(0,1,0,1)	(1,0,0,0)	(1,1,1,1)	(1,0,0,0)	(1,1,1,1)	(1,0,1,0)	(1,0,1,0)	(1,0,1,0)
S_{10}	(0,0,0,1)	(0,1,1,1)	(0,2,2,2+λ)	(0,1,1,0)	(0,0,0,1)	(1,1,1,1)	(1,1,1,1)	(1,1,1,0)	(0,0,0,1)	(1,1,1,1)	(1,1,1,1)	(2+λ,2,2,0)	(1,0,0,1)	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)
S_{11}	(0,1,0,1)	(1,1,0,1)	(0,1,1,0)	(0,1,1,0)	(0,1,0,1)	(1,1,0,1)	(1,1,1,0)	(1,1,1,0)	(0,1,0,1)	(1,0,0,0)	(2+λ,2,2,0)	(1+λ,1,1,0)	(2,1,0,1)	(1,0,0,0)	(1,0,0,0)	(1,0,0,0)
S_{12}	(0,1,0,1)	(0,2,1,1)	(1,1,0,2)	(1,1,1,1)	(0,1+3λ,λ,1+2λ)	(0,1,1,0)	(1,1,1,1)	(1,1,2,0)	(λ,1+2λ,0,1+3λ)	(1,1,1,1)	(1,0,0,1)	(2,0,1,1)	(1,1,1,1)	(1+2λ,λ,1+3λ,0)	(1+3λ,0,1+2λ,λ	(1,0,1,0)
S_{13}	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0,)	(λ,1+2λ,0,0)	(1,1,0,0)	(1,0,0,0)	(1,0,0,0)	(1+2λ,1+3λ,λ,0)	(1+λ,λ,λ,0)	(1+2λ,0,λ,0)	(1+λ,0,λ,0)
S_{14}	(0,1+λ,0,λ)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(0,1+2λ,0,λ)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(λ,1+λ,0,λ)	(1,1,0,0)	(1,0,0,0)	(1,0,0,0)	(1+3λ,1+2λ,0,λ)	(1+2λ,λ,0,0)	(1,0,0,0)	(1,0,0,0)
S_{15}	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(0,1,0,0)	(0,1,0,0)	(1,1,0,0)	(1,1,0,0)	(λ,1+λ,0,0)	(1,1,0,0)	(1,0,0,0)	(1,0,0,0)	(1,1,0,0)	(1+λ,λ,0,0)	(1,0,0,0)	(1,0,0,0)

Table.4

S_0	(0,0,0,0)	(0,0,−1,1)	(0,0,1,−1)	(0,0,0,0)	(0,0,0,0)	(0,0,−1,1)	(0,0,0,0)	(0,0,−1,1)
S_1	(0,−1,0,1)	2(−2−2λ,1+λ,1+λ,0)	(3+3λ,−1,−4−6λ,2+3λ)	2(−1−3λ,λ,1+3λ,−λ)	(2λ,−1−3λ,2λ,1−λ)	$(-12-4\lambda-\lambda^2,12+2\lambda,\lambda^2-2\lambda,4\lambda)$	(1,2+λ,−5−2λ,2+λ)	(1+λ)(−1,0,2,−1)
S_2	(0,1,0,−1)	(3+3λ,−4−6λ,−1,2+3λ)	(0,−1−λ,−1−λ,2+2λ)	2(λ,−1−3λ,−λ,1+3λ)	(λ,1,λ,−1−2λ)	(−3−2λ,3+3λ,λ,−2λ)	(2+λ,1,2+λ,−5−2λ)	(−1,3+3λ,2+3λ,−4−6λ)
S_3	(0,0,0,0)	2(−1−3λ,1+3λ,λ,−λ)	2(λ,−λ,−1−3λ,1+3λ)	(0,0,0,0)	2λ(1,−1,0,0)	2(1+λ)(−1,1,1,−1)	(0,0,0,0)	2(−λ,λ,1+3λ,−1−3λ)
S_4	(0,0,0,0)	(2λ,2λ,−1−3λ,1−λ)	(λ,λ,1,−1−2λ)	2λ(1,0,−1,0)	$\lambda^2(1+\lambda)(0,-1,-1,2)$	(0,0,−1,1)	(2λ,2λ,−λ,−3λ)	(0,0,−1,1)
S_5	(0,−1,0,1)	$(-12-4\lambda-\lambda^2,\lambda^2-2\lambda,12+2\lambda,4\lambda)$	(−3−2λ,λ,3+3λ,−2λ)	2(1+λ)(−1,1,1,−1)	(0,−1,0,1)	(0,0,0,0)	(0,0,0,0)	$(4\lambda,12+2\lambda,\lambda^2-2\lambda,-12-4\lambda-$
S_6	(0,0,0,0)	(1,−5−2λ,2+λ,2+λ)	(2+λ,2+λ,1,−5−2λ)	(0,0,0,0)	λ(2,−1,2,−3)	(0,0,0,0)	(1+λ)(2,1,1,−4)	(1+2λ,6+3λ,−2−λ,−5−4λ
S_7	(0,−1,0,1)	(1+λ)(−1,2,0,−1)	(−1,2+3λ,3+3λ,−4−6λ)	2(−λ,1+3λ,λ,−1−3λ)	(0,−1,0,1)	$(4\lambda,\lambda^2-2\lambda,12+2\lambda,-12-4\lambda-\lambda^2)$	(1+2λ,−2−λ,6+3λ,−5−4λ)	2(1+λ)(0,1,1,−2)
S_8	(0,0,0,0)	(0,2λ,−1−λ,1−λ)	(0,0,1,−1)	**2λ(0,1,0,−1)**	$\lambda^2(1+3\lambda,1+3\lambda,1-\lambda,-3-5\lambda)$	(λ,λ,−1−3λ,1+λ)	λ(2,2,−3,−1)	(λ,λ,−1−2λ,1)
S_9	(0,0,0,0)	(−4−5λ,−2−λ,6+3λ,1+2λ)	(6+3λ,1+2λ,−5−4λ,−2−λ)	(0,0,0,0)	λ(2,−1.5,1,−1.5)	(0,0,0,0)	(1+λ)(1,2,−4,1)	(2+λ,2+λ,−5−2λ,1)
S_{10}	(0,1,0,−1)	(3+3λ,−2λ,−3−2λ,λ)	$(12+2\lambda,4\lambda,-12-4\lambda-\lambda^2,\lambda^2-2\lambda)$	2(1+λ)(1,−1,−1,1)	(λ,1+λ,λ,−1−3λ)	(0,0,0,0)	(0,0,0,0)	(λ,−3−2λ,−2λ,3+3λ)
S_{11}	(0,1,0,−1)	(−4−6λ,2+3λ,3+3λ,−1)	2(1+λ)(1,0,−2,1)	2(1+3λ,−λ,−1−3λ,λ)	(2λ,1−λ,0,−1−λ)	(−2λ,λ,3+3λ,−3−2λ)	(−2−λ,1+2λ,−5−4λ,6+3λ)	(2+3λ,−1,−4−6λ,3+3λ)
S_{12}	(0,0,0,0)	2(4+2λ,−4−2λ,−1−λ,1+λ)	2(−1−λ,1+λ,4+2λ,−4−2λ)	(0,0,0,0)	$2\lambda^2(1+3\lambda,-1-3\lambda,-\lambda,\lambda)$	2(1+λ)(1,−1,−1,1)	(0,0,0,0)	2(1+λ,−1−λ,−4−2λ,4+2
S_{13}	(0,0,0,0)	(1,−1,0,0)	(−1−λ,1−λ,0,2λ)	2λ(0,−1,0,1)	(0,0,0,0)	(1,−1,0,0)	λ(−1.5,−1.5,1,2)	(1−λ,−1−3λ,2λ,2λ)
S_{14}	(0,0,0,0)	(1,−1−2λ,λ,λ)	(−1−3λ,1−λ,2λ,2λ)	2λ(−1,0,1,0)	$\lambda^2(1+3\lambda,-3-5\lambda,1+3\lambda,1-\lambda)$	(1+λ,−1−3λ,λ,λ)	λ(−1.5,−1.5,2,1)	**(1−λ,−1−λ,2λ,0)**
S_{15}	(0,0,0,0)	(1,−1,0,0)	(−1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(1,−1,0,0)	(0,0,0,0)	(1,−1,0,0)

S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}
(0,0,0,0)	(0,0,0,0)	(0,0,1,−1)	(0,0,1,−1)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
(0,−1−λ,2λ,1−λ)	(−5−4λ,6+3λ,−2−λ,1+2λ)	(3+3λ,−3−2λ,−2λ,λ)	(−4−6λ,3+3λ,2+3λ,−1)	2(4+2λ,−1−λ,−4−2λ1+λ)	(1,0,−1,0)	(1,λ,−1−2λ,λ)	(1,0,−1,0)
(0,1,0,−1)	(6+3λ,−5−4λ,1+2λ,−2−λ)	$(12+2\lambda,-12-4\lambda-\lambda^2,4\lambda,\lambda^2-2\lambda)$	2(1+λ)(1,−2,0,1)	2(−1−λ,4+2λ,1+λ,−4−2λ)	(−1−λ,0,1−λ,2λ)	(−1−3λ,2λ,1−λ,2λ)	(−1,0,1,0)
2λ(0,0,1,−1)	(0,0,0,0)	2(1+λ)(1,−1,−1,1)	2(1+3λ,−1−3λ,−λ,λ)	(0,0,0,0)	2λ(0,0,−1,1)	2λ(−1,1,0,0)	(0,0,0,0)
$\lambda^2(1+3\lambda,1-\lambda,1+3\lambda,-3-5\lambda)$	λ(2,1,−1.5,−1.5)	(λ,λ,1+λ,−1−3λ)	(2λ,0,1−λ,−1−λ)	$2\lambda^2(1+3\lambda,-\lambda,-1-3\lambda,\lambda)$	(0,0,0,0)	$\lambda^2(1+3\lambda,1+3\lambda,-3-5\lambda,1-\lambda)$	(0,0,0,0)
(λ,−1−3λ,λ,1+λ)	(0,0,0,0)	(0,0,0,0)	(−2λ,3+3λ,λ,−3−2λ)	2(1+λ)(1,−1,−1,1)	(1,0,−1,0)	(1+λ,λ,−1−3λ,λ)	(1,0,−1,0)
λ(2,−3,2,−1)	(1+λ)(1,−4,2,1)	(0,0,0,0)	(−2−λ,−5−4λ,1+2λ,6+3λ)	(0,0,0,0)	λ(−1.5,1,−1.5,2)	λ(−1.5,2,−1.5,1)	(0,0,0,0)
(λ,−1−2λ,λ,1)	(2+λ,−5−2λ,2+λ,1)	(λ,−2λ,−3−2λ,3+3λ)	(2+3λ,−4−6λ,−1,3+3λ)	2(1+λ,−4−2λ,−1−λ,4+2λ)	(1−λ,2λ,−1−3λ,2λ)	(1−λ,2λ,−1−λ,0)	(1,0,−1,0)
(0,0,0,0)	λ(1,2,−1.5,−1.5)	(0,0,1,−1)	(2λ,2λ,1−λ,−1−3λ)	$2\lambda^2(-\lambda,1+3\lambda,\lambda,-1-3\lambda)$	$\lambda^2(1-\lambda,1+3\lambda,-3-5\lambda,1+3\lambda)$	$\lambda^2(1+\lambda)(-1,0,2,-1)$	(0,0,0,0)
λ(1,−1.5,2,−1.5)	(1+λ)(−4,1,1,2)	(0,0,0,0)	(−5−2λ,1,2+λ,2+λ)	(0,0,0,0)	λ(−3,2,−1,2)	λ(−1,2,−3,2)	(0,0,0,0)
(0,1,0,−1)	(0,0,0,0)	(0,0,0,0)	$(\lambda^2-2\lambda,-12-4\lambda-\lambda^2,4\lambda,12+2\lambda)$	2(1+λ)(−1,1,1,−1)	(−1−3λ,λ,1+λ,λ)	(−1,0,1,0)	(−1,0,1,0)
(2λ,1−λ,2λ,−1−3λ)	(−5−2λ,2+λ,1,2+λ)	$(\lambda^2-2\lambda,4\lambda,-12-4\lambda-\lambda^2,12+2\lambda)$	(2+2λ,−1−λ,−1−λ,0)	2(−4−2λ,1+λ,4+2λ,−1−λ)	(−1−2λ,λ,1,λ)	(−1,0,1,0)	(−1,0,1,0)
$2\lambda^2(-\lambda,\lambda,1+3\lambda,-1-3\lambda)$	(0,0,0,0)	2(1+λ)(−1,1,1,−1)	2(−4−2λ,4+2λ,1+λ,−1−λ)	(0,0,0,0)	$2\lambda^2(\lambda,-\lambda,-1-3\lambda,1+3\lambda)$	$2\lambda^2(-1-3\lambda,1+3\lambda,\lambda,-\lambda)$	(0,0,0,0)
$\lambda^2(1-\lambda,-3-5\lambda,1+3\lambda,1+3\lambda)$	λ(−3,−1,2,2)	(−1−3λ,1+λ,λ,λ)	(−1−2λ,1,λ,λ)	$2\lambda^2(\lambda,-1-3\lambda,-\lambda,1+3\lambda)$	$\lambda^2(1+\lambda)(2,-1,-1,0)$	$\lambda^2(-3-5\lambda,1+3\lambda,1-\lambda,1+3\lambda)$	(0,0,0,0)
$\lambda^2(1+\lambda)(-1,2,0,-1)$	λ(−1,−3,2,2)	(−1,1,0,0)	(−1,1,0,0)	$2\lambda^2(-1-3\lambda,\lambda,1+3\lambda,-\lambda)$	$\lambda^2(-3-5\lambda,1-\lambda,1+3\lambda,1+3\lambda)$	(0,0,0,0)	(0,0,0,0)
(0,0,0,0)	(0,0,0,0)	(−1,1,0,0)	(−1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)

REFERENCES

1-Axelrod, R.and Dion, D.(1988) The further evolution of cooperation, Science 242, 1385-1390

2-Banks, J.S and Sundarm, R.K.(1990), Repeated games, finite automata and complexity, Games and Economic Behavior 2, 97-117.

3-Binmore, K.G. and Simuelson, L.(1992) Evolutionary stability in repeated games played by finite automata, Journal of Economic Theory 57, 278-305.

4-Miller, J.H (1989), The evolution of automata in the Repeated Prisoner's Dilemma, working paper of the Santa Fe Institute, Economic Research program.

5-Nowak, M.And Sigmund, K. Chaos and the evolution of cooperation, Proc. Natl. Acad. Sci. USA 90, 5091-5095 (1993).

6- Nowak, M., Sigmund, K, and El-Sedy, E. (1995), Automata, repeated games and noise, Journal of Mthematical Biology 33, 703-722.

7- Rubinstein, A. (1986), Finite automata play the repeated Prisoner's Dilemma, Journal of Economic Theory 39, 83-96.

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A POINTWISE ERGODIC THEOREM

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Abstract. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on $(X, \mathcal{A}, \mu)$. Necessary and sufficient conditions are given in order that for any f in L_1 the average $\frac{1}{n}\sum_{i=0}^{n-1} f \circ \varphi^i(x)$ converges almost everywhere.

INTRODUCTION AND RESULTS.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on $(X, \mathcal{A}, \mu)$, that is, $A \in \mathcal{A}$ and $\mu(A) = 0$ implies $\mu(\varphi^{-1}A) = 0$. We consider the operator T, acting on measurable functions,

$$Tf(x) = f(\varphi x) .$$

Associated with T we have the averages

$$M_n(T)f = \frac{1}{n}\sum_{i=0}^{n-1} T^i f .$$

Since T maps L_∞ to L_∞ and φ is nonsingular, the adjoint operator S acting on $L_1(\mu)$ can be defined by the relation

$$\int gTfd\mu = \int fSgd\mu \quad ,$$

$f \in L_\infty$, $g \in L_1$. As in [3], in order to extend the domain of S to the space $M^+(\mu)$ of all nonnegative extended real valued measurable functions on X, fix any $f \in M^+(\mu)$ and take $\{f_n\} \subset L_1^+(\mu)$ such that $f_n \uparrow f$ a.e. on X. We then define

$$Sf = \lim_n Sf_n \qquad \text{a.e. on } X .$$

It is easily checked that by this process S can be uniquely extended to an operator on $M^+(\mu)$ satisfying $S(\alpha f + \beta g) = \alpha Sf + \beta Sg$, $0 \leq \alpha, \beta < \infty$. In the sequel, S will be understood to be defined on $M^+(\mu)$ in this manner and we write

$$M_n(S)f = \frac{1}{n}\sum_{i=0}^{n-1} S^i f .$$

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The purpose of this paper is the following theorem:

Theorem 1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on X. Then the following conditions are equivalent:

A) For any $f \in L_1^+(\mu)$, $\lim_n M_n(T)f$ exists and is finite a.e. on X.

B) S satisfies the mean ergodic theorem in $L_1(\mu)$ and, further, for any $f \in L_1^+(\mu)$, $\lim_n M_n(S)(fv_0)$ exists and is finite a.e. on X, where v_0 is the pointwise and L_1–norm limit of $M_n(S)1$.

We will need the following well-known fact (see, e.g. [4])

Lemma 1. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and φ a nonsingular transformation on X. The following are equivalent:

(i) For any $f \in L_\infty$, $M_n(T)f$ converges almost everywhere.

(ii) For any $A \in \mathcal{A}$, $\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \mu(\varphi^{-i}(A))$ exists.

(iii) S satisfies the mean ergodic theorem in L_1.

Throughout this paper χ_A will be the characteristic function of the set A and we will consider two sets as 'equal' if they agree up to a set of measure zero. A measurable set A will be called *invariant* if $T\chi_A = \chi_A$ a.e.. We denote by $\mathcal{J}$ the σ–field of invariant sets.

THE PROOFS.

In order to prove Theorem 1 we will make some previous considerations. First, we observe that by virtue of Lemma 1 we may and do suppose that φ satisfies:

$$\text{For any } A \in \mathcal{A} \text{ there exists } \overline{\mu}(A) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} \mu(\varphi^{-i}(A)) \ .$$

By the Vitali-Hahn-Saks theorem $\overline{\mu}$ is a measure. It is easy to see that $\overline{\mu}$ is absolutely continuous with respect to μ, invariant under φ and $\overline{\mu}(A) = \mu(A)$, $A \in \mathcal{J}$.

Let $v_0 = \frac{d\overline{\mu}}{d\mu}$, $C = \{x : v_0(x) > 0\}$ and $D = X \setminus C$. We have $\mu(C \setminus \varphi^{-1}(C)) = 0$ and hence we may suppose that $C \subset \varphi^{-1}(C)$. Then the set $D_0 = \bigcap_{n\geq 0} \varphi^{-n}(D)$ is invariant and $\mu(D_0) = \overline{\mu}(D_0) = 0$. Thus we have

$$X = \bigcup_{n\geq 0} \varphi^{-n}(C) \ . \tag{1}$$

It is also easy to see that the validity of L_1−mean ergodic theorem for S implies

$$v_0 = \lim_n M_n(\dot{S})1 \ .$$

We prove the following:

Lemma 2. Let $h \in M^+(\mu)$ such that $h^*(x) = \lim_n M_n(T)h(x)$ exists and is finite a.e. on X. Then $h \in L_1(\overline{\mu})$ if and only if $h^* \in L_1(\mu)$.

Proof. If h is in $L_1(\overline{\mu})$, then by Birkhoff's classical ergodic theorem $h^* \in L_1(\overline{\mu})$ and we have

$$\int h d\overline{\mu} = \int h^* d\overline{\mu} = \int h^* d\mu \ ,$$

where the last equality follows from the fact that h^* is $\mathcal{J}$−measurable together with $\mu = \overline{\mu}$ on $\mathcal{J}$.

Conversely, assume $h^* \in L_1(\mu)$ and let $\{h_n\}$ be a sequence of nonnegative simple functions increasing to h. By (1) and the Lebesgue bounded convergence theorem, for all $A \in \mathcal{A}$ we have

$$\int \chi_A^* d\mu = \lim_n \int M_n(T)\chi_A d\mu = \int \chi_A d\overline{\mu} \ .$$

Hence

$$\int h^* d\mu \geq \lim_n \int h_n^* d\mu = \lim_n \int h_n d\overline{\mu} = \int h d\overline{\mu} \ .$$

Proof of Theorem 1. A) $\Rightarrow$ B). Let f be a function in $L_1^+(\mu)$ and $f^* = \lim_n M_n(T)f$. We consider for each natural N the set $J_N = \{x : f^* \leq N\}$. Since $J_N \in \mathcal{J}$, $(f\chi_{J_N})^* = \chi_{J_N} f^*$ μ−a.e., and from Lemma 2 it follows that $f\chi_{J_N}.v_0 \in L_1(\mu)$.

Then Lemma 1 and the fact that the validity of the L_1−mean ergodic theorem for S implies the validity of the pointwise ergodic theorem for S (see, e.g. [2]) give the almost everywhere convergence of $M_n(S)(f\chi_{J_N} v_0)$. From the relation

$$S(f\chi_{J_N} v_0) = \chi_{J_N} S(fv_0)$$

it follows that $M_n(S)(f\dot{v}_0)$ converges a.e. on J_N. Letting $N \uparrow \infty$, we obtain B).
B) $\Rightarrow$ A). Since for all $f \in L_1(\mu)$

$$|S(\chi_C f)| \leq \chi_C S|f| \qquad \mu - \text{a.e. on } X \ ,$$

S can be considered to be a positive linear contraction on $L_1(C, \mu)$.
Let $\mathcal{J}_C$ be the σ−field of invariant subsets of C. Using, for instance, the Chacon-Ornstein theorem and the identification of the limit function (see [1], p.41) it

follows that for each $h \in L_1(C,\mu)$ there exists a $\mathcal{J}_C$−measurable function $R(h)$ such that

$$\hat{h} = \lim M_n(S)h = R(h)v_0 \qquad \mu - \text{a.e. on } C \ .$$

Furthermore, we have

$$\int_K h d\mu = \int_K R(h) v_0 d\mu \ , K \in \mathcal{J}_C \ .$$

Now, let $f \in L_1^+(\mu)$ and set $f_n = \min\{f, n\}$, for each natural n. Then

$$\widehat{f_n v_0} = R(f_n v_0) v_0 \leq \widehat{f v_0} < \infty \qquad \mu - \text{a.e. on } C \ ,$$

where $\widehat{fv_0} = \lim\limits_n M_n(S)(fv_0)$.
We take $K_N = \{x \in C : \sup\limits_n R(f_n v_0) \leq N\}$. It follows that

$$\int_{K_N} f v_0 d\mu = \lim_n \int_{K_N} f_n v_0 d\mu \leq N \int_{K_N} v_0 d\mu \ .$$

Therefore, $f \in L_1(K_N, \overline{\mu})$ and by Birkhoff's classical ergodic theorem $M_n(T)f$ converges to a finite limit a.e. on K_N.
Since $K_N \uparrow C$, A) follows from (1).

References

[1] A.M. GARSIA, Topics in Almost Everywhere Convergence, Markham, Chicago 1970.

[2] Y. ITO, Uniform integrability and the pointwise ergodic theorem, Proc. Amer. Math. Soc. 16 (1965), 222-227.

[3] R. SATO, On pointwise ergodic theorems for positive operators, Studia Math. 97 (2) (1990), 71-84.

[4] R. SATO, Pointwise ergodic theorems in Lorentz Spaces, Studia Math. 144 (3) (1995), 227-236.

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HARDY-ORLICZ SPACES AND HÖRMANDER 'S MULTIPLIERS

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Abstract: We consider Hörmander 's multipliers of fractional order on Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$. The main tools we used are the atomic and molecular decompositions of these spaces.

1. Introduction

In this paper we study multipliers for the Hardy-Orlicz spaces $H_w(\mathbb{R}^n)$. We consider Hörmander's multipliers of order $t > 0$, where t is not necessarely an integer number. In [5], Taibleson and Weiss, proved that the functions m satisfying a Hörmander's type condition (see (1.2)) are multipliers for the classical Hardy spaces $H^p(\mathbb{R}^n)$, $0 < p \leq 1$. There, they use different techniques to deal with the cases t an integer and t real and non-integer (see theorems 4.2 and 4.9).

The purpose of this work is, on one side, to extend these results to the contect of Orlicz spaces. On the other side we present an approach that allows to deal simultaneously with all positive real values of t. Our main tools in this setting are the atomic and molecular decomposition of the Hardy-Orlicz spaces given in [3] and [6].

In order to introduce the spaces $H_w(\mathbb{R}^n)$ we first give some definitions.

Let g be a positive function defined on $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$. We shall say that g is of lower type $m \geq 0$ (respectively, upper type m) if there exists a positive constant C such that

$$g(st) \leq Ct^m g(s)$$

for every $0 < t \leq 1$ (respectively, $t \geq 1$).

Given g, a function of positive lower type l, we define

$$g^{-1}(s) = \sup\{t : g(t) \leq s\}.$$

Assume that w is a function of positive lower type l and upper type $d \leq 1$.

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Let $j \in I\!N$ such that $jl > 1$. We define

$$H_w = H_w(I\!R^n) = \left\{ f \in \mathcal{S}' : \int_{I\!R^n} w\left(\frac{f_j^*(x)}{\lambda^{1/l}}\right) dx < \infty \right\},$$

where f_j^* is the j-maximal function of a distribution $f \in \mathcal{S}'$, the dual space of the class of Schwartz functions (see [1]). We denote

$$||f||_{H_w} = \inf\left\{ \lambda > 0 : \int_{I\!R^n} w\left(\frac{f_j^*(x)}{\lambda^{1/l}}\right) dx \leq 1 \right\}$$

It can be seen that H_w is a complete topological vector space with respect to the quasi-distance induced by $||\ ||_{H_w}$. Moreover H_w is continuously included in $\mathcal{S}'$. Clearly, when $w(t) = t^p$, $0 < p \leq 1$, $H_w(I\!R^n) = H^p(I\!R^n)$. Also it can be proved that for every $f \in H_w$, $\widehat{f}$ is a continuous function on $I\!R^n$ which satisfies

$$\text{(1.1)} \qquad \begin{cases} |\widehat{f}(x)| \leq C\dfrac{w^{-1}(|x|^n)}{|x|^n}||f||_{H_w}^{1/l}, & x \neq 0 \\ \widehat{f}(0) = 0 \end{cases}$$

where C is a constant independent of f, see [5] for the case $H^p(I\!R^n)$.

Suppose that m is a measurable function such that the function $m\widehat{f}$ belongs to $\mathcal{S}'$ whenever $f \in H_w$. We say that m is a multiplier on H_w iff there is a constant $C > 0$ satisfying

$$||(m\widehat{f})^{\vee}||_{H_w} \leq C||f||_{H_w}$$

for all $f \in H_w$.

The Hörmander condition is given in terms of the difference operator which is defined by

$$\Delta_h u(x) = u(x) - u(x-h),$$

where u is a real valued function on $I\!R^n$ and $h \in I\!R^n$. We denote

$$\Delta_h^0 u = u \quad \text{and} \quad \Delta_h^k u = \Delta_h^{k-1}\Delta_h u, \quad k \in I\!N.$$

We say that a function m satisfies the Hörmander condition for $t > 0$ if m is bounded, $|m(x)| \leq A$, and for some integer $\bar{t} > t$ and all integers k, we have

$$\text{(1.2)} \qquad 2^{k(2t-n)} \int_{|h|<2^{k-1}} |h|^{-2t} \int_{2^k<|x|\leq 2^{k+1}} |\Delta_h^{\bar{t}} m(x)|^2 dx \frac{dh}{|h|^n} \leq A^2.$$

It can be proved that if t is an integer, condition (1.2) is equivalent to

$$R^{2|\beta|-n} \int_{R<|x|\leq 2R} |D^\beta m(x)|^2 dx \leq A^2$$

for $0 \leq |\beta| \leq t$ and all $R > 0$.

The main result in this paper is the following:

Theorem (1.3). *Assume that w is a function of positive lower type l and upper type $d \leq 1$. Suppose that m satisfies the Hörmander condition for $t > n(2/l - 3/2)$. Then, there is a constant $C > 0$, independent of m, such that*

$$\|(m\hat{f})^\vee\|_{H_w} \leq CA^l \|f\|_{H_w} \tag{1.4}$$

for every $f \in H_w$. When $w(t) = t^p$, $p \in (0,1]$ we have (1.4) with $t > n\ (1/l - 1/2)$.

The proof of theorem (1.3) is developed in section 3. As principal tools we use the atomic and molecular decompositions of H_w which are contained in section 2.

2. Atomic and molecular decompositions of H_w

In this section we shall give the definitions of the atomic and molecuar Hardy-Orlicz spaces and state some of their properties used in the next section. The proofs of these results can be found in [6] and [3]. As in those papers in the sequel we shall assume that:
(2.1) w is a function of positive lower type l and upper type $d \leq 1$, ρ is the function defined by $\rho(t) = t^{-1}/w^{-1}(t^{-1})$ and $N = [n(1/l - 1)]$, where $[x]$ stands for the biggest integer less than or equal to x.

The following definition will be usefull to introduce the atomic and molecular spaces.

Definition (2.2). Suppose that $\mathbf{b} = \{b_j\}_{j\in I\!N_0}$ is a sequence of functions in $L^2(I\!R^n)$, and $\mathbf{c} = \{c_j\}_{j\in I\!N_0}$ is a sequence of positive constants such that

$$\sum_j c_j w(||b_j||_2 c_j^{-1/2}) = L < \infty, \tag{2.3}$$

where $I\!N_0$ denotes the set of non-negative integers. We define

$$\Lambda(\mathbf{b},\mathbf{c}) = \inf\left\{\lambda > 0 : \sum_j c_j w\left(\frac{||b_j||_2 c_j^{-1/2}}{\lambda^{1/l}}\right) \leq 1\right\} \tag{2.4}$$

We observe that

$$\sum_j c_j w\left(\frac{||b_j||_2 c_j^{-1/2}}{\Lambda(\mathbf{b},\mathbf{c})^{1/l}}\right) \leq \tilde{C},$$

where $\tilde{C}$ is the upper type constant of w.

Definition (2.5). Let $\eta \in I\!N_0$, $\eta \geq N$. A (ρ,η) atom is a real valued function a on $I\!R^n$ satisfying

$$\int_{I\!R^n} a(x)x^\beta dx = 0 \tag{2.6}$$

for every multi-index $\beta = (\beta_1,\ldots,\beta_n)$ such that $|\beta| = \beta_1 + \ldots + \beta_n \leq \eta$, where $x^\beta = x_1^{\beta_1}x_2^{\beta_2}\ldots x_n^{\beta_n}$,
(2.7) the support of a is contained in a ball B and

$$||a||_2|B|^{-1/2} \leq [|B|\rho(|B|)]^{-1}. \tag{2.8}$$

Clearly, when $w(t) = t^p$, $p \in [0,1]$, we have that $\rho(t) = t^{1/p-1}$ and a (ρ,η) atom is a $(p,2,\eta)$ atom in the usual sense (see [5]).

Definition (2.9). We define $H^{\rho,\eta} = H^{\rho,\eta}(I\!R^n)$ as the linear space of distributions f on $\mathcal{S}$ which can be represented by

$$f(\psi) = \sum_j b_j(\psi), \tag{2.10}$$

where $\{b_j\}$ is a sequence of (ρ, η) atoms such that there exists a sequence of balls $\{B_j\}$ satisfying $\mathrm{supp}(b_j) \subset B_j = B(x_j, r_j)$ and (2.3) with $c_j = |B_j|$. We denote $\mathbf{b} = \{b_j\}$, $\mathbf{B} = \{|B_j|\}$ and let

$$||f||_{H^{\rho,\eta}} = \inf \ \Lambda(\mathbf{b}, \mathbf{B}),$$

where $\Lambda(\cdot,\cdot)$ is as in (2.4) and the infimum is taken over all possible representations of f of the form (2.10).

The definition of a molecule in the context of Hardy-Orlicz spaces is the following.

Definition (2.11). Assume that ϵ is admissible, that is $\epsilon > 0$ for the case $w(t) = t^p$ and $\epsilon > 1/l - 1$ for a general w. Let $x_0 \in I\!R^n$. A (ρ, ϵ) molecule centered at x_0 is a real valued function M on $I\!R^n$ satisfying

$$(2.12) \qquad ||M||_2 ||M\rho(|\cdot - x_0|^n)| \cdot - x_0|^{n(\epsilon+1/2)}||_2 \leq C$$

and

$$(2.13) \qquad \int_{I\!R^n} M(x) x^\beta dx = 0$$

for every multi-index β such that $|\beta| \leq N$.

We observe that when $w(t) = t^p$, $p \in (0,1]$, a (ρ, ϵ) molecule is a $(p, 2, \epsilon + 1/p - 1/2)$ molecule in the usual sense (see [2] and [5]).

Remark (2.14). It is not difficult to see that condition (2.12) implies that $Mx^\beta \in L^1$ for every β, $|\beta| \leq N$ and consequently $\widehat{M}$ has continuous derivatives up to the order N. Moreover, we get

$$D^\beta \widehat{M}(\xi) = [-2\pi i x^\beta M(\cdot)]^\wedge(\xi), \qquad \xi \in I\!R^n.$$

From this, we clearly have that if M satisfies (2.12), then (2.13) is equivalent to

$$D^\beta \widehat{M}(0) = 0 \ , \qquad |\beta| \leq I\!N.$$

Given M, a (ρ, ϵ) molecule centered at x_0, and B, a ball with the same center, we denote

$$M^B = M \mathcal{X}_B$$

$$M^{CB} = \frac{\rho(|\cdot - x_0|^n)|\cdot - x_0|^{n(\epsilon+1/2)} M \mathcal{X}_{CB}}{\rho(|B|)|B|^{\epsilon+1/2}}.$$

Definition (2.15). Assume that ϵ is admissible. We define $\mathcal{M}_{\rho,\epsilon} = \mathcal{M}_{\rho,\epsilon}(I\!R^n)$, as the linear space of distributions f on S which can be represented by

(2.16) $$f(\psi) = \sum_j M_j(\psi),$$

where $\{M_j\}$ is a sequence of (ρ, ϵ) molecules centered at $\{x_j\}$, such that there exists a sequence of balls $\{B_j\} = \{B(x_j, r_j)\}$ satisfying

$$\sum_j |B_j| w(||M_j^{B_j}||_2 |B_j|^{-1/2}) + \sum_j |B_j| w(||M_j^{CB_j}||_2 |B_j|^{-1/2}) < \infty.$$

Let $\mathbf{M^B} = \{M_j^{B_j}\}$, $\mathbf{M^{CB}} = \{M_j^{CB_j}\}$ and $\mathbf{B} = \{|B_j|\}$. We define

$$||f||_{\mathcal{M}_{\rho,\epsilon}} = \inf(\Lambda(\mathbf{M^B}, \mathbf{B}) + \Lambda(\mathbf{M^{CB}}, \mathbf{B})) ,$$

where $\Lambda(\cdot,\cdot)$ is as in (2.4) and the infimum is taken over all possible representations of f of the form (2.16).

We finally remark that both spaces $H^{\rho,\eta}$ and $\mathcal{M}_{\rho,\epsilon}$ coincide with the Hardy-Orlicz spaces H_w.

3. Proof of the main result.

In order to prove theorem (1.3) we first give two technical results which proofs are contained in [5], necessary modifications can be carried out.

Proposition (3.1). *Assume that b is a function belonging to $L^2(I\!R^n)$ with vanishing moments up to the order $\eta \in I\!N_0$ and $supp(b) \subset B = B(0, r)$. Then for every $h \in I\!R^n$, $0 \le k \le \eta$, $\delta \in I\!R$ and $E > 0$ we get*

(3.2) $$||(\Delta_h^k \hat{b})^2||_r \le C||b||_2^2 |B|^{\frac{2k}{n}+1-\frac{1}{r}} |h|^{2k}, \quad r \in [1, \infty]$$

and

$$(3.3)\qquad \begin{cases} |\tau_{\delta h}\Delta_h^k\widehat{b}(x)| \le C||b||_2|B|^{\frac{\eta+1}{n}+\frac{1}{2}}|h|^{\eta+1} & \text{if } |x| < E|h| \\ |\tau_{\delta h}\Delta_h^k\widehat{b}(x)| \le C||b||_2|B|^{\frac{\eta+1}{n}+\frac{1}{2}}|h|^{k}|x|^{\eta+1-k} & \text{if } |x| \ge E|h|, \end{cases}$$

where C is a positive constant which does not depend on b, h, and x; and $\tau_h f(x) = f(x+h)$.

We observe that by (3.3) with $\delta = k = 0$, we have

$$(3.4)\qquad |\widehat{b}(x)| \le C||b||_2|B|^{\frac{\eta+1}{n}+\frac{1}{2}}|x|^{\eta+1}$$

for every $x \in I\!R^n$.

Theorem (3.5). *Suppose that m satisfies the Hörmander condition for $t > n/2$. Then there exists a constant C, indepent of m, such that for all integer k,*

$$(3.6)\qquad 2^{k(2\gamma-\frac{n}{r})}\int_{I\!R^n}|h|^{-2\gamma}\Big[\int_{2^k<|x|\le 2^{k+1}}|\Delta_h^{\overline{\gamma}}m(x)|^{2r}dx\Big]^{1/r}\frac{dh}{|h|^n} \le C^2A^2$$

whenever $r = 1$ or $n/r > n - 2(t-\gamma)$, $\gamma \in I\!R^+$, and $\overline{\gamma}$ an integer greater than γ. Furthermore, m is bounded and continuous on $I\!R^n - \{0\}$ and $||m||_\infty \le CA$.

Let us remark that, since the results stated in the following are invariant under change of equivalent functions, in proving them we shall assume without loss of generality that w is in addition continuous and strictly increasing.

Lemma (3.7). *Assume that m satisfies the Hörmander condition for t such that $\epsilon \equiv t/n + 1/2 - 1/l$ is admissible. Suppose that b is a function belonging to $L^2(I\!R^n)$ with vanishing moments up to the order $[t]+1$ and $supp(b) \subset B = B(0,r)$. Then $(m\widehat{b})^\vee$ is a (ρ,ϵ) molecule centered at zero and satisfies*

$$(3.8)\qquad ||(m\widehat{b})^\vee||_2 \le CA||b||_2 \quad \text{and}$$

$$(3.9)\qquad ||(m\widehat{b})^\vee|\cdot|^t||_2 \le CA||b||_2|B|^{t/n}\quad .$$

Proof. Since $t/n+1-1/l>0$ and ρ has upper type $1/l-1$, we can write
$$||(m\hat{b})^{\vee}(\cdot)\rho(|\cdot|^n)|\cdot|^{n(\frac{t}{n}+1-\frac{1}{l})}||_2$$
$$\leq ||(m\hat{b})^{\vee}(\cdot)\rho(|\cdot|^n)|\cdot|^{n(\frac{t}{n}+1-\frac{1}{l})}\mathcal{X}_{|x|\leq 1}(\cdot)||_2+||(m\hat{b})^{\vee}(\cdot)|\cdot|^t\rho(|\cdot|^n)|\cdot|^{n(1-\frac{1}{l})}\mathcal{X}_{|x|\geq 1}(\cdot)$$
$$\leq C(||(m\hat{b})||_2+||(m\hat{b})^{\vee}|\cdot|^t||_2)\ .$$
Then, by (2.14), in order to prove that $(m\hat{b})^{\vee}$ is a (ρ,ϵ) molecule centered at zero, it will be enough to check (3.8), (3.9) and

$$D^{\beta}(m\hat{b})(0)=0 \quad ,\ |\beta|\leq N. \tag{3.10}$$

From the boundedness of m we clearly have (3.8). Let us prove (3.9). Suppose that $\bar{t}=[t]+1$. Then, using the identities
$$\int_{I\!R^n}|(m\hat{b})^{\vee}(x)|^2|x|^{2t}=C\int_{I\!R^n}\frac{||\Delta_h^{\bar{t}}(m\hat{b})(\cdot)||_2^2}{|h|^{n+2t}}dh\ ,$$
which proof is contained in [4] (p. 140), and
$$\Delta_h^{\bar{t}}(fg)=\sum_{k+j=\bar{t}}\binom{\bar{t}}{j}(\tau_{-kh}\Delta_h^{j}f)(\Delta_h^{k}g)\quad ,$$
we need to show that
$$\int_{I\!R^n}\frac{||\tau_{-kh}\Delta_h^{j}\hat{b}(\cdot)\Delta_h^{k}m(\cdot)||_2^2}{|h|^{n+2t}}dh=\left(\int_{|h|<|B|^{-1/n}}+\int_{|h|\geq|B|^{-1/n}}\right)\int_{I\!R^n}\frac{||\tau_{-kh}\Delta_h^{j}\hat{b}(\cdot)\Delta_h^{k}m(\cdot)||_2^2}{|h|^{n+2t}}$$
$$\equiv I_1+I_2\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}$$
for $k+j=\bar{t}$. It is clear that
$$I_2\leq CA^2||b||_2^2\int_{|h|\geq|B|^{-1/n}}\frac{1}{|h|^{n+2t}}dh\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}\ ,$$
because $||m||_\infty\leq CA$. Let us estimate I_1. We first consider the case $k=0$. From (3.2) with $r=1$ we easily obtain
$$I_1\leq CA^2||b||_2^2|B|^{\frac{2\bar{t}}{n}}\int_{|h|<|B|^{-1/n}}|h|^{2(\bar{t}-t)-n}dh\leq CA^2||b||_2^2|B|^{\frac{2t}{n}}.$$

For the estimate of I_1 for the case $k \geq 1$ we choose an integer K such that $2^{nK} < |B|^{-1} \leq 2^{n(K+1)}$ and we get

$$I_1 = \sum_{v=-\infty}^{K-1} \int_{|h|\leq 2^{v+1}} \int_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$+ \sum_{v=-\infty}^{K-1} \int_{2^{v+1}<|h|<|B|^{-1/n}} \int_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$+ \sum_{v=K}^{\infty} \int_{|h|<|B|^{-1/n}} \int_{2^v<|x|\leq 2^{v+1}} \frac{|\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$= S_1 + S_2 + S_3$$

Estimate for S_1. Let z be a nonnegative constant. Aplying the fact that $|B|^{-1/n}|h|^{-1}$ 1 and (3.3) it follows that

$$S_1 \leq |B|^{\frac{-2z}{n}} \sum_{v=-\infty}^{K-1} \int_{|h|\leq 2^{v+1}} \sup_{|x|\in(2^v,2^{v+1}]} |\tau_{-kh}\Delta_h^j \hat{b}(x)|^2 \cdot \frac{1}{|h|^{n+2t+2z}} \int_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^2 dx$$

$$\leq C|B|^{\frac{2(\bar{t}+1)}{n}+1-\frac{2z}{n}} \|b\|_2^2 \sum_{v=-\infty}^{K-1} 2^{v2(\bar{t}+1-j)} \int_{|h|\leq 2^{v+1}} \frac{1}{|h|^{n+2(t+z-j)}} \int_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^2 dx$$

Taking $z = \frac{1}{2}$ when t is an integer and $k = 1$, and $z = 0$ in other case, we can apply (3.5) and we obtain that S_1 is bounded by

$$C|B|^{2\frac{(\bar{t}+1)}{n}+1-\frac{2z}{n}} \|b\|_2^2 \sum_{v=-\infty}^{K-1} 2^{v(2\bar{t}+2-2t-2z+n)} \leq C\|b\|_2^2 |B|^{\frac{2t}{n}},$$

because $2\bar{t} + 2 - 2t - 2z + n > 0$ and $2^K \sim |B|^{-1/n}$.

Estimate for S_3. Take z as in the estimate of S_1. Since $t > n/2$, we can choose $r \geq 1$ such that

$$\frac{n}{r} > n - 2j + 2z$$

and

$$2(t + z - j) - \frac{n}{r} > 0.$$

Thus, from Hölder's inequality, (3.5) and (3.2), we get that S_3 is less than or equal to

$$|B|^{-\frac{2z}{n}} \sum_{v=K}^{\infty} \int_{|h|<|B|^{-1/n}} \left(\int_{I\!R^n} |\tau_{-kh}\Delta_h^j \widehat{b}(x)|^{2r'} dx\right)^{1/r'} \frac{1}{|h|^{n+2t+2z}} \left(\int_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^{2r} dx\right.$$

$$\leq |B|^{\frac{-2z}{n}+\frac{2j}{n}+\frac{1}{r}} ||b||_2^2 \sum_{v=K}^{\infty} \int_{|h|<|B|^{-1/n}} \frac{1}{|h|^{n+2(t+z-j)}} \left(\int_{2^v<|x|\leq 2^{v+1}} |\Delta_h^k m(x)|^{2r} dx\right)^{1/r} dh$$

$$\leq CA^2 ||b||_2^2 |B|^{\frac{-2z}{n}+\frac{2j}{n}+\frac{1}{r}} ||b||_2^2 \sum_{v=K}^{\infty} 2^{-v(2(t+z-j)-\frac{n}{r})}$$

$$\leq CA^2 ||b||_2^2 |B|^{\frac{2t}{n}}.$$

because $2^k \sim |B|^{-1/n}$.
Estimate for S_2. By Tonelli, we can write

$$S_2 \leq \int_{|x|\leq 2^k} \int_{|x|<|h|<|B|^{-1/n}} \frac{|\tau_{-kh}\Delta_h^j \widehat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dh dx$$

$$= \int_{|h|\leq 2^k} \int_{|x|<|h|} \frac{|\tau_{-kh}\Delta_h^j \widehat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$+ \int_{2^k<|h|<|B|^{-1/n}} \int_{|x|\leq 2^k} \frac{|\tau_{-kh}\Delta_h^j \widehat{b}(x)|^2 |\Delta_h^k m(x)|^2}{|h|^{n+2t}} dx dh$$

$$= I_3 + I_4.$$

From (3.3) it is clear that

$$I_3 \leq CA^2 |B|^{\frac{2(\bar{t}+1)}{n}+1} ||b||_2^2 \int_{|h|\leq 2^k} |h|^{2(\bar{t}+1)-2t} dh$$

$$\leq CA^2 ||b||_2^2 |B|^{\frac{2t}{n}},$$

since $|B|^{-1/n} \sim 2^K$. On the other hand, applying (3.2) with $r = \infty$, we have that

$$I_4 \leq CA^2 \int_{2^k<|h|<|B|^{-1/n}} ||b||_2^2 |B|^{\frac{2j}{n}} |h|^{2j-n-2t} dh \leq CA^2 ||b||_2^2 |B|^{\frac{2t}{n}},$$

which concludes the proof of (3.9).

Finally, in order to prove (3.10) we proceed as in the context of the spaces $H^p(I\!R^n)$ applying (2.14), (3.4) and the restriction on t.

Proof of theorem (1.3): Clearly it is sufficient to prove that there exists a constant C independent of m such that

$$(3.11) \qquad ||(m\hat{f})^\vee||_{\mathcal{M}_{\rho,\epsilon}} \leq CA^l ||f||_{H^{\rho,\bar{t}}}$$

for every $f \in H_w$, with $\bar{t}$ and ϵ as in the previous lemma.. Let $f \in H_w$ and assume that b $= \{b_j\}$ is a sequence of multiples of $(\rho, \bar{t})$ atoms such that $f = \sum_j b_j$ in $\mathcal{S}'$. Since $m \in L^\infty$, applying (1.1) it is clear that

$$(m\hat{f})^\vee = \sum_j (m\hat{b}_j)^\vee = \sum_j (m\widehat{\tau_{x_j} b_j})^\vee(\cdot - x_j) \qquad \text{in } \mathcal{S}'.$$

By (3.7) we have that $M_j = (m\widehat{\tau_{x_j} b_j})^\vee(\cdot - x_j)$ is a (ρ, ϵ) molecule centered at x_j which satisfies

$$(3.12) \qquad ||M_j||_2 \leq CA||b_j||_2 \quad \text{and}$$

$$(3.13) \qquad ||M_j| \cdot - x_j|^t||_2 \leq CA||b_j||_2 |B_j|^{t/n}.$$

Let α be a positive real constant. Following the notation of (2.15), by (3.12), we obtain

$$(3.14) \qquad \sum_j |B_j| w \left(\frac{||M_j^{B_j}||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right) \leq \sum_j |B_j| w \left(\frac{CA||b_j||_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}} \right).$$

On the other hand, applying (3.13) and the fact that ρ is of upper type $1/l - 1$ we get

$$\sum_j |B_j| w\left(\frac{\|M_j^{CB_j}\|_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}}\right) =$$

$$(3.15)\quad \begin{aligned} &= \sum_j |B_j| w\left(\frac{\|M_j \mathcal{X}_{CB_j} \rho(|\cdot - x_j|^n)|\cdot - x_j|^{n(1-1/l)}|\cdot - x_j|^t\|_2 |B_j|^{-1/2}}{\rho(|B_j|)|B_j|^{\frac{t}{n}+1-\frac{1}{l}}[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}}\right) \\ &\leq \sum_j |B_j| w\left(\frac{C\|M_j|\cdot - x_j|^t\|_2 |B_j|^{-1/2}}{|B_j|^{t/n}[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}}\right) \\ &\leq \sum_j |B_j| w\left(\frac{CA\|b_j\|_2 |B_j|^{-1/2}}{[\alpha\Lambda(\mathbf{b},\mathbf{B})]^{1/l}}\right). \end{aligned}$$

Since we can assume that the constants C in (3.14) and (3.15) coincide, taking $\alpha = (CA)^l$ it follows that

$$\|(m\hat{f})^\vee\|_{\mathcal{M}_{\rho,\epsilon}} \leq \Lambda(\mathbf{M^B},\mathbf{B}) + \Lambda(\mathbf{M^{CB}},\mathbf{B}) \leq CA^l\Lambda(\mathbf{b},\mathbf{B})$$

where C is a constant independent of m, which proves (3.11)[2].◊

References

[1] Fefferman, C. and Stein, E. M.: "*H^p spaces of several variables*", Acta Math. 129 (1972), 137-193.

[2] García Cuerva, J. and Rubio de Francia, J. L.: "*Weighted norm inequalities and related topics* ", North Holland, Amsterdam, New York, Oxford (1985).

[3] Serra, C.: "*Molecular characterization of Hardy-Orlicz spaces*", Revista de la Unión Matemática Argentina, Vol. 40 (1996), 203-217.

[4] Stein, E. M.: "*Singular integrals and differentiability properties of functions*", Princeton Univ. Press, Princeton, N. J. (1970).

[5] Taibleson, M. and Weiss, G.: "*The molecular characterization of certain Hardy spaces*", Asterisque, 77 (1980), 67-151.

[6] Viviani, B.: "*An atomic decomposition of the predual of BMO(ρ)*", Revista Mat. Iberoamericana, Vol. 3, Nros. 3 y 4 (1987), 401-425.

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A REMARK ON EULER'S CONSTANT

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ABSTRACT. Let x_0 be any real positive non-natural number which satisfies $\Gamma(x_0).k = \Gamma'(x_0)$ with k a rational number. We prove that either Euler's constant γ is trascendental or x_0 is irrational.

Define for $p, q \in N$,

$\alpha(p,q) := \sum_{i=1}^{\infty}\left(\frac{1}{qi} - \frac{1}{qi+p}\right)$ and $F(x) := \sum_{i=1}^{\infty}\left(\frac{x^{qi}}{qi} - \frac{x^{qi+p}}{qi+p}\right)$

Obviously $F(1) = \alpha(p,q)$ and $\frac{dF}{dx} = x^{q-1}\frac{(1-x^p)}{(1-x^q)}$. Thus $\alpha(p,q) = \int_0^1 x^{q-1}\frac{(1-x^p)}{(1-x^q)}dx$ and one obtains, for example, $\alpha(1,2) = 1 - ln2$, $\alpha(1,3) = 1 - \frac{ln3}{2} - \frac{\pi}{6\sqrt{3}}$, etc. Indeed one can compute $\alpha(p,q)$ in closed form with the following formula due to Gauss ([1] pg. 35):

$$\alpha(p,q) = -\frac{1}{2q}\pi cot(\frac{p}{q}\pi) - \frac{1}{q}ln(q) + \frac{S}{q} + \frac{1}{p} \tag{1}$$

where $S = \sum_{r=1}^{(q-1)/2} cos(2\pi rp/q)\ ln[4sin^2(\pi r/q)]$, $(q\ odd)$,

$S = \sum_{r=1}^{(q-2)/2} cos(2\pi rp/q)\ ln[4sin^2(\pi r/q)] + (-1)^p ln2$, $(q\ even)$.

Lemma 1. *$\alpha(p,q) - \frac{1}{p} \neq 0$ for $p, q \in N$, $0 < \frac{p}{q} < 1$.*

Proof. Suppose $\alpha(p,q) = \frac{1}{p}$. Then as $0 < p < q$, $\frac{1}{p} = \int_0^1 x^{q-1}\frac{(1-x^p)}{(1-x^q)}dx \leqslant \int_0^1 x^{q-1}dx = \frac{1}{q}$, a contradiction. ∎

The following theorem, proved in 1966, is due to Baker (see [2] pg.11):

Baker's Theorem. *$e^{\beta_0}.\theta_1^{\beta_1}\ldots\theta_n^{\beta_n}$ is trascendental for any non-zero algebraic numbers $\beta_0, \ldots, \beta_n, \theta_1, \ldots, \theta_n$.*

We use this theorem to prove the following result.

Theorem 1. *$\alpha(p,q)$ is trascendental for every pair $p, q \in N$, $\frac{p}{q}$ non-integer.*

Proof. There is no loss of generality if we assume p, q coprime. It is enough to prove the theorem for $0 < \frac{p}{q} < 1$, because $\alpha(p,q)$ and $\alpha(p + q.n, q)$, $n \in N$, differ by a rational number. Thus assume p, q are coprime and verify $0 < \frac{p}{q} < 1$. Moreover one can assume $\frac{p}{q} \neq \frac{1}{2}$ for $\alpha(1,2) = 1 - ln2$ and $ln2$ is trascendental by Lindemann's theorem ([2], pg. 6).

Recall that the set of algebraic numbers is a field. First observe that $sin(\frac{p}{q}\pi)$ and $cos(\frac{p}{q}\pi)$ are algebraic because $sin(\frac{1}{q}\pi)$ and $cos(\frac{1}{q}\pi)$ are algebraic, and this last assertion follows from De Moivre formula $e^{inx} = (cos\, x + isin\, x)^n$ with $x = \frac{1}{q}\pi$ and $n = q$.
Thus from (1) for one sees that $\alpha(p,q) = \pi.\zeta_0 + \sum_{j=1}^{n} \delta_j log(\zeta_j) + \frac{1}{p}$ with $\zeta_0, \ldots, \zeta_n$, $\delta_1, \ldots, \delta_n$ algebraic and non-zero.
Assume that $\alpha(p,q)$ is algebraic. Then $\beta_0 = \frac{\alpha(p,q)-1/p}{\zeta_0} i = i\pi + \sum_{j=1}^{n} \frac{i\delta_j log(\zeta_j)}{\zeta_0}$ is algebraic and non-zero by lemma 1. Therefore

$$e^{\beta_0}.\zeta_1^{-i(\frac{\delta_1}{\zeta_0})}\zeta_2^{-i(\frac{\delta_2}{\zeta_0})}\ldots\ldots\zeta_n^{-i(\frac{\delta_n}{\zeta_0})} = -1$$

which contradicts Baker's theorem. ■
The point $x_0 \in R$ stands for a non integer positive number which satisfies $\Gamma(x_0).k = \Gamma'(x_0)$ where k is a rational number. Then we have
Theorem 2. *Either x_0 is irrational or γ is trascendental.*
Proof. If x_0 is irrational then the theorem is true. Thus assume $x_0 = p/q$ is a positive rational non-integer number. Recall the well-known formula $\sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+x}\right) - \frac{1}{x} = \frac{\Gamma'(x)}{\Gamma(x)} + \gamma$. Then, replacing x by x_0 in this formula we get $q\alpha(p,q) - q/p = k + \gamma$ and therefore γ is trascendental by theorem 1.■
NOTE: One such point x_0 could be the point where the minimun of $\Gamma(x)$ is attained.

REFERENCES

1. Harold T. Davis, *The Summation of Series*, The Principia Press of Trinity University (1962).
2. Alan Baker, *Transcendental Number Theory*, Cambridge Univesety Press (1979).

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Homogeneous (2,0)-geodesic Submanifolds of Euclidean Spaces

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Abstract

Under the hypothesis that an almost complex submanifold M^n of R^N is $(2,0)$-geodesic and homogeneous, a formula for the canonical covariant derivative of the second fundamental form of the submanifold is obtained. As a consequence of this formula, it is proved that if the submanifold is full then the first normal space coincides with the whole normal space. Other consequence is obtained under more restrictive conditions.

1 Introduction and main results

Let (M, g, J) be a connected Riemannian manifold with metric g and an almost complex structure J. We are not assuming, at least at this point, that the manifold is Hermitian i.e. $g(JX, JY) = g(X, Y)$. Let N be another Riemannian manifold and $\varphi : M \to N$ be an isometric immersion. As usual, we shall denote by α the second fundamental form of the immersion φ. Let $T(M)$ denote the tangent bundle of M and let $T^c(M)$ be its complexification. The almost complex structure J, extended to $T^c(M)$ induces a decomposition of this bundle into its eighenspaces

$$T^c(M) = T^c(M)^{(1,0)} \oplus T^c(M)^{(0,1)}$$

which in turn induces a decomposition of the complexified second fundamental form $\alpha^c$of the isometric immersion φ. This is of course defined as

$$\alpha^c(X_1 + iY_1, X_2 + iY_2) = \alpha(X_1, X_2) - \alpha(Y_1, Y_2) + i[\alpha(X_1, Y_2) + \alpha(Y_1, X_2)]$$

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and if $Z_1,\ Z_2 \in T^c(M)$ then we have, for $k=1,2$,

$$Z_k = \frac{1}{2}(Z_k - iJZ_k) + \frac{1}{2}(Z_k + iJZ_k) = Z_k^{(1,0)} + Z_k^{(0,1)}.$$

Then

$$\begin{aligned}\alpha^c(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(1,0)}\right) + \alpha^c\left(Z_1^{(1,0)}, Z_2^{(0,1)}\right)\\ &\quad + \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right) + \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right).\end{aligned}$$

It is usual to define now

$$\begin{aligned}\alpha^{(2,0)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(1,0)}\right)\\ \alpha^{(0,2)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right)\\ \alpha^{(1,1)}(Z_1, Z_2) &= \alpha^c\left(Z_1^{(1,0)}, Z_2^{(0,1)}\right) + \alpha^c\left(Z_1^{(0,1)}, Z_2^{(1,0)}\right).\end{aligned}$$

The isometric immersion φ is called (i,k)-geodesic if $\alpha^{(i,k)} \equiv 0$.

In the present paper we want to assume that φ is a $(2,0)$-geodesic.

Now recall that, due to the almost complex structure J_p, the tangent space $T_p(M)$ is a complex vector space which is isomorphic to the *holomorphic tangent space* $T^c(M)^{(1,0)}$ by the correspondence $X \mapsto \frac{1}{2}(X - iJ_pX)$. Then, for $X,\ Y \in T_p(M)$ we get, computing by definition

$$\begin{aligned}&\alpha^c(X - iJX, Y - iJY) =\\ &= \alpha(X,Y) - \alpha(JX,JY) + i\left[\alpha(X,JY) + \alpha(JX,Y)\right]\\ &= \alpha^c\left(X^{(1,0)}, Y^{(1,0)}\right) = \alpha^{(2,0)}(X,Y)\end{aligned}$$

Then the condition $\alpha^{(2,0)} \equiv 0$ is clearly amounts to

$$\begin{aligned}(i)\quad & \alpha(X,Y) - \alpha(JX,JY) = 0\\ (ii)\quad & \alpha(X,JY) + \alpha(JX,Y) = 0\end{aligned}$$

and it is clear that (i) and (ii) are equivalent. Then φ is $(2,0)$-geodesic if and only if

$$\alpha(X,Y) = \alpha(JX,JY) \qquad \forall X,\ Y \in T(M). \tag{1}$$

The objective of the present paper is to present the following two results concerning $(2,0)$-geodesic isometric immersions.

Theorem 1 *Let M be a compact homogeneous almost complex Riemannian manifold and $\varphi : M \rightarrow N$ an isometric $(2,0)$-geodesic immersion which is substantial or full (i.e. $\varphi(M)$ is not contained in any proper totally geodesic submanifold of N). Assume furthermore that the immersion φ has the property that its second fundamental form α satisfies Codazzi's equation $\left(\left(\overline{\nabla}_X \alpha\right)(Y,Z) = \left(\overline{\nabla}_Y \alpha\right)(X,Z)\right.$ where $\overline{\nabla}\alpha$ denotes the usual covariant derivative of the second fundamental form α). Then, at each point, the first normal space of the immersion coincides with the whole normal space. i.e. the space generated by the image of the second fundamental form coincides with the normal space.*

If the Riemannian manifold N is R^n then any isometric immersion has the property that its second fundamental form satisfies Codazzi's equation. In a general Riemannian manifold N this may not be the case.

By a homogeneous almost complex Riemannian manifold we mean a Riemannian manifold M supporting a transitive action of a Lie group G of isometries. and having an invariant almost complex structure J which is *not necessarily compatible with the metric* (when this compatibility exists it is customary to say that the manifold is Hermithian).

Let us denote by $\langle .,. \rangle$ the Riemannian metric in the ambient manifold N. In general an isometry g of the group G does not extend to N but it follows easily from the above theorem, that the necessary and sufficient condition for the existence of these extensions is the invariance, by the group G, of the tensor $\Psi(X,Y,Z,W) = \langle \alpha(X,Y), \alpha(Z,W) \rangle$ (see for instance [7]).

The presence of the transitive action of the Lie group G on M yields the existence on M of a canonical affine connection (see [4] or [3]), usually denoted by ∇^c. The invariance of the metric induced by $\langle .,. \rangle$ on M, by the action of the group G, implies that $\nabla^c \langle .,. \rangle = 0$ i.e. the connection ∇^c is compatible with the metric on M.

Let ∇ denote the Riemannian connection on M associated to the metric and let $D(X,Y) = \nabla_X Y - \nabla^c_X Y$ be the difference tensor. Both, the tensor D and the almost complex structure J, are invariant by the action of the group G and hence $\nabla^c D = 0 = \nabla^c J$. Even when the connection ∇^c is compatible with the Riemannian metric it has, in general, non zero torsion and it is easy to see that it has the form $T(X,Y) = D(Y,X) - D(X,Y)$.

As in [6] and [2] we say that the canonical connection ∇^c satisfies **Axiom 6** (with respect to the immersion φ) if for each $p \in M$ and every $X,\ Y,\ Z \in$

$T_p(M)$ the second fundamental form of φ satisfies the identity

$$\alpha_p(T(X,Y),Z) = \alpha_p(Y, D(X,Z)) - \alpha_p(X, D(Y,Z)). \qquad (2)$$

There are plenty of compact manifolds M and isometric immersions $\varphi : M \rightarrow R^N$ such that M admits a canonical connection ∇^c which satisfies Axiom 6. In fact if M is an R-space (also called orbit of an s-representation or real flag manifold) and φ is its canonical imbedding then, for any of the possible canonical connections, Axiom 6 holds (see [6] or [2]).

The following consequence of the proof of Theorem 1 shows that in the case that $N = R^n$, the fact that Axiom 6 holds for a $(2,0)$-geodesic embedding φ of a compact homogeneous almost complex manifold, *implies that M is an R-space and in fact φ must be its canonical imbedding.*

Theorem 2 *Let M be a compact homogeneous almost complex Riemannian manifold and $\varphi : M \rightarrow R^n$ an isometric $(2,0)$-geodesic embedding which is substantial or full (i.e. $\varphi(M)$ is not contained in any proper totally geodesic submanifold of N). Assume that the canonical connection satisfies Axiom 6 with respect to φ. Then M is an R-space and φ is its canonical embedding.*

This result generalizes Theorem 4 in [1, p. 88].

The proof of these two results is contained in the next section.

2 Proof of the results.

Proof of Theorem 1.

Let $\varphi : M \rightarrow R^N$ be the $(2,0)$-geodesic isometric immersion and recall that in [5] a "canonical" covariant derivative of the second fundamental form was introduced by the formula

$$(\nabla^c_X \alpha)(Y,Z) = \nabla^\perp_X(\alpha(Y,Z)) - \alpha(\nabla^c_X Y, Z) - \alpha(Y, \nabla^c_X Z).$$

This covariant derivative is the key ingredient in the characterization of general R-spaces obtained in [4] (see also [2]).

By recalling the definition of the Riemannian covariant derivative of the second fundamental form we obtain immediately

$$(\nabla^c_X \alpha)(Y,Z) = \left(\overline{\nabla}_X \alpha\right)(Y,Z) + \alpha(D(X,Y),Z) + \alpha(Y, D(X,Z)). \qquad (3)$$

Since the second fundamental form α of the immersion φ satisfies Codazzi's equation, by interchanging the letters X and Y and substracting we get

$$\begin{aligned} & (\nabla^c_Y\alpha)(X,Z) - (\nabla^c_X\alpha)(Y,Z) \\ = \; & \alpha(T(X,Y),Z) - [\alpha(Y,D(X,Z)) - \alpha(X,D(Y,Z))]. \end{aligned} \tag{4}$$

This formula replaces Codazzi's equation for the canonical covariant derivative of the second fundamental form α.

Now since our immersion φ is an isometric $(2,0)$-geodesic immersion we have by the condition (1)

$$\alpha(JX,Y) = -\alpha(X,JY). \tag{5}$$

and this yields very easily

$$(\nabla^c_Y\alpha)(JX,Z) = -(\nabla^c_Y\alpha)(X,JZ) \tag{6}$$

Now starting with the identity (4) we write

$$\begin{aligned} & (\nabla^c_Y\alpha)(X,Z) = \\ & = (\nabla^c_X\alpha)(Y,Z) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z)) \\ & = -(\nabla^c_X\alpha)(Y,J^2Z) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z)) \end{aligned}$$

because $J^2 = -I$.

Then by (6)

$$\begin{aligned} & (\nabla^c_Y\alpha)(X,Z) = \\ & (\nabla^c_X\alpha)(JY,JZ) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z)). \end{aligned}$$

Now we may change $(\nabla^c_X\alpha)(JY,JZ)$ using again (4) and then the last equation becomes

$$\begin{aligned} & (\nabla^c_Y\alpha)(X,Z) = \\ & = (\nabla^c_{JY}\alpha)(X,JZ) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z)) \\ & +\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ)). \end{aligned}$$

By using (6) now we get

$$\begin{aligned} & (\nabla^c_Y\alpha)(X,Z) = \\ & = -(\nabla^c_{JY}\alpha)(JX,Z) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z)) \\ & +\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ)) \end{aligned}$$

and (4) again yields

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z) =\\&= -(\nabla^c_Z\alpha)(JX,JY) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ))\\&-\alpha(T(Z,JX),JY) + \alpha(JX,D(Z,JY)) - \alpha(Z,D(JX,JY))\,.\end{aligned}$$

Once more (6) implies

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z) =\\&= (\nabla^c_Z\alpha)(X,J^2Y) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ))\\&-\alpha(T(Z,JX),JY) + \alpha(JX,D(Z,JY)) - \alpha(Z,D(JX,JY))\,.\end{aligned}$$

Using again the identity $J^2 = -I$ we have

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z) =\\&= -(\nabla^c_Z\alpha)(X,Y) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ))\\&-\alpha(T(Z,JX),JY) + \alpha(JX,D(Z,JY)) + \alpha(Z,D(JX,JY))\end{aligned}$$

and (4) once again yields

$$\begin{aligned}&(\nabla^c_Y\alpha)(X,Z) =\\&= -(\nabla^c_Y\alpha)(X,Z) + \alpha(T(X,Y),Z) - \alpha(Y,D(X,Z)) + \alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ))\\&-\alpha(T(Z,JX),JY) + \alpha(JX,D(Z,JY)) - \alpha(Z,D(JX,JY))\\&-\alpha(T(Y,X),Z) + \alpha(X,D(Y,Z)) - \alpha(Y,D(X,Z))\,;\end{aligned}$$

which obviously becomes

$$\begin{aligned}&2(\nabla^c_Y\alpha)(X,Z) =\\&= 2\alpha(T(X,Y),Z) - 2\alpha(Y,D(X,Z)) + 2\alpha(X,D(Y,Z))\\&+\alpha(T(JY,X),JZ) - \alpha(X,D(JY,JZ)) + \alpha(JY,D(X,JZ))\\&-\alpha(T(Z,JX),JY) + \alpha(JX,D(Z,JY)) - \alpha(Z,D(JX,JY))\,.\end{aligned}\tag{7}$$

In the particular case in which $X = Y = Z$ (7) reduces to

$$\begin{aligned}&(\nabla^c_X\alpha)(X,X) =\\&= \alpha(T(JX,X),JX) - \alpha(X,D(JX,JX)) + \alpha(JX,D(X,JX))\,.\end{aligned}\tag{8}$$

By considering formulas (7) and (3) we see immediately that the covariant derivative of the second fundamental form can be written as a linear combination of elements of the first normal space (which, by definition, is the subspace of the normal space generated by the image of the second fundamental form). This clearly implies that the first normal space coincides with the normal space of our immersion φ. and completes the proof of Theorem 1. ■

Proof of Theorem 2.

It follows immediately from Axiom 6 (formula (2)) and formula (8) that, for each point $p \in M$ and each $X \in T_p(M)$,

$$(\nabla^c_X \alpha)(X, X) = 0. \tag{9}$$

Furthermore it follows from (4) that the canonical covariant derivative of the second fundamental form satisfies the identity

$$(\nabla^c_Y \alpha)(X, Z) = (\nabla^c_X \alpha)(Y, Z)$$

for each point $p \in M$ and each $X, Y, Z \in T_p(M)$. This easily implies that the canonical covariant derivative of the second fundamental form vanishes identically on M and since φ is an isometric embedding into a Euclidean space, it follows from [4] that M is an R-space and φ is its canonical embedding. This completes the proof of Theorem 2. ■

References

[1] Ferus, D.: Symmetric submanifolds of Euclidean spaces, *Math. Ann.* **247** (1980), 81-93.

[2] Hulett, E. and Sánchez, C. U.: An algebraic characterization of R-Spaces, *Geometriae Dedicata* 67: 349-365, 1997.

[3] Kowalaski, O.: *Generalized Symmetric Spaces*, Lecture Notes in Math **805**, Springer Verlag, New York 1980.

[4] Olmos, C. and Sánchez, C. U.: A geometric characterization of the orbits of s-representations *J. Reine Angew. Math.* **420** (1991), 195-202.

[5] Sánchez, C. U.: A characterization of extrinsik k-symmetric submanifolds of R^N. Rev. Unión Mat. Argentina. **38** (1992), 1-15.

[6] Sánchez, C. U., Dal Lago, W., García A. and Hulett, E.: On some properties which characterize symmetric and general R-spaces *Differential Geometry and its Applications* 7 (1997), 291-302.

[7] Spivak, M.: *A Comprehensive Introduction to Differential Geometry,* Publish or Perish 1979

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TWO-WEIGHT INEQUALITIES FOR CERTAIN MAXIMAL FRACTIONAL OPERATORS ON SPACES OF HOMOGENEOUS TYPE

by

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Presentado por Carlos Segovia Fernández

Abstract: The study of mean oscillation properties for the fractional integral is naturally conneted with the study of boundedness properties for the composition of the sharp maximal function with the fractional integral. Here, an operator that generalizes that composition on spaces of homogeneous type is considered. Sufficient conditions on pairs of weights are given for which strong and weak weighted inequalities hold for that operator. The work includes a study about the necessity of the conditions on the weights.

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§1. Introduction and statements of the main results

The spaces of homogeneous type were introduced by R. Coifman and G. Weiss in [CW] and they were studied and used by several authors (see [AM], [BS], [C], [MS1], [MS2], [MT], [SW], [W]). Let us recall some definitions and properties relative to them.

Let X be a set. A function $d : X \times X \to \mathbb{R}_0^+$ is called a quasi-distance on X if the following conditions are satisfied:

(1.1) for every x and y in X, $d(x,y) = 0$ if and only if $x = y$,
(1.2) for every x and y in X, $d(x,y) = d(y,x)$ and
(1.3) there exists a constant K such that

$$d(x,y) \leq K(d(x,z) + d(z,y))$$

holds for every x, y and z in X.

The subsets $\{(x,y) : d(x,y) < \varepsilon\}$ of $X \times X$ define a base of metrizable uniform structure on X. Moreover, from this fact, it can be proved that always it is possible to find a distance δ, defined on X, and a number $\alpha \geq 1$ such that d is equivalent to δ^α, i.e.: there exist two constants, D_1 and D_2, such that

$$D_1 \delta(x,y)^\alpha \leq d(x,y) \leq D_2 \delta(x,y)^\alpha \tag{1.4}$$

holds for every x and y in X (see [MS2]).

Let μ be a positive measure on a σ-algebra of subsets of X which contains the balls $B(x,r) = \{y : d(x,y) < r\}$, for every x in X and every finite positive r. We assume that μ satisfy a doubling condition, that is, there exists a constant D such that

$$0 < \mu(B(x,2r)) \leq D\mu(B(x,r)) < \infty \tag{1.5}$$

holds for every ball B in X.

A structure (X, d, μ), with d and μ as above, is called a space of homogeneous type. By keeping in mind (1.4), we can assume (replacing d by δ^α, if it would be necessary) that d is a continuous quasi-distance.

A space of homogeneous type (X, d, μ) is named normal if there exist four constants A_1, A_2, K_1 and K_2, $A_1 \leq A_2$, $K_2 \leq 1 \leq K_1$, such that

(1.6)	$A_1 r \leq \mu(B(x,r))$	if $r \leq K_1 \mu(X)$,
(1.7)	$B(x,r) = X$	if $r > K_1 \mu(X)$,
(1.8)	$A_2 r \geq \mu(B(x,r))$	if $r \geq K_2 \mu(\{x\})$, and
(1.9)	$B(x,r) = \{x\}$	if $r < K_2 \mu(\{x\})$,

holds for every x in X and $r > 0$.

Let w be a positive and locally integrable function defined on a space of homogeneous type (X, d, μ). We denote by $w(E)$ the measure with density w with respect to the measure μ, i.e.: $w(E) = \int_E w \, d\mu$. The density w will be called a weight with respect to μ. We shall say that a pair of weights (u, v) belongs to the class $A(p, q), 1 \leq p \leq \infty$ and $1 \leq q < \infty$, if there exists a constant C such that

$$\left(\frac{u^q(B)}{\mu(B)}\right)^{\frac{1}{q}} \left(\frac{v^{-p'}(B)}{\mu(B)}\right)^{\frac{1}{p'}} \leq C \tag{1.10}$$

holds for every ball B in X, where $p' = p/(p-1)$. In particular, if $q = \infty$ the condition $(u, v) \in A(p, \infty)$ becomes

$$(\operatorname{ess}\sup_B u) \left(\frac{v^{-p'}(B)}{\mu(B)}\right)^{\frac{1}{p'}} \leq C \tag{1.11}$$

for every ball B.

Let (X, d, μ) be a space of homogeneous type. It is not difficult to see that the function $\rho : X \times X \to I\!R_0^+$ defined as

$$\rho(x, y) = \begin{cases} (\mu(B(x, d(x, y))) + \mu(B(y, d(x, y))))/2 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{1.12}$$

satisfies (1.1), (1.2) and (1.3). If there exists $\alpha \geq 1$ such that $\rho^{\frac{1}{\alpha}}$ is a distance we define $\delta = \rho^{\frac{1}{\alpha}}$. In the case that such α does not exist, we reason as before to obtain δ and α such that (1.4) holds. With this choice of δ and α, we introduce, for each γ in $(0, 1)$, the function

$$K_\gamma(x, z, y) = \begin{cases} \delta(x, y)^{\alpha(\gamma-1)} - \delta(z, y)^{\alpha(\gamma-1)} & \text{if } x \neq y \text{ and } z \neq y \\ \mu(\{x\})^{\gamma-1} - \delta(z, y)^{\alpha(\gamma-1)} & \text{if } x = y \text{ and } z \neq y \\ \delta(x, y)^{\alpha(\gamma-1)} - \mu(\{z\})^{\gamma-1} & \text{if } x \neq y \text{ and } z = y \\ 0 & \text{if } x = y = z \end{cases} \tag{1.13}$$

for x, z and y in X. Now, for each γ and each $s \geq 1$, we define the following operator

$$T_\gamma^s f(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)^2} \int_B \int_B \left| \int_X K_\gamma(x, z, y) f(y) d\mu(y) \right|^s d\mu(z) \, d\mu(x)\right)^{\frac{1}{s}} \tag{1.14}$$

for every $x \in X$ and every measurable function f defined on X, where the sup is taken over all the balls B in X containing x.

In the euclidean case (i.e.:$X = I\!R^n$ with the usual distance and the Lebesgue measure), this operator appears naturally connected to the study of mean oscillation properties for the fractional integral I_γ, defined as

$$I_\gamma f(x) = \int_{I\!R^n} \frac{f(y)}{|x-y|^{n(1-\gamma)}}\, dy,$$

for $0 < \gamma < n$. In fact, this study involves, formally, the study of the behaviour of

$$\left(\sup_B \frac{1}{|B|^2} \int_B \int_B |I_\gamma f(x) - I_\gamma f(y)|^s \, dx\, dy\right)^{\frac{1}{s}} \tag{1.15}$$

where the sup is taken over all balls B in $I\!R^n$, for $1 \le s < \infty$ (see, for instance, [MW], p. 269). A more correct mathematical formulation (in order to avoid some problems relative to the convergence of $I_\gamma f$) implies to replace $I_\gamma f(x) - I_\gamma f(y)$ by

$$\int_{I\!R^n} \left(\frac{1}{|x-z|^{n(1-\gamma)}} - \frac{1}{|y-z|^{n(1-\gamma)}}\right) f(z)\, dz$$

in (1.15). But the kernel between parenthesis in the above integral coincides with the euclidean case of (1.13) for $\alpha = n$.

In a general space of homogeneous type (X, d, μ), an extension of the fractional integral can be defined as

$$I_\gamma f(x) = \int_X Q_\gamma(x,y) f(y)\, d\mu(y) \tag{1.16}$$

with

$$Q_\gamma(x,y) = \begin{cases} \delta(x,y)^{\alpha(\gamma-1)} & \text{if } x \neq y \\ \mu(\{x\})^{\gamma-1} & \text{if } x = y \end{cases}$$

for $0 < \gamma < 1$, where δ and α are as in (1.13). So, obviously, the operator associated to the corresponding version of (1.15) is exactly our operator T_γ^s.

The operator T_γ^s was first considered by E. Harboure, R. Macías and C. Segovia, in the euclidean case, in [HMS2], in order to get the boundedness of the fractional operator I_γ from weighted $L^{\frac{n}{\alpha}}$ into weighted BMO. From this result, the authors, as an application of a theorem of extrapolation, proved weigted L^p-norm inequalities for T_γ^s. The purpose of this work is to extend those results to the general setting of spaces of homogeneous type. Our first main result is the following theorem.

(1.17) Theorem: *Let (X, d, μ) be a space of homogeneous type and let $0 < \gamma < 1$. If $(a, b) \in A(1/\gamma, \infty)$, then, for each s in $[1, 1/(1-\gamma))$, there exists a constant C, independent of f, such that*

$$\operatorname{ess}\sup_{x \in X} \left(a(x) T_\gamma^s f(x)\right) \le C \left(\int_X \left(|f(x)|\, b(x)\right)^{\frac{1}{\gamma}} d\mu(x)\right)^\gamma, \tag{1.18}$$

for every measurable function f.

The techniques that we are going to use in order to prove the above theorem are extensions of those used in [HMS2] for the euclidean case. In particular, we will need to know that the left hand side of (1.18) behaves like

$$(1.19)\quad \sup_B \left(ess \sup_{x\in B} \quad a(x) \left(\frac{1}{\mu(B)^2} \int_B \int_B \left| \int_X K_\gamma(x,z,y) f(y)\, d\mu(y) \right|^s d\mu(z)\, d\mu(x) \right)^{\frac{1}{s}} \right)$$

where the sup is taken over all the balls B in X. The proof of this fact follows a similar reasoning, with obvious changes, to that given for the euclidean case (see [HMS1]) and it is omitted here. With this result, Theorem (1.15) can be considered as a result on boundedness of fractional integrals on spaces of homogeneous type. Moreover it is easy to see that Theorem 7, p. 269, in [MW] of B. Muckenhoupt and R. Wheeden, for fractional integral operators in $I\!R^n$, is the euclidean case for $s = 1$ of (1.17). Actually, the techniques in [HMS2] for the euclidean case of (1.17) have been taken from [MW].

Now, we state an extrapolation theorem which will allow us to derive further results about T_γ^s from (1.17). Let us first introduce some notation. For (X, d, μ) be a space of homogeneous type, we denote by $\mathcal{M}$ the set of measurable functions defined on X, and by $\mathcal{M}_0$ the subset of bounded functions. Now, we state the theorem

(1.20) **Theorem:** *Let T be an operator defined on $\mathcal{M}_0$ with values in $\mathcal{M}$. Let us assume that T satisfies*

(1.21) $|T(\lambda f)| = |\lambda||Tf|$ *and* $|T(f+g)| \le |Tf| + |Tg|$ *for every scalar λ and every f and g in $\mathcal{M}_0$*

(1.22) *for a fix pair of numbers r and $\beta, 1 \le r < \beta \le \infty$, and for every pair of such that (a^r, b^r) in $A(\beta/r, \infty)$ the operator T satisfies*

$$ess \sup_{x\in X} \quad (a(x)|Tf(x)|) \le C \left(\int_X (|f(x)| b(x))^\beta \, d\mu(x) \right)^{\frac{1}{\beta}},$$

for any f in $\mathcal{M}_0$, and where C is a finite constant independent of f (for $\beta = \infty$, the left member of the above inequality becomes $ess \sup_{x\in X}(b(x)|f(x)|)$)

Then, for any p, $r < p < \beta$, $1/q = 1/p - 1/\beta$ and $(u^r, v^r) \in A(p/r, q/r)$, there exists a constant C, independent of f, such that

$$u^q(\{x \in X : |T(f(x)| > \lambda\}) \le C \left(\lambda^{-p} \int_X (|f(x)| v(x))^p \, d\mu(x) \right)^{\frac{q}{p}}$$

holds for every $\lambda > 0$.

This theorem can be proved using an argument similar to that of the euclidean case, with only minor modifications. For the euclidean case, see [HMS2].

Now, from theorems(1.17) and (1.20), we easily obtain

(1.23) **Theorem:** *Let (X,d,μ) be a space of homogeneous type and let $0<\gamma<1$, $1<p<1/\gamma$ and $1/q=1/p-\gamma$. If $(u,v)\in A(p,q)$, then, for each s in $[1,1/(1-\gamma))$, there exists a constant C, independent of f, such that*

$$(1.24)\qquad u^q\left(\{x\in X:\ T^s_\gamma f(x)>\lambda\}\right)\le C\left(\lambda^{-p}\int_X(|f(x)|v(x))^p d\mu(x)\right)^{\frac{q}{p}}$$

holds for every $\lambda>0$ and every measurable function f.

Note that in theorems (1.17) and (1.23) we only give sufficient conditions on the weights to ensure that (1.18) and (1.24) hold. One can wonder whether or not they are also neccesary. The answer in both cases is negative, as we can see from following example.

(1.25) *Example*: Let $X=\{0,1\}$, $d(x,y)=|x-y|$ and μ be the measure defined as $\mu(\{0\})=\mu(\{1\})=1$. It is clear that (X,d,μ) is a space of homogeneous type. On the other hand, it is obvious that ρ, defined as in (1.12), gives that $\rho(x,y)=1$, if $x\neq y$, and $\rho(x,y)=0$, if $x=y$, so it is a distance. Therefore, we have $\delta=\rho$ in (1.17), which implies $K_\gamma\equiv 0$, and, as a consequence, $T^s_\gamma f\equiv 0$ for every function f. Now it is evident that (1.18) and (1.24) hold for every pair of weights, in particular, we can take $a\equiv u\equiv 1$ and $b\equiv v\equiv 0$. Since that pair is not in $A(p,q)$ for every p and q, with $1\le p\le\infty$ and $1\le q<\infty$, we have, as we said, that the condition on the weights is not neccessary in (1.17) neither in (1.23).

The above example proves that the reverse implications for (1.15) and (1.21) do not hold in the general case. However, in a more restrictive class of spaces, the normal spaces, we can obtain a result very close to that. In fact, we have

(1.26) **Theorem:** *Let (X,d,μ) be a normal space of homogeneous type. There exist a constant K_0, only depending on the constants of the space, such that*

(1.27) *if (1.18) holds, then the pair (a,b) satisfies*

$$(\operatorname{ess}\sup_B a)\left(\frac{1}{\mu(B)}\int_B b^{-\frac{1}{1-\gamma}}\right)^{1-\gamma}\le C$$

for every ball B with finite radius less than or equal to $K_1K_0^{-1}\mu(X)$.

(1.28) *if* (1.24) *holds, then the pair* (u,v) *satisfies*

$$\left(\frac{1}{\mu(B)}\int_B u^q\right)^{\frac{1}{q}}\left(\frac{1}{\mu(B)}\int_B v^{-p'}\right)^{\frac{1}{p'}}\le C$$

for every ball B *with finite radius less or equal than* $K_1K_0^{-1}\mu(X)$.

In each occurrance, K_1 *is the constant of (1.6) and* C *depends only on the constants of the space and the constant involved in the assumptions.*

From the above theorem, it follows clearly that the reverse implications of (1.17) and (1.23) hold whenever (X,d,μ) is a normal space with $\mu(X)=\infty$. But, when $\mu(X)<\infty$, a result like (1.26) is the best that one can expect without further assumptions. Example (1.25) can be used again to see this. In fact, it is very easy to check that the space (X,d,μ) involved is normal, with constants $K_1 = K_2 = 1$ and $A_2 = A_1^{-1} = 2$. As (1.26) and the example suggest, for $\mu(X)<\infty$, the difficulty to get the necessity of the conditions on the weights relies on the existence of points with too large measure. Actually, we can prove

(1.29) **Corollary:** *Let* (X,d,μ) *be as in (1.24) and such that* $\mu(\{x\})\le 2KK_1K_2^{-1}K_0^{-1}\mu(X)$ *for every* x *in* X, *where* K_0 *is the same constant of the theorem, and* K, K_1 *and* K_2 *are the constants of (1.3), (1.6) and (1.8), respectively. With these assumptions we get*

(1.30) *if* (1.18) *holds, then* $(a,b)\in A(1/\gamma,\infty)$;

(1.31) *if* (1.22) *holds, then* $(u,v)\in A(p,q)$.

The proofs of (1.17), (1.26) and (1.29) are in the next section.

§2. Proofs

The proof of theorem (1.17) requires the following result concerning a weak type inequality for the operator I_γ, defined in (1.16). This result extends a well known property for the usual fractional integral in $\mathbb{R}^n$.

(2.1) **Lemma:** *Let* $0<\gamma<1$. *The operator* I_γ *is of weak type* $(1,(1-\gamma)^{-1})$, *i.e.: there exists a constant* C, *such that*

$$\mu(\{x\in X:|I_\gamma f(x)|>\lambda\})\le C\left(\frac{1}{\lambda}\int_X |f|\,d\mu\right)^{\frac{1}{1-\gamma}}, \tag{2.2}$$

holds for every $\lambda>0$ *and every* f *in* $L^1(X,d\mu)$.

Proof. It is clear, from the definition of Q_γ, that we only need to prove (2.2) for I_γ defined using the kernel $K_\gamma(x,y) = \mu(\bar{B}(x,d(x,y)))^{\gamma-1}$, where $\bar{B}(x,r)$ denotes the set $\{y \in X : d(x,y) \leq r\}$, instead of Q_γ. Let $R > 0$. We define

$$I_\gamma^i f(x) = \int_X K_\gamma^i(x,y) f(y) d\mu(y) \quad , i = 1,2 \ ,$$

where $K_\gamma^1(x,y) = K_\gamma(x,y)\chi_{\{(x,y):\mu(\bar{B}(x,d(x,y)))<R\}}$ and $K_\gamma^2(x,y) = K_\gamma(x,y) - K_\gamma^1(x,y)$. Now, let $y \in X$. Let us consider the sets $\Omega_j = \{x \in X : \mu(\bar{B}(y,d(x,y))) < 2^{-j-1}R\}$ for $j = 0,1,\ldots.$ By defining $R_j = \sup\{d(y,x) : \mu(\bar{B}(y,d(y,x))) < 2^{-j-1}R\}$, where the sup is taken over all $x \in X$, it can be proved that $\Omega_{j+1} \subset \bar{B}(y,R_j)$ and $\mu(\bar{B}(y,R_j)) \leq C2^{-j}R$ (see [MT], Lemma (2.5), p. 9). Then, we get

$$\int_X K_\gamma^1(x,y)\, d\mu(x) \leq C \sum_{j=0}^{\infty} \int_{\Omega_{j+1}-\Omega_j} K_\gamma^1(y,x) d\mu(x)$$

$$\leq C \sum_{j=0}^{\infty} \left(\frac{2^j}{R}\right)^{1-\gamma} \mu(\Omega_{j+1})$$

$$\leq CR^\gamma \sum_{j=0}^{\infty} 2^{-j\gamma} = CR^\gamma$$

The above inequality and Tonelli's theorem allow us to obtain

$$\int_X |I_\gamma^1 f(x)|\, d\mu(x) \leq \int_X \int_X K_\gamma^1(x,y)|f(y)|\, d\mu(y)\, d\mu(x)$$

$$(2.3) \qquad = \int_X |f(y)| \left(\int_X K_\gamma^1(x,y)\, d\mu(x)\right) d\mu(y)$$

$$\leq CR^\gamma \int_X |f(y)|\, \mu(y).$$

On the other hand, we have

$$(2.4) \qquad \int_X |I_\gamma^2 f(x)|\, d\mu(x) \leq R^{\gamma-1} \int_X |f(y)|\, d\mu(y).$$

Finaly, given $\lambda > 0$, (2.2) follows from the obvious inequality

$$\mu(\{x \in X : |I_\gamma f(x)| > \lambda\}) \leq \mu(\{x \in X : |I_\gamma^1 f(x)| > \frac{\lambda}{2}\}) + \mu(\{x \in X : |I_\gamma^2 f(x)| > \frac{\lambda}{2}\})$$

and the estimates (2.3) and (2.4) with $R = \lambda^{\frac{1}{\gamma-1}}$.■

Proof of Theorem (1.17). As we said after the statement of the theorem in § 1, the left member of (1.18) is equivalent to (1.19). Hence we only need to prove that (1.19) is bounded by the right member of (1.18). To do that we first notice that the expression between the inner parenthesis in (1.19) is bounded by the sum of the following terms

$$\left(\frac{1}{\mu(B)^2}\int_B\int_B\left|\int_{\tilde{B}}K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \tag{2.5}$$

and

$$\left(\frac{1}{\mu(B)^2}\int_B\int_B\left|\int_{X-\tilde{B}}K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \tag{2.6}$$

Here we denote with $\tilde{B}$ to the ball with same center that B and radius equal to $2K$ times the radius of B, where K is the constant of (1.3).

Now, we consider the extension of the fractional integral operator I_γ defined in (1.16). From lemma (2.1) and Kolmogorov's and Hölder's inequalities, we have that (2.5) is bounded by

$$\begin{aligned} 2\left(\frac{1}{\mu(B)}\int_B |I_\gamma(f\chi_{\tilde{B}})|^s\, d\mu\right)^{\frac{1}{s}} &\leq C\frac{1}{\mu(\tilde{B})^{1-\gamma}}\int_{\tilde{B}}|f|\,d\mu \\ &\leq C\left(\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}} b^{\frac{-1}{1-\gamma}}d\mu\right)^{1-\gamma}\left(\int_X (|f|b)^{\frac{1}{\gamma}}\,d\mu\right)^{\gamma} \end{aligned} \tag{2.7}$$

for each s in $[1\,,(1-\gamma)^{-1})$, where C is independent of f and B. On the other hand, from the definitions of K_γ and δ and applying the mean value theorem, we have that

$$\begin{aligned} |K_\gamma(x,y,z)| &= \left|\delta(x,y)^{\alpha(\gamma-1)} - \delta(z,y)^{\alpha(\gamma-1)}\right| \\ &\leq C\frac{\left|\delta(x,y)^{\alpha(1-\gamma)} - \delta(z,y)^{\alpha(1-\gamma)}\right|}{\mu(B(x_0,d(x_0,y)))^{2(1-\gamma)}} \\ &\leq C\frac{\delta(x,z)}{\mu(B(x_0,d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} \\ &\leq C\frac{\mu(B)^{\frac{1}{\alpha}}}{\mu(B(x_0,d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} \end{aligned}$$

holds for every x, z in B and every y in $X - \tilde{B}$, where x_0 is the center of B and C is independent of B. Therefore, (2.6) is bounded by

(2.8)
$$C\mu(B)^{\frac{1}{\alpha}} \int_{X-\tilde{B}} \frac{|f(y)|}{\mu(B(x_0, d(x_0,y)))^{1-\gamma+\frac{1}{\alpha}}} d\mu(y)$$

$$\leq C\mu(B)^{\frac{1}{\alpha}} \left(\int_X (|f(y)|\, b(y))^{\frac{1}{\gamma}}\, d\mu(y) \right)^{\gamma} \left(\int_{X-\tilde{B}} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu(B(x_0, d(x_0,y)))^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y) \right)^{1-\gamma}$$

Note that $\mu(B(x_0, d(x_0,y))) \geq R_0$, with $R_0 = C\mu(B)$ for every y in $X - \tilde{B}$, where C is independent of B. Now, let $\Omega_j = \left\{ y \in X : \mu\left(\overline{B}(x_0, d(x_0,y))\right) < 2^j R_0 \right\}$ for $j = 0, 1, \ldots$, where the sets $\overline{B}(x,r)$ are defined as in the proof of Lemma (2.1). By using Lemma (2.5) of [MT], as in the above Lemma, we get $\Omega_{j+1} \subset \bar{B}(x_0, R_j)$ and $\mu(\bar{B}(x_0, R_j)) \leq C 2^{j+1} R_0$, where C only depends on the constants of the space and $R_j = \sup_{y \in X} \{ d(x_0, y) : \mu(\bar{B}(x_0, d(x_0,y))) \leq 2^{j+1} R_0 \}$. Then, we can bound the integral over $X - \tilde{B}$ in the right member of (2.8) by

$$C \int_{X-\Omega_0} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu\left(\bar{B}(x_0, d(x_0,y))\right)^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y)$$

$$= C \sum_{j=0}^{\infty} \int_{\Omega_{j+1}-\Omega_j} \frac{b(y)^{-\frac{1}{1-\gamma}}}{\mu\left(\bar{B}(x_0, d(x_0,y))\right)^{1+\frac{1}{\alpha(1-\gamma)}}} d\mu(y)$$

$$\leq C \sum_{j=0}^{\infty} (2^j R_0)^{-1-\frac{1}{\alpha(1-\gamma)}} \int_{\Omega_{j+1}} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y)$$

$$\leq C \sum_{j=0}^{\infty} (2^j R_0)^{-\frac{1}{\alpha(1-\gamma)}} \frac{1}{\mu(B(x_0, 2R_j))} \int_{B(x_0, 2R_j)} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y).$$

Now, combining this inequality with (2.8), we get that (2.6) is bounded by a constant times the following expresion

$$\left(\sum_{j=0}^{\infty} \frac{2^{-\frac{j}{\alpha(1-\gamma)}}}{\mu\left(B(x_0, 2R_j)\right)} \int_{B(x_0, 2R_j)} b(y)^{-\frac{1}{1-\gamma}}\, d\mu(y) \right)^{1-\gamma} \left(\int_X (|f(y)|\, b(y))^{\frac{1}{\gamma}}\, d\mu(y) \right)^{\gamma}$$

Finally, since $B \subset \bar{B}(x_0, R_j)$ for every $j \geq 0$, from the hypothesis on (a, b) we obtain the wished boundedness for (1.19) from the above result and the estimate (2.7) of (2.5). This concludes the proof.∎

In order to prove Theorem (1.26) we need the next lemma.

(2.9) **Lemma:** *Let (X,d,μ) be a normal space of homogeneous type. For each γ in $(0,1)$ and each $s \geq 1$, there exist two constants, K_0 and C, depending only on s, γ and the constants of the space such that for every ball $B = B(x_0,R)$ satisfying $K_2(2K)^{-1}\mu(\{x_0\}) \leq R \leq K_1K_0^{-1}\mu(X)$, where K, K_1 and K_2 are the constants of (1.3), (1.6) and (1.8), respectively, the inequality*

$$
\left(\frac{1}{\mu(B^*)^2}\int_{B^*}\int_{B^*}\left|\int_B K_\gamma(x,z,y)f(y)d\mu(y)\right|^s d\mu(z)d\mu(x)\right)^{\frac{1}{s}} \geq C\frac{1}{\mu(B)^{1-\gamma}}\int_B f(x)d\mu(x) \tag{2.10}
$$

holds with $B^* = B(x_0,K_0R)$ for every non negative function f.

Proof. Let $B = B(x_0,R)$ be a ball in X and let θ be fixed in $(0,A_1A_2^{-1})$, where A_1 and A_2 are the constants of (1.6) and (1.8). From these conditions, it follows that $B(x_0,L\theta^{-1}R) - B(x_0,LR) \neq \emptyset$ whenever $K_2\mu(\{x_0\}) \leq LR \leq \theta K_1\mu(X)$, where L is a constant to be chosen later. Let x_1 be in the above annulus. Using (1.3) it is not difficult to prove that

$$
\left(\frac{L}{K^2}-\frac{1}{K}-1\right)R \leq d(z,y) \leq \left(K+K^2\left(\frac{L}{\theta}+1\right)\right)R \tag{2.11}
$$

holds for every y in B and every z in $B(x_1,R)$. On the other hand, we know that there exist two constants, D_1 and D_2, such that

$$
D_1\mu(B(x,d(x,y))) \leq \delta(x,y)^\alpha \leq D_2\mu(B(x,d(x,y))) \tag{2.12}
$$

holds for every x and y in X, where δ is the quasi-distance of (1.13). Then, from this relation, (2.11), (1.6) and (1.8), and choosing $L = K^2(A_2D_2A_1^{-1}D_1^{-1}4K + K^{-1} + 1)$, we get

$$
\begin{aligned}
\mu(B(y,d(z,y))) &\geq A_1\left(\frac{L}{K^2}-\frac{1}{K}-1\right)R \\
&\geq A_1\frac{A_2}{A_1}\frac{D_2}{D_1}4KR \\
&\geq 2\frac{D_2}{D_1}\mu(B(y,2KR)),
\end{aligned} \tag{2.13}
$$

for every $y \in B$ and every $z \in B(x_1, R)$, whenever x_0 and R satisfy.

$$\frac{K_2}{2K}\mu(\{x_0\}) \leq R \leq (K + K^2(L\theta^{-1}+1))^{-1}K_1\mu(X)$$

Therefore, under this restriction, from (2.12), (2.13) and definition of K_γ, we have

$$K_\gamma(x,z,y) \geq (D_2\mu(B(y,2KR)))^{\gamma-1} - (D_1\mu(B(y,d(z,y))))^{\gamma-1}$$

$$\geq (D_2\mu(B(y,2KR)))^{\gamma-1}\left(1-2^{\gamma-1}\right)$$

$$\geq C\mu(B)^{\gamma-1},$$

for every x and y in B and every z in $B(x_1,R)$. Finally, (2.10) follows inmediately from the above inequality and the fact that $B(x_1,R) \subset B^* = B(x_0, K_0R)$ with $K_0 = K + K^2(L\theta^{-1}+1)$.∎

Proof of Theorem (1.26). First let us see (1.27). Let $B = B(x_0,R)$ be a ball such that $R \leq K_1K_0^{-1}\mu(X)$, where K_0 is the same constant of the above lemma. If $K_2(2K)^{-1}\mu(\{x_0\}) \leq R$ from that lemma, (1.18) and the equivalence between (1.19) and the left member of (1.18), it follows inmediately that

$$(\operatorname{ess}\sup_B a)\frac{1}{\mu(B)^{1-\gamma}}\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu \leq C\left(\int_B\left(b_k^{-\frac{1}{1-\gamma}}b\right)^{\frac{1}{\gamma}}d\mu\right)^{\gamma}$$

$$\leq C\left(\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu\right)^{\gamma}$$

holds for every $k \in \mathbb{N}$, where $b_k = b + 1/k$. Thus, we get

$$(\operatorname{ess}\sup_B a)\left(\frac{1}{\mu(B)}\int_B b_k^{-\frac{1}{1-\gamma}}\,d\mu\right)^{1-\gamma} \leq C,$$

and letting $k \to \infty$ we obtain the wished inequality. Now, suppose $R < K_2(2K)^{-1}\mu(\{x_0\}) \leq K_1K_0^{-1}\mu(X)$. If $K_1K_0^{-1}\mu(X) < K_2\mu(\{x_0\})$ the result clearly follows from the above case since, according (1.9), $B(x_0,R) = \{x_0\}$ for every $R \leq K_2\mu(\{x_0\})$. On the other hand, if $K_2\mu(\{x_0\}) \leq K_1K_0^{-1}\mu(X)$, we get the result taking $\tilde{R}$ such that $K_2(2K)^{-1}\mu(\{x_0\}) \leq \tilde{R} \leq K_2\mu(\{x_0\})$ and applying the first case again since $B(x_0,R) = B(x_0,\tilde{R}) = \{x_0\}$. This completes the proof of (1.27).

In order to proof (1.28), let us consider a ball $B = B(x_0,R)$ such that, as before, $R \leq K_1K_0^{-1}\mu(X)$. If $K_2(2K)^{-1}\mu(\{x_0\}) \leq R$, from lemma (2.6), the definition of T_γ^s we have

$$T_\gamma^s f(x) > C\frac{1}{\mu(B)^{1-\gamma}}\int_B f\,d\mu,$$

for every $x \in B$ and every non negative function f. Then, by taking $f = v_k^{-p'}\mathcal{X}_B$, where $v_k = v + 1/k$, $k \in I\!N$, inequality (2.4) allows us to obtain

$$u^q(B) \leq C\left(\lambda^{-p}\int_B \left(v_k^{-p'}v\right)^p d\mu\right)^{\frac{q}{p}}$$

$$\leq C\mu(B)^{(1-\gamma)q}\left(\int_B v_k^{-p'}d\mu\right)^{-\frac{q}{p'}}$$

for every $k \in I\!N$. This yields

$$\frac{u^q(B)}{\mu(B)^{(1-\gamma)q}}\left(\int_B v_k^{-p'}d\mu\right)^{-\frac{q}{p'}} \leq C$$

and letting $k \to \infty$ we achieve the inequality of (1.28) in the considered case. Finally, in the other cases, the result can be obtained by applying a reasoning as in the proof of (1.27).■

The proof of Corollary (1.29) requires the following lemma concerning to the geometry of the spaces of homogeneous type.

(2.14) **Lemma:** *Let (X, d, μ) be a space of homogeneous type and let θ belonging to $(0,1)$. There exist a number N only depending on θ and the constants of the space, such that, for each $x_0 \in X$ and each $R > 0$, it can be founded a set $\{x_i\}_{i\in I}$ satisfying*

(2.15) *$x_i \in B(x_0, R)$ for every $i \in I$,*

(2.16) *$B(x_0, R) \subset \cup_{i\in I}B(x_i, \theta R)$,*

(2.17) *the cardinal of I is less or equal than N.*

Proof. The lemma is a straightforward consequence of the fact that, given θ in $(0,1)$, there exists a number N, only depending on θ and the constants of the space, such that in each ball B the amount of points whose mutual distance are bigger or equal than θ times the radius of B is, at most, N (see [CW], p. 68).■

Proof of Corollary (1.29). If $\mu(X) = \infty$, (1.30) and (1.31) are obvious from (1.27) and (1.28), respectively. So, let us assume $\mu(X) < \infty$. In order to prove (1.30), since (1.27) holds, we just need to show that (1.18) implies the inequality of (1.27) in the case $B = X$. Let x_0 be a point in X and let $R = 2K_1\mu(X)$ where K_1 is the constant of (1.18). Then $B(x_0, R) = X$. Now, applying lemma (2.14) with $\theta = (2K_0)^{-1}$, where K_0 is the constant in (1.24), we obtain a finite family of balls, $\{B_i\}_{i\in I}$ with radius $K_1K_0^{-1}\mu(X)$, such that $B(x_0, R) \subset \cup_{i\in I}B_i$. Moreover, from (1.18) and (2.9), it follows that

$$(\operatorname*{ess\,sup}_X a)\frac{1}{\mu(B_i)^{1-\gamma}}\int_{B_i} b_k^{-\frac{1}{1-\gamma}}d\mu \leq C\left(\int_{B_i} b_k^{-\frac{1}{1-\gamma}}d\mu\right)^{\gamma}$$

holds for every $k \in I\!N$ and every $i \in I$, where $b_k = b + 1/k$. By reasoning as in the proof of (1.26), we get

$$(ess \sup_X a)\left(\frac{1}{\mu(B_i)}\int_{B_i} b^{-\frac{1}{1-\gamma}}d\mu\right)^{1-\gamma} \leq C$$

for every $i \in I$. Therefore, since μ verifies (1.5), an standard argument allows us to obtain

$$(ess \sup_X a)\left(\frac{1}{\mu(X)}\int_X b^{-\frac{1}{1-\gamma}}d\mu\right)^{1-\gamma} \leq C$$

This concludes the proof of (1.30). Let us see (1.31). As before, we only need to consider the case $B = X$. Let $\{B_i\}_{i\in I}$ be as above. From the definition of T_γ^s and Lemma (2.6), it follows that there exists a constant C, only depending on s, γ and the constant of the space, such that

$$\frac{C}{\mu(B_i)^{1-\gamma}}\int_{B_i} v_k^{-p'} < T_\gamma^s\left(\mathcal{X}_{B_i}v_k^{-p'}\right)(x)$$

holds for every $k \in I\!N$ and every $i \in I$ and every $x \in X$, where $v_k = v + 1/k$. Then, from (1.22) with λ equal to the left member of the above inequality, we have

$$\frac{u^q(X)}{\mu(B_i)^{(1-\gamma)q}}\left(\int_{B_i} v_k^{-p'}\right)^{\frac{q}{p'}} \leq C$$

for every $i \in I$. Then, taking $k \to \infty$, we get

$$\frac{u^q(X)}{\mu(B_i)^{(1-\gamma)q}}v_k^{-p'}(B_i)^{\frac{q}{p'}} \leq C$$

for any $i \in I$. Finally, applying an argument like the used in the proof of (1.28), it follows that

$$\frac{u^q(X)}{\mu(X)^{(1-\gamma)q}}v_k^{-p'}(X)^{\frac{q}{p'}} \leq C$$

holds, as we wanted to prove.∎

References

[AM] Aimar, H. and Macías, R.: "*Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type*", Proc. Amer. Math. Soc. 91 (1984), 213-216.

[BS] Bernardis, A. and Salinas, O.: "*Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type*", Studia Math. 108 (1994), 201-207.

[C] Calderón, A.: "Inequalities for the maximal function relative to a metric", Studia Math. 57 (1976), 297-306

[CW] Coifman, R and Weiss, G.: "*Analyse harmonique non-conmutative certains espaces homogènes*", Lectures Notes in Math., 242, Springer-Verlag 1971.

[HMS1] Harboure, E.; Macías, R. and Segovia, C.: "*Extrapolation results for classes of weights*", Amer. J. of Math. 110 (1988), 383-397.

[HMS2] Harboure, E.; Macías, R. and Segovia, C.: "*An extrapolation theorem for pairs of weights*", to appear in Revista de la Unión Matemática Argentina.

[MS1] Macías, R. and Segovia, C.: "*A decomposition into atoms of distributions on spaces of homogeneous type*", Advances in Math. 33 (1979), 271-309.

[MS2] Macías, R. and Segovia, C.: "*Lipschitz functions on spaces of homogeneous type*", Advances in Math. 33 (1979), 257-270.

[MT] Macías, R. and Torrea, J.: "*L^2 and L^p boundedness of singular integrals on non necessarly normalized spaces of homogeneous type*", Cuadernos de Matemática y Mecánica, No. 1-88, PEMA-INTEC-GTM, Santa Fe (Argentina).

[MW] Muckenhoupt, B. and Wheeden, R.: "*Weighted norm inequalities for fractional integrals*", Trans. of the A.M.S. 192 (1974), 261-274.

[SW] Sawyer, E. and Wheeden, R.: "*Wheighted inequalities for fractional integrals on euclidean and homogeneous spaces*", Amer. J. Math. 114 (1992), 813-874.

[W] Wheeden, R.: "*A characterization of some weighted norm inequalities for the fractional maximal function*", Studia Math. 107 (1993) 257-272.

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A REMARK ON NUMBERS WITH POWERS IN A POINT-LATTICE

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ABSTRACT. We prove by using elementary methods that if the positive powers of a given complex nonreal number b belong to a point-lattice Λ then they belong also to the point-lattice L generated by 1 and b and b is a quadratic integer. This settles the following question. Let D be a finite set of rational integers that contains 0 and 1. If the set of values of polynomials with coefficients in D evaluated at b is included in Λ, is it true or not that it is part of L?

I. INTRODUCTION. We shall assume that $b=b_1+ib_2$ is a fixed complex number with $|b|>1$, b_2=Im(b)>0. The point-lattice L:=$[b,1]=\{mb+n;m,n\in Z\}$ is naturally associated with b. Let be $u=u_1+iu_2$, $v=v_1+iv_2$, $v_2\neq 0$ and $\Lambda:=[v,u]=\{mv+nu:m,n\in Z\}$ the point-lattice generated by the *linearly independent* numbers u,v. Define P:= $\{b^k;k=1,2,\dots\}$. The following result holds (cf. [1]):

THEOREM 1. If u=1 then $P\subset\Lambda\Rightarrow P\subset L$.▪

II. THE MAIN RESULT. We shall prove the following generalization of this theorem.

THEOREM 2. If $b^j\in\Lambda$ for j≥N then b is a quadratic integer and $P\subset L$.▪

If $b^{j+N}\in\Lambda$ for j=0,1,2 and $b^{j+N}=m_ju+n_jv$ then

(1) $b^j=m_j\tilde{u}+n_j\tilde{v}$ for j=0,1,2,... with $\tilde{u}=b^{-N}u$ and $\tilde{v}=b^{-N}v$.

So, we can assume without loss of generality that N=0.

We begin with two auxilliary results.

PROPOSITION 1. If $b^j\in\Lambda$ for j ≥ 0 then $|b|^2\in$Z.▪

PROOF. We know that for j=1,2,..., $b^{j-1}=m_{j-1}v+n_{j-1}u$ with m_{j-1}, n_{j-1} rational integers. Any three of these equations is a homogeneous system in 1,u and v. Then we have for any j=1,2,...

(2) $$\begin{vmatrix}-b^{j-1} & m_{j-1} & n_{j-1}\\ -b^j & m_j & n_j\\ -b^{j+1} & m_{j+1} & n_{j+1}\end{vmatrix}=0.$$

If we *define*: $A_j=m_{j-1}n_j-m_jn_{j-1}$, $B_j=m_{j-2}n_j-m_jn_{j-2}$, then $A_j\neq 0$ and

(3) $$b^2A_j-bB_{j+1}+A_{j+1}=0.$$

Since the coefficients in (3) are real and b is not real, we must have

(4) $$2\,\mathrm{Re}(b) = \frac{B_{j+1}}{A_j} \text{ and } |b|^2 = \frac{A_{j+1}}{A_j} \text{ for j=1,2,....}$$

Multiplying the last identities from j=1 to k, one gets

(5) $$|b|^{2k} = \frac{A_{k+1}}{A_1}.$$

Thus $A_1|b|^{2k}$ is an integer for any k. Therefore $|b|^2$ must be an integer, QED.

PROPOSITION 2. If $b^j \in \Lambda$ for $j \geq 0$ then there are rational integers α_j, β_j such that

(6) $$A_1 b^j = \alpha_j b + \beta_j. \blacksquare$$

PROOF. Regarding the identities $b^{j-1} = m_{j-1}v + n_{j-1}u$ for j=1,2,k as a homogeneous system in $1,u,v$, one gets that

(7) $$\begin{vmatrix} -1 & m_0 & n_0 \\ -b & m_1 & n_1 \\ -b^k & m_k & n_k \end{vmatrix} = 0$$

This yields the thesis with $\alpha_j = m_0 n_j - m_j n_0$ and $\beta_j = m_j n_1 - m_1 n_j$, QED.

PROOF OF THEOREM 2. From (3) and proposition 1 one gets

(8) $$b^2 = \frac{p}{q} b - k,$$

where $k = |b|^2$, p,q coprime integers. Using (8) one can prove by induction on j that

(9) $$b^j = \left(\left(\frac{p}{q}\right)^{j-1} + \sum_{h<j-1} \sigma_{jh} \left(\frac{p}{q}\right)^h \right) b + \sum_{h<j-1} \lambda_{jh} \left(\frac{p}{q}\right)^h$$

where small greek letters represent rational integers.

Comparing (6) and (9) we get

(10) $$\alpha_j = A_1 \left(\left(\frac{p}{q}\right)^{j-1} + \sum_{h<j-1} \sigma_{jh} \left(\frac{p}{q}\right)^h \right) = \text{rational integer for all } j > 2.$$

Thus, $\alpha_j = A_1 (p^{j-1} + q\gamma_j)/q^{j-1}$. This can only hold for q=1, QED.

COROLLARY. There is an integer K=K(b,u,v,N) such that if $b^j \in \Lambda$ for K+N≥j≥N then b is a quadratic integer.$\blacksquare$

III. NECESSARY CONDITIONS FOR $b^2 \in [1,b]$. In this section we assume that $b^2 = mb + n$, $m,n \in Z$, ($n \neq 0$). We obtain from this hypothesis that

(11) $$k := |b|^2 = -n \in \mathbf{N}, \ \mathrm{Re}(b) = m/2 \in Z/2$$

THEOREM 2. If $b, b^2 \in \Lambda = [u,v]$ with $u = u_1 + u_2 i$ and $v = v_1 + v_2 i$ then

$$u_1, v_1 \in Q, \ \frac{u_2}{b_2}, \frac{v_2}{b_2} \in Q. \blacksquare$$

PROOF. Solving the following system

(12) $$b = m_0 u + n_0 v \ , \ b^2 = mb - k = m_1 u + n_1 v$$

for u and v, we get that

$$A_1 u = (n_1 - mn_0)b + n_0 k \ , \ A_1 v = (mm_0 - m_1)b - m_0 k \ .$$

Taking real and imaginary parts and using (11) we obtain

$$A_1 u_1 = (n_1 - mn_0)m/2 + n_0 k \ , \ A_1 u_2 = (n_1 - mn_0)b_2$$

$$A_1 v_1 = (mm_0 - m_1)m/2 - m_0 k \ , \ A_1 v_2 = (mm_0 - m_1)b_2 \ ,$$ QED.

NB. The results of the present paper should be compared with Theorem 1 of [3].

REFERENCES.

[1] Benedek A. and Panzone R., On complex bases for number systems with the digit set {0,1}, Actas 4to Congreso "Dr. A.A.R. Monteiro", (1997)17-32.
[2] Hardy G. H. and Wright E. M., AN INTRODUCTION TO THE THEORY OF NUMBERS, Oxford, (1960).
[3] Indlekofer K.-H., Kátai I. and Racskó P., Some remarks on generalized number systems, Acta Sci. Math (Szeged), 57(1993)543-553.

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Interpretability into Łukasiewicz Algebras

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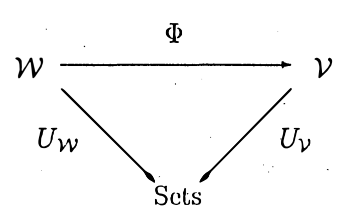
Abstract

In this paper we give a characterization of all the interpretations of the varieties of bounded distributive lattices, De Morgan algebras and Lukasiewicz algebras of order m in the variety of Lukasiewicz algebras of order n.
In the case of distributive lattices we give a structure theorem that is generalized to De Morgan algebras and to Łukasiewicz algebras of order m. In the last two cases we also give the number of such interpretations.

1 Introduction

We say that a variety $\mathcal{V}$ is *interpretable* in a variety $\mathcal{W}$, in symbols, $\mathcal{V} \leq \mathcal{W}$, if for each $\mathcal{V}$-operation $F_t(x_1, \ldots, x_n)$ there is a $\mathcal{W}$-term $f_t(x_1, \ldots, x_n)$ such that if $\langle A, G_t \rangle$ is in $\mathcal{W}$, then $\langle A, f_t^A \rangle$ is in $\mathcal{V}$. Intuitively, $\mathcal{V} \leq \mathcal{W}$ means that all algebras in $\mathcal{W}$ can be turned into an algebra in $\mathcal{V}$ by defining the $\mathcal{V}$-operations applying a uniform procedure. This notion of interpretation differs from that used by logicians in that the universe of the algebra remains the same. It was first proposed in [7] and later developed in [5]; for more details and information the reader is referred to the latter monograph.
Another way of thinking about this notion is the following. The above relation defines a functor $\Phi : \mathcal{W} \longrightarrow \mathcal{V}$ which commutes with the underlying set functors, i.e.:



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is commutative; here $U_{\mathcal{V}} : \mathcal{V} \longrightarrow Sets$ and $U_{\mathcal{W}} : \mathcal{W} \longrightarrow Sets$ are the so called forgetful functors which assign to each algebra its universe. Each functor Φ is called an *interpretation of* $\mathcal{V}$ *in* $\mathcal{W}$.

If $\boldsymbol{A} = \langle A; G_t\rangle$ is any algebra and for each $\mathcal{V}$–operation $F_t(x_1,\ldots,x_n)$ there is a term $f_t(x_1,\ldots,x_n)$ in the language of $\boldsymbol{A}$ such that $\langle A; f_t^{\boldsymbol{A}}\rangle$ is in $\mathcal{V}$, the terms $f_t(x_1,\ldots,x_n)$ define an interpretation of $\mathcal{V}$ in $\mathcal{V}(\boldsymbol{A})$, the variety generated by the algebra $\boldsymbol{A}$. One only has to observe that the evaluation of any term in an algebra $\boldsymbol{B}$ in $\mathcal{V}(\boldsymbol{A})$, is determined by its evaluation in $\boldsymbol{A}$ and that both $\langle A; G_t\rangle$ and $\langle B; G_t\rangle$ satisfy the same equations. We sometimes say that $\mathcal{V}$ is interpretable in $\boldsymbol{A}$ and if Φ is the functor, we say that $\Phi(\boldsymbol{A})$ is an interpretation of $\mathcal{V}$ in $\mathcal{V}(\boldsymbol{A})$. This fact is particularly useful if we want to interpret a variety $\mathcal{V}$ in a variety $\mathcal{W}$ that is generated by a single algebra.

In this paper we will study what are all the possible interpretations of the varieties of bounded distributive lattices, De Morgan algebras and Lukasiewicz algebras of order m in the variety $\mathcal{L}_n$ of Lukasiewicz algebras of order n. As we know, this variety is generated by a single algebra, the n element chain, which is a semi–primal algebra. These are the main facts used in the proofs.

The results in sections 3 and 4 are included in [6], the author's doctoral dissertation *Interpretations between Varieties of Algebraic Logic.* The general presentation and most of the proofs are different from the ones that appear there.

2 Definitions and Preliminaries

Throughout this paper $\mathcal{D}_{01}$ will stand for the variety of bounded distributive lattices, $\mathcal{DM}$ the variety of De Morgan algebras, i.e., the class of all algebras $\langle A\,;\,+,\,\cdot\,,\,',\,\mathbf{0},\mathbf{1}\rangle$ whose similarity type is (2,2,1,0,0) and such that $\langle A,\ +,\ \cdot\ ,\ \mathbf{0},\ \mathbf{1}\rangle$is in $\mathcal{D}_{01}$ and satisfies

1. $(x+y)' = x'\,y'$,
2. $(x\cdot y)' = x' + y'$,
3. $x'' = x$,

The term x' is called the *quasi-complement* of x. Also, x and x' are said to be *conjugates.* The variety $\mathcal{L}_n$ of Lukasiewicz algebras of order n is the class of all algebras $\langle A\,;\,+,\cdot,\,',\,\sigma_1,\ldots,\sigma_{n-1},\,\mathbf{0},\,\mathbf{1}\rangle$ of type (2,2,1,... ,1,0,0) such that $\langle A\,;\,+,\cdot,\,',\,\mathbf{0},\,\mathbf{1}\rangle$ is a De Morgan algebra and for $1\leq i\leq n-1$,

1. $\sigma_i(x+y) = \sigma_i(x)+\sigma_i(y)$ and $\sigma_i(x\cdot y) = \sigma_i(x)\cdot\sigma_i(y)$,
2. $\sigma_i(x) + (\sigma_i(x))' = \mathbf{1}$ and $\sigma_i(x)\cdot(\sigma_i(x))' = \mathbf{0}$,
3. $\sigma_i(\sigma_j(x)) = \sigma_j(x)$, for $1\leq j\leq n-1$,
4. $\sigma_i(x') = (\sigma_{n-i}(x))'$,
5. $\sigma_i(x)\cdot\sigma_j(x) = \sigma_i(x)$, for $i\leq j\leq n-1$,
6. $x+\sigma_{n-1}(x) = \sigma_{n-1}(x)$ and $x\cdot\sigma_1(x) = \sigma_1(x)$,
7. $y\cdot(x+(\sigma_i(x))'+\sigma_{i+1}(y)) = y$, for $i\neq n-1$.

These axioms are not independent. The reader is referred to [2], [1], [3] and [4] for more information about these classes of algebras.
The following four properties of Lukasiewicz algebras will be used extensively in section 5. The first two are immediate from axioms (1), (5) and (1), respectively. The fourth one was introduced in the original definition of Lukasiewicz algebras instead of axioms (6) and (7); its proof appears in [3].

Lemma 2.1.

(L_1) $\sigma_i(0) = 0$ *and* $\sigma_i(1) = 1$, *for* $1 \leq i \leq n-1$.

(L_2) $\sigma_1(x) \leq \cdots \leq \sigma_{n-1}(x)$.

(L_3) If $x \leq y$, *then for* $1 < i \leq n-1$, $\sigma_i(x) \leq \sigma_i(y)$.

(L_4) If $\sigma_i(x) = \sigma_i(y)$, *for* $1 \leq i < n$, *then* $x = y$.

We will now define a very important Lukasiewicz algebra.

Definition. Let $n = \{0, 1, \ldots, n-1\}$. We define the algebra

$$\mathcal{N} = \langle n; +, \cdot, ', \sigma_1, \ldots, \sigma_{n-1}, \mathbf{0}, \mathbf{1} \rangle,$$

where

$$\begin{aligned} x + y &= \max\{x, y\}, \\ x \cdot y &= \min\{x, y\}, \\ m' &= n - 1 - m, \quad \text{for each } m \in n, \\ \mathbf{0} &= 0, \\ \mathbf{1} &= n - 1, \end{aligned}$$

and for $1 \leq i \leq n-1$

$$\sigma_i(m) = \begin{cases} 1 & \text{if } i \leq m, \\ 0 & \text{if } i > m. \end{cases}$$

It is easy to check that $\mathcal{N}$ is in $\mathcal{L}_n$. The next theorems give some of the most important features of Lukasiewicz algebras that we will use in the sequel. Their proofs and much more can be found in [1], [2] and [3].

Theorem 2.2. (Cignoli) [3]
Let $\boldsymbol{L} \in \mathcal{L}_n$, $n \geq 2$ *and* L *of cardinality greater than 1. Then the following are equivalent.*

1. *$\boldsymbol{L}$ is a chain.*
2. *$\boldsymbol{L}$ is an $\mathcal{L}_n$–subalgebra of $\mathcal{N}$.*
3. *$\boldsymbol{L}$ is subdirectly irreducible.*

Corollary 2.3. *The variety $\mathcal{L}_n$ is generated by the algebra $\mathcal{N}$.*

This corollary has a very important consequence. As we said in the introduction, any interpretation of a variety $\mathcal{V}$ in $\mathcal{L}_n$ is determined by an interpretation of $\mathcal{V}$ in $\mathcal{N}$, that is to say, by defining new term–defined operations f_t, for each $\mathcal{V}$–operation F_t, such that $\hat{\mathcal{N}} = \langle n; f_t^{\mathcal{N}} \rangle \in \mathcal{V}$.

Theorem 2.4. *$\mathcal{N}$ is a semi–primal algebra.*

As we know, in a semi–primal algebra all functions that preserve subuniverses can be represented by term functions. In the following theorem we will state this precisely in the special cases which we will use, that of unary and binary functions.

Theorem 2.5. *If $f : n \longrightarrow n$ is such that for all $a \in n$, $f(a) \in \{0, a, a', 1\}$, then there exists a term $\varphi(x)$ such that*

$$\varphi^{\mathcal{N}}(x) = f(x).$$

If $g : n \times n \longrightarrow n$ is such that for all $a, b \in n$, $g(a, b) \in \{0, a, a', b, b', 1\}$, then there exists a term $\gamma(x, y)$ such that

$$\gamma^{\mathcal{N}}(x, y) = g(x, y).$$

Lemma 2.6. *For any $a, b \in n$ and any $\mathcal{L}_n$–term $\alpha(x)$ or $\beta(x, y)$,*

$$\alpha^{\mathcal{N}}(a) \in \{0, a, a', 1\} \qquad \text{and} \qquad \beta^{\mathcal{N}}(a, b) \in \{0, a, a', b, b', 1\}.$$

Proof. Simply observe that $\{0, a, a', 1\}$ and $\{0, a, a', b, b', 1\}$ are subuniverses of $\mathcal{N}$. □

Corollary 2.7. *If $a \notin \{0, 1\}$ and $a = \beta^{\mathcal{N}}(b, c)$ for some term $\beta^{\mathcal{N}}(x, y)$, then either $b \in \{a, a'\}$ or $c \in \{a, a'\}$.*

3 Interpreting $\mathcal{D}_{01}$ in $\mathcal{L}_n$

We will let $\hat{\mathcal{N}} = \langle n; \oplus, \odot, \hat{0}, \hat{1}\rangle$ be an interpretation of $\mathcal{D}_{01}$ in $\mathcal{L}_n$, that is, $x \oplus y$ and $x \odot y$ are binary $\mathcal{L}_n$-terms, $\hat{0}$ and $\hat{1}$ are $\mathcal{L}_n$–constant terms such that $\langle n; \oplus, \odot, \hat{0}, \hat{1}\rangle$ is a bounded distributive lattice.

Notice that while theorem 2.5 gives us a lot of flexibility, lemma 2.6 restricts the possible values of $x \oplus y$ and $x \odot y$. As for the constants, $\{\hat{0}, \hat{1}\} = \{0, 1\}$.

We will prove several lemmas that will enable us to determine some special cases and a general structure theorem. The strategy is to use lemma 2.6 and the fact that $\hat{\mathcal{N}}$ is a distributive lattice to determine the possible values of the term functions defined by the terms $x \oplus y$ and $x \odot y$.

Throughout this paper, the following well known property of distributive lattices will be used without explicitly mentioning it. If $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$, then $b = c$.

All the lemmas in this section refer to the lattice $\hat{\mathcal{N}}$. The first ten deal with the cases when $\hat{1}$ is join–reducible and $\hat{0}$ is meet–reducible. The next three are the cases when $\hat{0}$ is meet–reducible, when $\hat{1}$ is join–reducible and when there are some other meet–reducible and join–reducible elements. The main theorem 3.12 summarizes all these.

Lemma 3.1. *There is at most one pair of conjugates $a, a' \in n$, different from 0 and 1, such that $a \oplus a' = \hat{1}$.*

Proof. Assume there exists $a, b \in n$, a, b different from 0 and 1, a and b not conjugates, such that $a \oplus a' = \hat{1}$ and $b \oplus b' = \hat{1}$. By lemma 2.6 and since $\mathcal{N}$ is a lattice, this implies that $a \odot a' = \hat{0}$ and $b \odot b' = \hat{0}$.
Assume $a \oplus b = \hat{1}$. Then multiplying by a', we get $(a \odot a') \oplus (b \odot a') = b \odot a' = a'$ and then $a \odot b = a'$, so $b = (a \odot b) \oplus (a' \odot b) = (a \odot b) \oplus a'$ and then Corollary 2.7 forces $a \odot b = b'$. But then $b = b \oplus (a \odot b) = \hat{1}$, a contradiction, so $a \oplus b \neq \hat{1}$.
Assume either $a \oplus b = a$ or $a \oplus b = b$. In this case either $a' \oplus b = a' \oplus (a \oplus b) = \hat{1}$ or $a \oplus b' = (a \oplus b) \oplus b' = \hat{1}$, and this is the same as case 1. interchanging the roles of a and a' or those of b and b'.
Assume either $a \oplus b = a'$ or $a \oplus b = b'$. In this case either $a \oplus a' = a \oplus b \neq \hat{1}$ or $b \oplus b' = a \oplus b \neq \hat{1}$.
Since obviously $a \oplus b \neq \hat{0}$, under the hypotheses $a \oplus b$ cannot be defined, so we may conclude that there is at most one pair of conjugate elements a and a' such that $a \oplus a' = \hat{1}$. □

Lemma 3.2. *There is no element different from* 0 *and* 1 *that covers or is covered by more than two elements.*

Proof. Suppose $a \notin \{0, 1\}$ covers three different elements b, c, d. That is $a = b \oplus c = b \oplus d = d \oplus c$.
From Corollary 2.7, we may assume w.l.o.g. that $b = a'$ and $c \neq a'$, $d \neq a'$, but then $d \oplus c = a$, contradicting lemma 2.6.
A dual argument shows that a is not covered by more than two elements. □

Lemma 3.3. *If there exist three elements a, b and c different from $\hat{1}$ such that $\hat{1} = a \oplus b = b \oplus c = c \oplus a$, then $n = 8$.*
Dually, if there exist three elements a, b and c different from $\hat{0}$ such that $\hat{0} = a \odot b = b \odot c = c \odot a$, then $n = 8$.

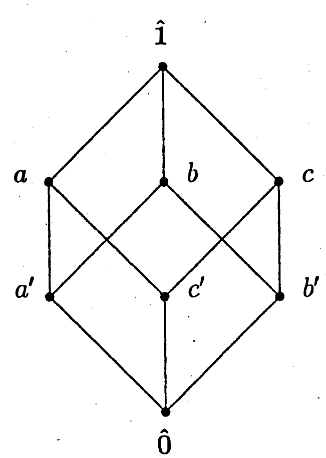


Diagram 1

Proof. Let us assume that there exist three such elements a, b and c as in Diagram 1. ($\hat{1}$ is not necessarily a cover of a, b and c.)

Suppose $a \odot b = \hat{0}$. Then $a = a \odot \hat{1} = a \odot (b \oplus c) = a \odot c$, so $a \oplus c \neq \hat{1}$, a contradiction, so $a \odot b \in \{a', b'\}$. Similarly, $a \odot c \in \{a', c'\}$ and $b \odot c \in \{b', c'\}$. Moreover the three products are all different or else $a = b$, $a = c$ or $b = c$. In particular this also implies that no two of them are conjugates and that $a \neq a'$, $b \neq b'$ and $c \neq c'$. So we have at least eight elements.
Let $a \odot b = a'$, then by the last remarks, $a \odot c = c'$ and this implies $b \odot c = b'$. Similarly, if $a \odot b = b'$, then $b \odot c = c'$ and $a \odot c = a'$, that is, the choice of $a \odot b$ (or of one of the others) determines the values of $a \odot c$ and of $b \odot c$ and we get the lattice in Diagram 1, (or one with b', a' and c' instead of a', c' and b', respectively.)
Let us now assume that $n > 8$ and let d be different from all of the above.
Suppose $a \oplus d = \hat{1}$. Then of course $a \odot d \notin \{a, d, \hat{1}\}$ and also $a \odot d \notin \{a', \hat{0}\}$, or else $d \in \{b, b'\}$, (or $d \in \{c, c'\}$.) Thus $a \odot d = d'$.
Now $b \oplus d \neq \hat{1}$, or else the same argument would show that $b \odot d = d'$ and this leads to $a = b$. Similarly, $c \oplus d \neq \hat{1}$.
Also, $b \oplus d \neq d$, or else $c \oplus d = \hat{1}$ and $b \oplus d \neq d'$, or else $a = a \oplus d' = a \oplus (b \oplus d) = \hat{1}$. So $b \oplus d = b$ and similarly $c \oplus d = c$. But then $\hat{0} = a \odot (b \odot c) = a \odot ((b \oplus d) \odot (c \oplus d)) = a \odot ((b \odot c) \oplus d) = a \odot d = d'$, a contradiction, thus $a \oplus d \neq \hat{1}$. Similarly we prove that $b \oplus d \neq \hat{1}$ and $c \oplus d \neq \hat{1}$.

Suppose now that $a \oplus d \in \{d, d'\}$. Then $d \oplus b = \hat{1}$ or $d' \oplus b = \hat{1}$, a contradiction. Finally, the only choice is $a \oplus d = a$, so multiplying this by b', we get $d \odot b' = \hat{0}$. But then $d \oplus b' \notin \{\hat{1}, \hat{0}, d, b'\}$. Also, $d \oplus b' \neq b$, or else $d = a'$ and $d \oplus b' \neq d$, or else $a \oplus d = \hat{1}$. Since there is no possible value for $a \oplus d$, such an element cannot exist and $n = 8$.
The proof of the dual is similar. □

Lemma 3.4. *Assume there exists an $a \notin \{0, 1\}$ such that $a \oplus a' = \hat{1}$. If $a \oplus b = \hat{1}$ for some $b \notin \{a, a', 1, 0\}$, then the subalgebra of $\hat{\mathcal{N}}$ generated by a and b is the lattice in Diagram 2 (a).*

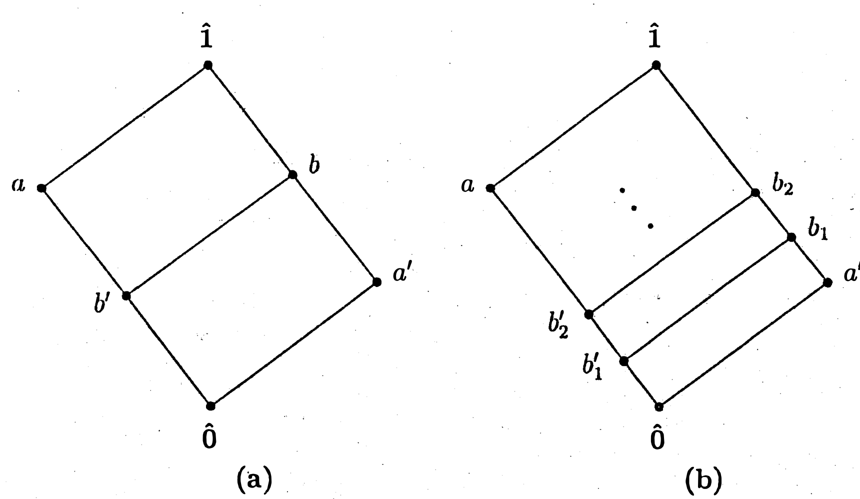


Diagram 2

Proof. Let $b \notin \{0,1,a,a'\}$. Since $a \oplus b = \hat{1}$, $a \odot b \notin \{a,b,\hat{1}\}$. Also $a \odot b \neq \hat{0}$, or else $b = a'$ $a \odot b \neq a'$, or else $a \oplus a' = a \neq \hat{1}$, so $a \odot b = b'$ and thus $a \odot b' = b'$. But then $a' \odot b' = a' \odot (a \odot b') = \hat{0}$.
A similar dual argument shows that $a' \oplus b' = b$, which completes the proof of our lemma. □

Corollary 3.5. *If* $\hat{1}$ *is a cover of* a *and* a'*, then* $n = 4$.

Lemma 3.6. *Let* $n \neq 4, 8$*. Assume there exists an* $a \notin \{0,1\}$*, such that* $a \oplus a' = \hat{1}$*. If* $a \oplus b = \hat{1}$ *for some* $b \notin \{0,1,a,a'\}$*, then for all* $c \notin \{0,1,a,a',b,b'\}$*, either*

$$a \oplus c = \hat{1} \quad \text{and} \quad a \odot c = c' \quad \text{or}$$
$$a \oplus c' = \hat{1} \quad \text{and} \quad a \odot c' = c,$$

and thus $\hat{\mathcal{N}}$ *is the lattice in Diagram 2 (b). The intermediate elements need not exist.*

Proof. By lemma 3.4, the lattice generated by a and b is the lattice in Diagram 2 (a). Let $c \notin \{0,1,a,a',b,b'\}$
If $a \oplus c = \hat{1}$, then by lemma 3.4, the subalgebra of $\hat{\mathcal{N}}$ generated by a and c is the lattice in Diagram 2 (a), with b replaced by c, that is, $a \odot c = c'$.
Since $n \neq 8$, $b \oplus c \neq \hat{1}$, so either $b \oplus c = b$ or $b \oplus c = c$, in which case either $b \oplus c' = c$ and $b \odot c' = b'$ or $b' \oplus c = b$ and $b' \odot c = c'$, respectively. Since this is the case with any other element d such that $a \oplus d = \hat{1}$, the theorem follows.
If $a \oplus c = a$, then $a' \odot c = \hat{0}$ and thus $a' \oplus c \neq \hat{1}$, since the latter would entail $a = c$. So $a' \oplus c = c'$ and thus $a \oplus c' = \hat{1}$ and we are back in the previous case.
If either $a \oplus c = c$ or $a \oplus c = c'$, then there is an element between a and $\hat{1}$. We may assume it is c. But then $b \oplus c = \hat{1}$ and since $c > a > c'$, $b \odot c = c'$ is the only possibility for $b \odot c$, but this is clearly impossible since in that case $a = c$. □

Lemma 3.7. *If* a *and* b *are not conjugates,* $a \oplus b = \hat{1}$ *and* $a \odot b = \hat{0}$*, then neither* $a = a'$ *nor* $b = b'$.

Proof. Suppose $a \oplus b = \hat{1}$, $a \odot b = \hat{0}$ and $a = a'$. Then $b \neq b'$, since there is only one element x such that $x = x'$.
If $b \oplus b' = \hat{1}$, then else $a = b'$, so either $b < b'$ or $b' < b$.
If $b < b'$, $a \oplus b' = \hat{1}$ and in that case $a \odot b' \neq \hat{0}$, or else $b' = b$. So $a \odot b' = b$, but then $a = a \oplus (a \odot b') = a \oplus b = \hat{1}$.
On the other hand, if $b' < b$, $a \odot b' = \hat{0}$ and the dual of the above argument provides a contradiction. □

Theorem 3.8. *If* a *and* b *are not conjugates, they are both different from* 0 *and* 1*,* $a \oplus b = \hat{1}$ *and* $a \odot b = \hat{0}$*, then* $n = 6$ *or* $n = 8$.

Proof. Notice that by lemma 3.7 we need at least six elements. Also, $a \oplus a' \neq \hat{1}$ and $b \oplus b' \neq \hat{1}$ or else a and b are conjugates.
By renaming if necessary, we may assume that $a \oplus a' = a$ and $b \oplus b' = b'$.
This implies that $a' \odot b = \hat{0}$, so $a' \oplus b \neq \hat{1}$ or else $a = a'$, contradicting lemma 3.7.
We can easily check that the subalgebra generated by a and b, is the one depicted in Diagram 3.

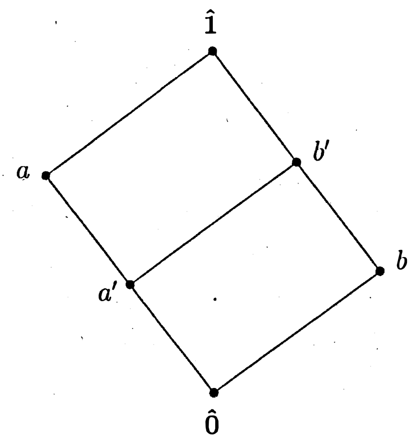
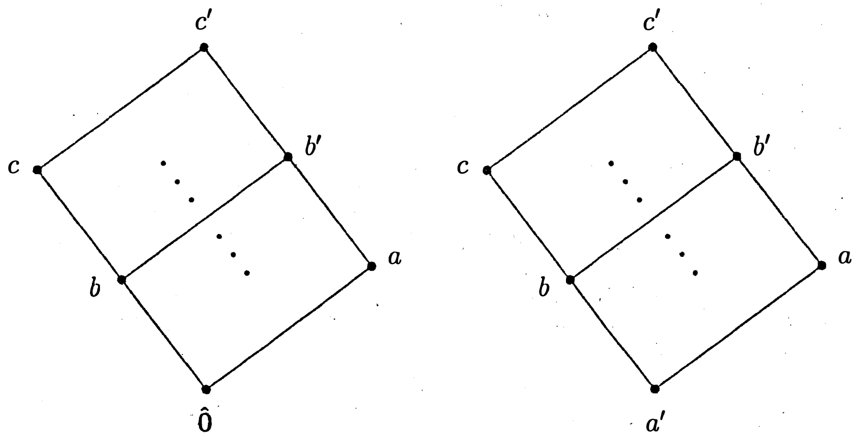


Diagram 3

This proves that if $n = 6$, there is a possible interpretation with the features of the hypothesis.
Let us now assume $n \geq 7$, so let c be different from all of the above. Suppose $a \oplus c = \hat{1}$. Using a now familiar argument, $a \odot c \notin \{\hat{1}, a, a', c\}$, the latter would imply $b' = c$. Also, $a \odot c \neq \hat{0}$, or else $c = b$, so the only possibility is $a \odot c = c'$. Multiplying by b', we get $a' \odot c = c' \odot b'$.
If $b' \oplus c = c$, then $c' \odot b' = a' \odot c = a' \odot (b' \oplus c) = a'$ and if $b' \oplus c = c'$, then $a' \odot c = c' \odot b' = (b' \oplus c) \odot b' = b'$. Both cases contradict lemma 2.6. The only possibility left is $b' \oplus c = \hat{1}$, so by lemma 3.3, $n = 8$.
Suppose $a \oplus c = c$. Then $b \oplus c = \hat{1}$, so $b \odot c = c'$ and similarly $b' \oplus c = \hat{1}$, so $b' \odot c = c'$ and this implies $b = b'$, a contradiction. We get a similar contradiction if we assume $a \oplus c = c'$ and since there are no other possibilities, the theorem is proved. □

Lemma 3.9. *Let $n \neq 6$, 8. If there exist elements a and b such that $a \odot b = \hat{0}$, $a \oplus b = b'$ and $a \oplus a' \neq \hat{1}$, then there exists an element $c \in n$ such that the interval $[\hat{0}, c']$ of $\hat{\mathcal{N}}$ is the lattice depicted in Diagram 4 (a) and c is the $\hat{\mathcal{N}}$-largest such an element, (that is, for any element d such that $d \odot a = \hat{0}$, $d \odot c = c$.) The intermediate elements need not exist. c' is meet–irreducible.*

Proof. If there is no $x \in n$ other than b such that $x \odot a = \hat{0}$, we let $c = b$.
Since $n \neq 6$, 8 and $a \oplus a' \neq \hat{1}$, there is no $x \in n$ such that $x \odot a = \hat{0}$ and $x \oplus a = \hat{1}$. We will now prove that there is no $x \in n$ such that $x \odot a = \hat{0}$ and $x \oplus a = a'$. If on the contrary there is one, since $n \neq 8$, $b \odot x \neq \hat{0}$ and obviously $b \odot x \neq b'$.
Suppose $b \odot x \in \{x, x'\}$, then $b \odot a' = b \odot (x \oplus a) = b \odot x \in \{x, x'\}$ and this contradicts lemma 2.6.
Suppose $b \odot x = b$, then $b' \oplus x = (a \oplus b) \oplus x = a \oplus (b \oplus x) = a \oplus x = a'$, which also contradicts lemma 2.6.
So if $x \odot a = \hat{0}$, then $x \oplus a \neq a'$ and thus $x \oplus a = x'$, as in the Diagram.



(a) (b)

Diagram 4

Now the set of all elements $x \in n$ such that $x \odot a = \hat{0}$ and $x \oplus a = x'$ has to be linearly ordered since if for two such elements x and y, $x \odot y \neq x, y$, then $x \odot y = \hat{0}$, contradicting the fact that $n \neq 8$. Take c to be the largest one. By Corollary 2.7, c' is meet–reducible. □

By duality, we can prove the following.

Corollary 3.10. *Let $n \neq 6, 8$. If there exist elements a and c such that $a \oplus c = \hat{1}$, $a \odot c = c'$, and $a \odot a' \neq \hat{0}$, then there exists an element $c \in n$ such that the interval $[c', \hat{1}]$ of $\hat{\mathcal{N}}$ is dual to the lattice depicted in Diagram 4 (a) and c is the $\hat{\mathcal{N}}$-least such an element. The intermediate elements need not exist. Also, c' is join–irreducible.*

Lemma 3.11. *If there exist a, c both different from $\mathbf{0}$ and $\mathbf{1}$ such that $a \oplus c = c'$ and $a \odot c = a'$, then the interval $[a', c']$ is the lattice in Diagram 4 (b).*
Moreover, if there is no element b such that $a \oplus b = \hat{1}$, then there exists the $\hat{\mathcal{N}}$–largest such an element c. The intermediate elements need not exist.

Proof. Let a and c be two such elements and let b be any other element in $[c', a']$.
Suppose $c \oplus b = b$. Then $c' \geq a \oplus b = a \oplus c \oplus b = c' \oplus b \geq c'$, so $c \oplus b = a'$, contradicting lemma 2.6. A similar contradiction is obtained if $a \oplus b = b'$.
So either $c \oplus b = c'$, in which case one obtains the lattice in Diagram 4 (b), or $c \oplus b = c$ and we obtain that lattice with b and b' interchanged.
If there is no element b such that $a \oplus b = \hat{1}$, the largest such an element c exists by an argument similar to the one used in lemma 3.9. □

For the main theorem of this section we will use the following notation. If $\mathcal{A}$ and $\mathcal{B}$ are two lattices, $\top_{\mathcal{A}}$ is the largest element of $\mathcal{A}$ and $\bot_{\mathcal{B}}$ is the least element of $\mathcal{B}$.

We define $\mathcal{A} \dagger \mathcal{B}$ as the lattice obtained by identifying $\top_{\mathcal{A}}$ and $\bot_{\mathcal{B}}$ and extending the order in the natural way, i.e. if $x,\ y \in A \cup B$ then

$$x \leq y \text{ iff } \begin{cases} x,\ y \in A, \text{ and } x \leq_{\mathcal{A}} y \\ x \in A,\ y \in B \\ x,\ y \in B, \text{ and } x \leq_{\mathcal{B}} y. \end{cases}$$

Theorem 3.12. *Let $n \neq 6,\ 8$. Then any interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$ is of the form*

$$\mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \cdots \dagger \mathcal{A}_m,$$

where for each $i \leq m$, $\mathcal{A}_i$ is either a chain or one of the lattices in Diagram 5. Conversely, each such lattice gives rise to an interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$. In each case the intermediate elements need not exist.

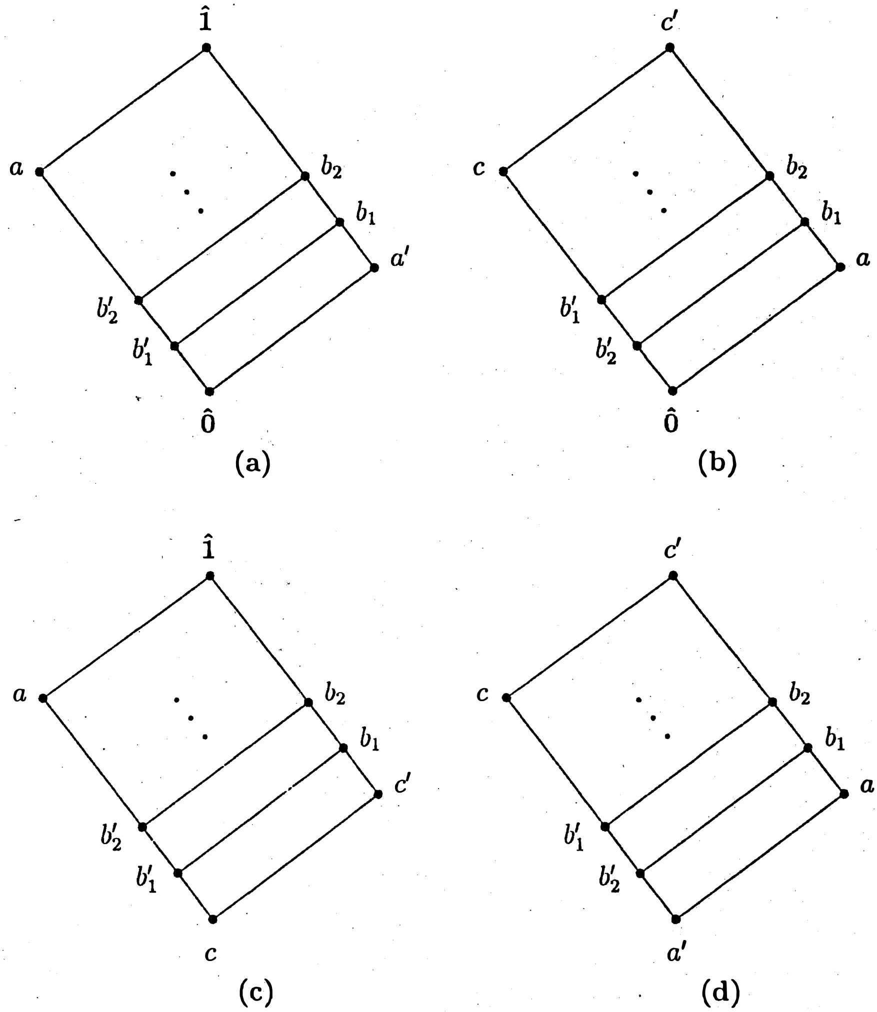


Diagram 5

Proof.
If there is no meet–reducible element in $\hat{\mathcal{N}}$, the interpretation is a chain.

If there exist meet–reducible elements, then there are several cases.

Case 1: The $\mathcal{N}$–least meet–reducible element is $\hat{0}$ and there is an element a such that $a \odot a' = \hat{0}$. Then by lemma 3.4, the interpretation is a lattice as in Diagram 5 (a).

Case 2: $\hat{0}$ is the $\mathcal{N}$–least meet–reducible element and there are elements a, b, not conjugates, such that $a \odot b = \hat{0}$.

Since $n \neq 6, 8$, $a \oplus b$ must be either a' or b'. We may assume w.l.o.g. that $a \oplus b = a'$. Then by lemma 3.9, there exists a $\mathcal{N}$–greatest element c such that the interval $[\hat{0}, c']$ is a lattice as in Diagram 5 (b). We let $c_1 = c'$ and $\mathcal{A}_1 = [\hat{0}, c_1]$. Observe that since c_1 must be meet irreducible, it has a unique cover.

1. If there is no meet–reducible element x such that $c_1 < x < \hat{1}$, we let $\mathcal{A}_2 = [c_1, \hat{1}]$ and thus $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2$. Observe that $\mathcal{A}_2$ is a chain.

2. If there is one, let c_2 be the $\mathcal{N}$–least meet–reducible element greater than c_1. We let $\mathcal{A}_2 = [c_1, c_2]$. Again $\mathcal{A}_2$ is a chain of length at least 2.

   Since c_2 is meet reducible, there exists an element a such that $a \odot c_2' = c_2$. Again we have two possibilities, either $a \oplus c_2' = \hat{1}$ or $a \oplus c_2' = a'$.

   (a) In the first case, by lemma 3.10 the interval $[c_2, \hat{1}]$ is a lattice as the one in Diagram 5 (c). Let $\mathcal{A}_3 = [c_2, \hat{1}]$ and $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \mathcal{A}_3$.

   (b) In the second case, by lemma 3.11, there exists the largest element c such that the interval $[c_2, c']$ is a lattice as the one in Diagram 5 (d). We let $c_3 = c'$ and $\mathcal{A}_3 = [c_2, c_3]$.

We can now continue as in the previous step, searching for c_4, the next meet–reducible element, if one exists, and proceed as we did with c_2. The process must eventually terminate and we have $\hat{\mathcal{N}} = \mathcal{A}_1 \dagger \mathcal{A}_2 \dagger \cdots \dagger \mathcal{A}_m$.

Case 3: $\hat{0}$ is not the $\mathcal{N}$–least meet–reducible element. Then since $\hat{0}$ is meet–irreducible, it has a single immediate successor. Let c_1 be the $\hat{\mathcal{N}}$–least meet–reducible element and define $\mathcal{A}_1 = [\hat{0}, c_1]$. $\mathcal{A}_1$ is a chain. Now proceed as in step 2 with c_1 in place of $\hat{0}$. This completes the proof of necessity.

That each such lattice gives rise to an interpretation of $\mathcal{D}_{01}$ in $\mathcal{N}$ is immediate from theorem 2.5. □

4 Interpreting $\mathcal{DM}$ in $\mathcal{L}_n$

We will now study the interpretations of the variety $\mathcal{DM}$ of De Morgan algebras in the variety of Łukasiewicz algebras. In this case we also have to interpret the unary operation $'$, the quasi–complement. Of course, since De Morgan algebras are distributive lattices, all that was proved in the previous section holds for them.

Throughout this section, we will let $\hat{\mathcal{N}} = \langle n\,; \oplus, \odot, ^{\ominus}, \hat{0}, \hat{1}\rangle$ be an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$, where $\langle n\,; \oplus, \odot, \hat{0}, \hat{1}\rangle$ is an interpretation of $\mathcal{D}_{01}$ in $\mathcal{L}_n$ as in section 3 and $^{\ominus}$ is a unary operation, the interpretation of the quasi–complement $'$.

The first lemma is a straightforward consequence of the definition of a quasi–complement and lemma 2.6.

Lemma 4.1.

1. *The quasi–complement* $^{\ominus}$ *is one–to–one.*

2. $\hat{0}^{\ominus} = \hat{1}$ *and* $\hat{1}^{\ominus} = \hat{0}$.

3. *If* $a^{\ominus} = a$, *then* $a'^{\ominus} = a'$.

4. *If* $a \leq b$, *then* $b^{\ominus} \leq a^{\ominus}$.

Theorem 4.2. *If* $n \neq 4,\ 6$, *then the underlying lattice of every interpretation of* $\mathcal{DM}$ *in* $\mathcal{L}_n$ *is a chain and the new quasi–complement coincides with the old one.*

Proof. Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$ and suppose there exist a meet–reducible element c. We have several cases.
If $c = \hat{0}$ and $\hat{0} = a \odot a'$, then underlying lattice of $\hat{\mathcal{N}}$ is like the lattice in Diagram 5 (a) and since $n \neq 4,\ 6$, there exist at least four elements b_1, b_2, b_1' and b_2' like the ones depicted in the Diagram. But then $b_1 \oplus b_2' = b_2$ and $b_1 \geq b_1^{\ominus}$. So $b_1 \geq b_1^{\ominus} \odot b_2'^{\ominus} = b_2^{\ominus} \in \{b_2, b_2'\}$ and this is not possible.
If $c = \hat{0}$ and $\hat{0} = a \odot b$, where a and b are not conjugates, then since $n \neq 6$, the underlying lattice of $\hat{\mathcal{N}}$ is like the lattice in Diagram 5 (b). In this case, since $b' > c'$, $\hat{1} = b^{\ominus} \oplus c^{\ominus} \in \{c', b'\}$.
If $c = \hat{0}$, $n = 8$ and the underlying lattice of $\hat{\mathcal{N}}$ is the one in Diagram 1, then $a \odot c = c'$, but taking quasi–complements, $a' \oplus c' = c \neq a$.
If $c > \hat{0}$. Then in the decomposition of $\hat{\mathcal{N}}$, $\mathcal{A}_1$ is a chain and $\mathcal{A}_2$ is either the lattice in Diagram 5 (c) or the one in 5 (d). These cases are similar to case 2.
So in any case we get a contradiction, thus there is no meet–reducible element and $\hat{\mathcal{N}}$ is a chain.
Assume now that there exists an a such that $a^{\ominus} = a$. By lemma 4.1.3, $a'^{\ominus} = a'$. So if $a < a'$ and by lemma 4.1.4, $a' = a'^{\ominus} < a^{\ominus} = a$, a contradiction. A similar contradiction arises if we assume that $a' < a$. This implies that $a = a'$ and thus $a^{\ominus} = a = a'$. Now since in a chain there is at most one element such that $a = a'$, for any other b, $b^{\ominus} = b'$. So for any x, $x^{\ominus} = x'$. □

In the following theorem we will prove that if $n = 4$, there are two possible definitions for the quasi–complement.

Theorem 4.3. *There are 8 interpretations of* $\mathcal{DM}$ *in* $\mathcal{L}_4$.

Proof. Let $n = \{0, a, a', 1\}$. Recall that the underlying lattice of the interpretation must be isomorphic to one of the lattices of Diagram 6.

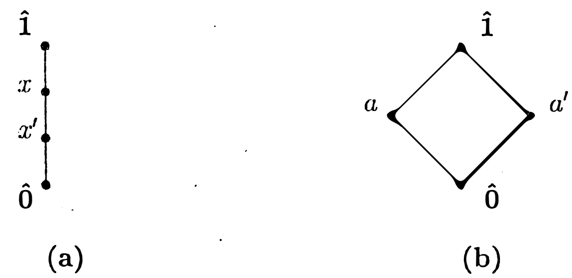


Diagram 6

If the lattice interpretation is the lattice in Diagram 6 (a), since $\hat{1}$ can be either **0** or **1**, we have two choices. For each of these, x can be either a or a' and that gives us 4 possibilities.
An argument similar to the one in the previous theorem shows that for all $x^{\ominus} = x'$. This gives us four interpretations.
If the lattice interpretation is the lattice in Diagram 6 (b), again $\hat{1}$ can be either **0** or **1**, so we have two choices, but in this case, by symmetry we have only one choice for a. For each of these there are two possible quasi-complements, namely,

$$x^{\ominus} = x'$$
$$x^{\ominus} = (\sigma_1(x))' + x(\sigma_3(x))'.$$

The first function defines the four element Boolean algebra. The second function assigns **0**, a, a' and **1** to **1**, a, a' and **0**, respectively. It is well known fact that these two are $\mathcal{DM}$ algebras. This provides the other four interpretations. □

Theorem 4.4. *There are 32 interpretations of* $\mathcal{DM}$ *in* $\mathcal{L}_6$.

Proof. Let $6 = \{\hat{\mathbf{0}}, a, b, a', b', \mathbf{1}\}$. If the underlying lattice of the interpretation is a chain, its first element $\hat{\mathbf{0}}$ has to be either **0** or **1**. For each of those, the second element can be filled by any of the four elements a, a', b or b', the third has only two possibilities since the others are determined by the previous selections and lemma 4.1.4. That gives us 16 possible interpretations.
If the underlying lattice of the interpretation is the lattice in Diagram 2 (a) and $b^{\ominus} = b$, then by lemma 4.1.4, $b'^{\ominus} > b^{\ominus}$, which would force $b'^{\ominus} = \hat{1}$, contradicting lemma 4.1.1, so $b^{\ominus} = b'$.But then $a^{\ominus} \odot b' = (a \oplus b)^{\ominus} = \hat{0}$, so $a^{\ominus} = a$. So for all x, $x^{\ominus} = x'$. The reader can easily check that the old quasi–complement works well. A similar analysis to that of the previous paragraph shows there are another 16 interpretations of this sort.
Finally, using the same arguments of Theorem 4.2, one can check that for the other possible underlying lattices for an interpretation, there is no acceptable definition for the quasi–complement. For instance, in the lattice in Diagram 3, $a \oplus b = a \oplus b' = \hat{1}$, so $a^{\ominus} \odot b^{\ominus} = a^{\ominus} \odot b'^{\ominus} = \hat{\mathbf{0}}$. But in this lattice this is possible only if $b^{\ominus} = b'^{\ominus} = b$, a contradiction. There are essentially three other underlying lattices. □

Theorem 4.5. *The number of interpretations of $\mathcal{DM}$ in $\mathcal{L}_n$ is*

$$2^{\frac{n}{2}}(\tfrac{n}{2}-1)! \quad \textit{if } n \textit{ is even, } n \neq 4,\ 6,$$
$$2^{\frac{n-1}{2}}(\tfrac{n-1}{2}-1)! \quad \textit{if } n \textit{ is odd.}$$

Proof. The proof is a straight forward generalization of the $n = 6$ case. One must observe that in the odd case, there is one single element c for which $c' = c$ and by lemma lemma 4.1.4 it must be assigned to the "midpoint" of the underlying lattice. □

5 Interpreting $\mathcal{L}_m$ in $\mathcal{L}_n$

In the previous section we proved that De Morgan interpretations are pretty tight. We will now extend those results to Lukasiewicz algebras, that is, we have to define the new unary operations $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_{m-1}$.
Throughout this section $\hat{\mathcal{N}} = \langle n\,; \oplus, \odot, ^{\ominus}, \hat{\sigma}_1 \ldots, \hat{\sigma}_{m-1}, \hat{0}, \hat{1}\rangle$ will be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$, where $\langle n\,; \oplus, \odot, ^{\ominus}, \hat{0}, \hat{1}\rangle$ is an interpretation of $\mathcal{DM}$ in $\mathcal{L}_n$ as in section 4 and the $\hat{\sigma}_i$'s are unary operations, the interpretation of the σ_i's. Of course this means that except for $n = 4$ and 6, $\langle n\,; \oplus, \odot, ^{\ominus}, \hat{0}, \hat{1}\rangle$ is a chain and the quasi–complements $^{\ominus}$ and $'$ coincide, so we will analize these two cases separately.

Lemma 5.1. *If $\hat{4}$ is an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_4$ and its underlying lattice is not a chain, then $\hat{4}$ is the four element Boolean algebra and for all x, $\hat{\sigma}_1(x) = \hat{\sigma}_2(x) = \cdots = \hat{\sigma}_{m-1}(x) = x$.*

Proof. Assume that the underlying lattice of the interpretation is the lattice in Diagram 6 (b) and that $a^{\ominus} = a$ and $a'^{\ominus} = a'$. Then $\hat{\sigma}_{m-1}(a) \neq a, a'$ or else $\hat{\sigma}_{m-1}(a) \oplus (\hat{\sigma}_{m-1}(a))^{\ominus} \in \{a, a'\}$, contradicting axiom (2).
By axiom (1), since $a \odot a' = \hat{0}$, $\hat{\sigma}_{m-1}(a) \odot \hat{\sigma}_{m-1}(a') = \hat{0}$, so either $\hat{\sigma}_{m-1}(a) = \hat{0}$ or $\hat{\sigma}_{m-1}(a') = \hat{0}$. But $\hat{\sigma}_{m-1}(a) = \hat{0}$ (and similarly $\hat{\sigma}_{m-1}(a') = \hat{0}$) is clearly impossible because by (L_2) we would have $\hat{\sigma}_1(a) = \hat{\sigma}_2(a) = \cdots = \hat{\sigma}_{m-1}(a) = \hat{0}$, that is to say, for all $i \leq m-1$, $\hat{\sigma}_i(a) = \hat{\sigma}_i(\hat{0})$, which in turn by ($L_4$) implies $a = \hat{0}$.
If the De Morgan interpretation is the four element Boolean algebra, then it is a well known fact that the only possibility for the $\hat{\sigma}_i$'s is the identity. See [3]. □

Lemma 5.2. *If $\hat{6}$ is an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_6$, then its underlying lattice is a chain.*

Proof. Suppose the lattice reduct of $\hat{6}$ is not a chain, then by Theorem 4.4, it is the one that appears in Diagram 2 (a) and $x^{\ominus} = x'$ for all x.
If $\hat{\sigma}_i(b) \in \{b, b'\}$, then $\hat{1} = \hat{\sigma}_i(b) \oplus (\hat{\sigma}_i(b))^{\ominus} = b \oplus b' = b$, so for any i, $\hat{\sigma}_i(b) \in \{\hat{0}, \hat{1}\}$. Similarly, $\hat{\sigma}_i(b') \in \{\hat{0}, \hat{1}\}$.
As in the previous lemma, $\hat{\sigma}_{m-1}(b) \neq \hat{0}$. So $\hat{\sigma}_{m-1}(b) = \hat{\sigma}_{m-1}(b') = \hat{1}$, which, by ($L_3$) and since $a \geq b'$, implies $\hat{\sigma}_{m-1}(a) = \hat{1}$. But then, $\hat{0} = \hat{\sigma}_{m-1}(a \odot a') = \hat{\sigma}_{m-1}(a) \odot \hat{\sigma}_{m-1}(a') = \hat{\sigma}_{m-1}(a')$, which as we know implies $a' = \hat{0}$, a contradiction. So there is no possible definition for $\hat{\sigma}_{m-1}(b)$ and the lattice reduct must be a chain. □

Lemma 5.3. *Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$ for which the underlying lattice is a chain. Then for $0 \leq i \leq m-1$, $\hat{\sigma}_i(x) \in \{\hat{0},\ \hat{1}\}$.*
If we let $\mu(a)$ be the least k such that $\hat{\sigma}_k(a) = \hat{1}$, then μ defines a one–to–one correspondence between the non-zero elements of $\hat{\mathcal{N}}$ and the $\hat{\sigma}_i$'s. Moreover, if the De Morgan reduct of $\hat{\mathcal{N}}$ is $\hat{0} = a_0 < a_1 < \cdots < a_{n-1} = \hat{1}$, setting $\mu(\hat{0}) = n$, $\mu(a_j) \geq n - j$.

Proof. We first observe that since the lattice reduct of $\hat{\mathcal{N}}$ is a chain, every element, in particular $\hat{1}$ is join irreducible, so by axiom (2), for all $i \leq m-1$ and any x, $\hat{\sigma}_i(x) \in \{\hat{0},\ \hat{1}\}$.
Next, recall that by (L_1), for all $i \leq n-1$, $\hat{\sigma}_i(\hat{0}) = \hat{0}$, so for $a \neq \hat{0}$, there exists some k such that $\hat{\sigma}_k(a) = \hat{1}$. If not, for all $i \leq m-1$, $\hat{\sigma}_i(a) = \hat{0} = \hat{\sigma}_i(\hat{0})$ and by (L_4), $a = \hat{0}$, a contradiction.
Let $a \neq \hat{0}$ and $b \neq \hat{0}$. We now observe that if $a \neq b$, then $\mu(a) \neq \mu(b)$. If not, by (L_2), for $r \geq \mu(a) = \mu(b)$, $\hat{\sigma}_r(a) = \hat{1} = \hat{\sigma}_r(b)$ and for $r < \mu(a)$, $\hat{\sigma}_r(a) = \hat{0} = \hat{\sigma}_r(b)$, so again using ($L_4$), we get $a = b$, a contradiction.
Suppose that $k = \mu(a_j) < n - j$, for some $0 < j < n$. Then $\hat{\sigma}_{k-1}(a_j) = \hat{0}$. This implies that $\hat{\sigma}_{k-1}(a_{j+1}) = \hat{1}$ or else by (L_2) and (L_3), $\hat{\sigma}_r(a_j) = \hat{\sigma}_r(a_{j+1})$, for all $1 \leq r < n$, and by (L_4), $a_j = a_{j+1}$. So $\mu(a_{j+1}) < n - j - 1$.
In a similar way we prove that for $s \leq k-1$, $\hat{\sigma}_{k-s}(a_{j+s}) = \hat{1}$, in particular, $\hat{\sigma}_1(a_{j+k-1}) = \hat{1}$, so by ($L_2$), $\hat{\sigma}_r(a_{j+k-1}) = \hat{1}$, for $1 \leq r < n$. But by (L_1) and (L_4), this implies that $a_{j+k-1} = a_{n-1}$, that is $j + k - 1 = n - 1$, contradicting our assumption.
This completes the proof that $\mu(a_j) \geq n - j$. □

Notice that the function μ determines the $\hat{\sigma}_i$'s as follows:

$$\hat{\sigma}_i(a_j) = \begin{cases} \hat{0} & \text{if } j < \mu(a_j), \\ \hat{1} & \text{if } j \geq \mu(a_j). \end{cases}$$

for all $1 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Also, since $\mu(x)$ is one-to-one and the number of non–zero elements of n is $n-1$, there has to be at least as many $\hat{\sigma}_i$'s. This provides another proof of our next Theorem 5.4.1.

Theorem 5.4.

1. *If $m < n$, there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*

2. *If m is even and n is odd, then there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*

Proof. One should observe that $\hat{\mathcal{N}}$ is an $\mathcal{L}_m$–algebra and it is a chain, so by Theorem 2.2, $\hat{\mathcal{N}}$ is a an $\mathcal{L}_m$–subalgebra of $\mathcal{M}$. This immediately implies that $n \leq m$. The second assertion follows from the fact that $\mathcal{M}$ does not have subalgebras of odd cardinality. □

Theorem 5.5. *Let $m \geq n$. Then the number of interpretations of $\mathcal{L}_m$ in $\mathcal{L}_n$ is determined as follows.*

1. *If m is even and n is odd there is no interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$.*

2. *In any other case, for each De Morgan interpretation in $\mathcal{L}_n$, there are $\binom{h(m)}{h(n)}$ interpretations of $\mathcal{L}_m$ in $\mathcal{L}_n$, where for any positive integer p,*

$$h(p) = \begin{cases} \frac{p}{2} - 1 & \text{if } p \text{ is even,} \\ \frac{p-1}{2} - 1 & \text{if } p \text{ is odd.} \end{cases}$$

3. *If $n = 4$ and $m \geq 4$, there are two more interpretations of $\mathcal{L}_m$ in $\mathcal{L}_4$.*

Proof. Let $\hat{\mathcal{N}}$ be an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$ such that the De Morgan reduct of the interpretation is the chain $\hat{0} = a_0 < a_1 < \cdots < a_{n-1} = \hat{1}$ and $a_j^{\ominus} = a_j' = a_{n-j}$. Our problem here is to count the number of possible functions μ defined in lemma 5.3. We know that they are one–to–one and $\mu(a_j) \geq n - j$, but that is not all we know. By axiom (4) μ has to have a symmetry with respect to the midpoint of $\mathcal{N}$ if n is odd or its midpoints if n is even. Recall that axiom (4) states that $\hat{\sigma}_i(a) = \hat{\sigma}_{m-i}(a^{\ominus})^{\ominus}$. In this case this means that $\hat{\sigma}_i(a_j) = (\hat{\sigma}_{m-i}(a_{n-j}))'$.

Case (1): This is Theorem 5.4, 2.

Case (2): Both m and n are even.
In this case $\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}}) = (\hat{\sigma}_{m-\frac{m}{2}}(a'_{\frac{n}{2}}))' = (\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}))'$, and since $a_{\frac{n}{2}-1} < a_{\frac{n}{2}}$, by ($L_2$), $\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}) \leq \hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}})$. These two imply that

$$\hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}-1}) = \hat{1} \quad \text{and} \quad \hat{\sigma}_{\frac{m}{2}}(a_{\frac{n}{2}}) = \hat{0}.$$

The information gathered so far is summarized in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}-1}$		$\cdots$		$\hat{\sigma}_{\frac{m}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{0}$			$\vdots$				$\hat{0}$		$\hat{0}$
a_1								$\hat{0}$		$\hat{1}$
$\vdots$								$\vdots$		$\vdots$
$a_{\frac{n}{2}-1}$								$\hat{0}$		$\hat{1}$
$a_{\frac{n}{2}}$				$\vdots$			?	$\hat{1}$		$\hat{1}$
							$\hat{1}$	$\hat{1}$		$\hat{1}$
$\vdots$					?	$\iddots$		$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$		$\hat{1}$
a_{n-1}	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$

Case (3): Both m and n are odd.
Then $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}}) = (\hat{\sigma}_{\frac{m+1}{2}}(a'_{\frac{n-1}{2}}))' = (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}))'$, but by ($L_2$), $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}) \geq \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}})$, so these two together with (L_3) imply

$$\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n-1}{2}}) = \hat{1} \quad \text{and} \quad \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n-1}{2}}) = \hat{0}.$$

Also, if $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n+1}{2}}) = \hat{0}$, by (L_4), $a_{\frac{n+1}{2}} = a_{\frac{n-1}{2}}$, a contradiction, so $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n+1}{2}}) = \hat{1}$. We summarize this in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}}$		$\cdots$		$\hat{\sigma}_{\frac{m-1}{2}}$	$\hat{\sigma}_{\frac{m+1}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{0}$			$\vdots$				$\hat{0}$	$\hat{0}$		$\hat{0}$
a_1	$\hat{0}$										$\hat{1}$
$\vdots$								$\vdots$			$\vdots$
$a_{\frac{n-1}{2}}$	$\hat{0}$							$\hat{0}$	$\hat{1}$	$\cdots$	$\hat{1}$
				$\vdots$			?	$\hat{1}$	$\hat{1}$		$\hat{1}$
							$\hat{1}$	$\hat{1}$	$\hat{1}$		$\hat{1}$
$\vdots$					?	$\iddots$		$\vdots$	$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\hat{1}$		$\hat{1}$
a_{n-1}	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$

Case (4): m is odd and n is even.

Then if $\hat{0} = \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) = (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}))'$, so $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}) = \hat{1}$ and thus by (L_3), $\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}}) \geq (\hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1})) = \hat{1}$, and also $\hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}-1}) \leq \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) = \hat{0}$. Putting these together, by (L_4), we get $a_{\frac{n}{2}} = a_{\frac{n}{2}-1}$, a contradiction. So

$$\begin{aligned} \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}-1}) &= \hat{0} = \hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}-1}) \\ \hat{\sigma}_{\frac{m-1}{2}}(a_{\frac{n}{2}}) &= \hat{1} = \hat{\sigma}_{\frac{m+1}{2}}(a_{\frac{n}{2}}). \end{aligned}$$

We summarize this in the following chart.

	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\cdots$	$\hat{\sigma}_{\frac{m-n}{2}}$		$\cdots$		$\hat{\sigma}_{\frac{m-1}{2}}$	$\hat{\sigma}_{\frac{m+1}{2}}$	$\cdots$	$\hat{\sigma}_{m-1}$
a_0	$\hat{0}$			$\vdots$				$\hat{0}$	$\hat{0}$		$\hat{0}$
a_1	$\hat{0}$							$\hat{0}$	$\hat{0}$		$\hat{1}$
$\vdots$								$\vdots$			$\vdots$
$a_{\frac{n}{2}-1}$								$\hat{0}$	$\hat{0}$		$\hat{1}$
$a_{\frac{n}{2}}$							?	$\hat{1}$	$\hat{1}$		$\hat{1}$
							$\hat{1}$	$\hat{1}$	$\hat{1}$		$\hat{1}$
$\vdots$					?	$\iddots$		$\vdots$	$\vdots$		$\vdots$
a_{n-2}	?	?	$\cdots$	?	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\hat{1}$		$\hat{1}$
a_{n-1}	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$	$\hat{1}$	$\hat{1}$	$\cdots$	$\hat{1}$

In the charts above we see that

$$\hat{\sigma}_i(a_j) = \begin{cases} \hat{0} & \text{if } 1 \leq j < n/2 \text{ and } 1 \leq i \leq m/2, \\ \hat{1} & \text{if } j \geq n/2 \text{ and } i \geq m/2. \end{cases}$$

Observe that by axiom (4), the values of $\hat{\sigma}_i(a_j)$ for $j < n/2$ and $i \geq (m+1)/2$, are determined by those of $\hat{\sigma}_i(a_j)$ for $j \geq n/2$ and $i < m/2$. Also, we must take into

account (L_2), (L_3) and (L_4), which imply that $\hat{\sigma}_i(a_j)$ must increase both with i and with j.
So in order to find all possible functions μ, one only has to determine how many "?" one has to replace by $\hat{0}$'s in the lower left hand side of the charts.
Assuming l is the number of rows and k is the number of columns, this is the same as the number of integers less than l which can be expressed as a sum of k positive integers, this number is $\binom{l}{k}$.

Conversely, by Theorem 2.5 any such partition defines an interpretation of $\mathcal{L}_m$ in $\mathcal{L}_n$. So for appropriate l and k, the number of interpretations equals the number of these partitions. Now it is a matter of determining the particular l's and k's in each of the three cases and the theorem follows. Notice that by (L_1) the last line in each chart is fixed.

If $n = 4$ and its De Morgan reduct is the four element Boolean algebra, then as we mentioned before, we have another interpretation if we let for all x, $\hat{\sigma}_1(x) = \hat{\sigma}_2(x) = \cdots = \hat{\sigma}_{m-1}(x) = x$. □

References

[1] Balbes, R. and Dwinger, P., *Distributive Lattices*, University of Missouri Press, 1974.

[2] Boicescu, V., Filipoiu, A., Georgescu, G. and Rudeanu, S., *Lukasiewicz–Moisil Algebras*, North–Holland, 1991.

[3] Cignoli, R. *Moisil Algebras*, Notas Matemáticas, Instituto de Matemáticas, Universidad Nacional del Sur, Bahía Blanca, **27**, (1970).

[4] Cignoli, R. and de Gallego, M. S., *The Lattice Structure of some Lukasiewicz Algebras*, Algebra Universalis **13**, (1981), 315-328.

[5] García, O.C. and Taylor W., *The Lattice of Interpretability Types*, Memoirs AMS, **50**, N° 305 (1984).

[6] Lewin, R.A., *Interpretations into Varieties of Algebraic Logic*, Ph.D. thesis, University of Colorado–Boulder, 1985.

[7] Neumann, W. D., *On Malcev Conditions* Journal of the Australian Math. Soc. **17**, (1974), 376-384.

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DENSITY OF PERIODIC GEODESICS IN THE UNIT TANGENT BUNDLE OF A COMPACT HYPERBOLIC SURFACE

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Abstract

Let S be a compact oriented surface of constant curvature -1 and let T^1S be the unit tangent bundle of S endowed with the canonical (Sasaki) metric. We prove that T^1S has dense periodic geodesics, that is, the set of vectors tangent to periodic geodesics in T^1S is dense in TT^1S.

Let M be a compact Riemannian manifold. M is said to have the DPG property (density of periodic geodesics) if the vectors tangent to periodic geodesics in M are dense in TM, the tangent bundle of M. A compact manifold is known to have this property if, for example, its geodesic flow is Anosov (see [1]), in particular if it is hyperbolic. In this note we prove that the unit tangent bundle of a compact oriented surface of constant curvature -1 shares with the surface the DPG property.

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Theorem *Let S be a compact oriented surface of constant curvature -1 and let T^1S be the unit tangent bundle of S endowed with the canonical (Sasaki) metric. Then T^1S has the DPG property.*

Remarks.

a) Geodesics in T^1S do not project necessarily to geodesics in S.

b) The unit tangent bundle of any compact oriented surface of constant curvature 0 or 1 has also the DPG property.

c) The geodesic flow of T^1S, which is a flow on T^1T^1S, is not Anosov.

d) T^1S may be written as $\Gamma \backslash PSl(2, \mathbf{R})$, where Γ is the fundamental group of S. In general, not every compact quotient of a Lie group endowed with a left invariant Riemannian metric has the DPG property.

The proof of the theorem and comments on the remarks can be found at the end of the article. Next, we give some preliminaries. Let H be the hyperbolic plane of constant curvature -1. Any oriented surface S of constant curvature -1 inherits from its universal covering H a canonical complex structure. If V is a smooth curve in TS, then V' will denote the covariant derivative along the projection of V to S. The geodesic curvature of a curve c in S with constant speed $\lambda \neq 0$ is defined by $\kappa(t) = \langle \dot{c}'(t), i\dot{c}(t) \rangle / \lambda^3$. We consider on T^1S the canonical (Sasaki) metric, defined by $\|\xi\|^2 = \|\pi_{*v}\xi\|^2 + \|\mathcal{K}(\xi)\|^2$ for $\xi \in T_vT^1S$, $v \in T^1S$, where $\pi : T^1S \to S$ is the canonical projection and $\mathcal{K}$ is the connection operator. Next, we recall from [7] a description of the geodesics of T^1H and some properties of curves in H of constant geodesic curvature.

Proposition 1 *Let V be a geodesic in T^1H and let $c = \pi \circ V$. Then $\|V'\| = \text{const}$, $\|\dot{c}\| = \text{const} =: \lambda$ and one of the following possibilities holds:*

a) If $\lambda = 0$, then V is a constant speed curve in the circle $T^1_{c(0)}H$.

b) If $\lambda \neq 0$, then the geodesic curvature κ of c with respect to the normal $i\dot{c}/\lambda$ is also constant and for $t \in \mathbf{R}$

$$V(t) = e^{-2\lambda\kappa t i} z\dot{c}(t), \tag{1}$$

where $z \in \mathbf{C}$ is such that $V(0) = z\dot{c}(0)$.

Conversely, each curve V in T^1H which satisfies (a) or (b) is a geodesic. Moreover, given a constant speed curve c in H with constant geodesic curvature, and $V_0 \in T^1_{c(0)}H$, there is a unique geodesic V in T^1H which projects to c and such that $V(0) = V_0$.

We recall from the proof of this proposition that if V is the geodesic in T^1S with initial velocity ξ, then $\lambda = \|\pi_{*V(0)}\xi\|$ and $\mathcal{K}(\xi) = -\lambda\kappa iV(0)$, in particular $\kappa = \pm\|\mathcal{K}(\xi)\|/\lambda$.

In the following we consider the upper half space model $H = \{x + iy \mid y > 0\}$ with the metric $ds^2 = (dx^2 + dy^2)/y^2$.

Lemma 2 *Let c be a complete curve in H of constant geodesic curvature κ. Given $\theta \in (0, \pi)$, let c_θ be the curve in H defined by $c_\theta(t) = e^t e^{i\theta}$.*

a) If $|\kappa| > 1$, the image of c is a geodesic circle of radius $|r|$ and length $|2\pi \sinh r|$, where $\coth r = \kappa$ (this implies that the length is $2\pi/\sqrt{\kappa^2 - 1}$).

b) If $|\kappa| = 1$, the image of c is a horocycle.

c) If $\kappa = \cos\theta$, the image of c is congruent to that of c_θ.

Let $G = PSl(2,\mathbf{R}) = \{g \in M_2(\mathbf{R}) \mid \det g = 1\}/\{\pm I\}$ and let $\mathfrak{g} = \{X \in M_2(\mathbf{R}) \mid \operatorname{tr} X = 0\}$ be its Lie algebra. Via the canonical action of G on H by Möbius transformations, G is the group of orientation preserving isometries of H. Hence, H may be identified with G/K, where $K = PSO(2)$ is the isotropy group at the point $i \in H$.
Consider the Cartan decomposition $\mathfrak{g} = \mathbf{R}Z \oplus \mathfrak{p}$, where $Z = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ spans the Lie algebra of K and $\mathfrak{p} = \{X \in \mathfrak{g} \mid X = X^t\}$. As usual we shall identify $T_{eK}H$ with $\mathfrak{p}$. Under this identification, the quasi-complex structure induced on $\mathfrak{p}$ is given by $\mathrm{ad}_Z : \mathfrak{p} \to \mathfrak{p}$ and $X_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{p}$ is a unit vector. One can show that G acts simply transitively and by isometries on T^1H. Hence, the map $\Phi : G \to T^1H$ defined by $\Phi(g) = g_{*K}(X_0)$ is a diffeomorphism which induces in G a left invariant metric. From now on we identify sometimes in this way G with T^1H. In particular, the unit tangent bundle of a surface $\Gamma\backslash S$ may be identified with $\Gamma\backslash G$.
Let S be an oriented surface of constant curvature -1 and let κ be a real number. The κ-flow on T^1S is defined by $\phi_t^\kappa(v) = \dot{c}_v^\kappa(t)$, where c_v^κ is the unique unit speed curve in H with constant geodesic curvature κ and initial velocity v. In particular, the 0-flow is the geodesic flow of S. Next, we obtain the κ-flow on T^1H using the identification $\Phi : G \to T^1H$, taking advantage of the group structure of G. Let L_h, R_h denote left and right multiplication by h, respectively, and set $Y_\kappa = X_0 + \kappa Z$.

Lemma 3 *If φ_t^κ denotes the κ-flow on T^1H, then for all t we have*

$$\varphi_t^\kappa \circ \Phi = \Phi \circ R_{\exp(tY_\kappa)}.$$

Proof. a) follows from the fact that $Y_\kappa = \frac{1}{2}\begin{pmatrix} 1 & \kappa \\ -\kappa & -1 \end{pmatrix}$ diagonalizes with eigenvalues $\pm a/2$, since $|\kappa| < 1$.

b) If a, h are as above, then $R_h \circ R_{\exp(atX_0)} = R_{\exp(tY_\kappa)} \circ R_h$ for all t. Therefore, Lemma 3 implies that $\varphi_t^\kappa = F \circ \varphi_{at}^0 \circ F^{-1}$ for all t. One checks that $F(\gamma v) = \gamma F(v)$ for all $\gamma \in \Gamma, v \in T^1 H$ and the existence of f is proved. The last assertion follows from straightforward computations.

c) We have that the κ-flow on $T^1 S$ is conjugate to a constant rate reparametrization of the geodesic flow of S, which is known to be Anosov and has dense periodic orbits by Theorem 3 of [1]. □

Lemma 5 *Let S be a compact oriented surface of constant curvature -1. Let c be a periodic constant speed curve in S of constant geodesic curvature κ_0, with $|\kappa_0| < 1$. Then, for each κ with $|\kappa| < 1$, there exists a periodic constant speed curve c_κ in S, of constant geodesic curvature κ, such that*

a) $c_{\kappa_0} = c$,

b) $c_\kappa(0)$ converges to $c(0)$ and $\dot{c}_\kappa(0)$ converges to $\dot{c}(0)$ for $\kappa \to \kappa_0$,

c) the function $\kappa \mapsto \kappa \operatorname{length}(c_\kappa)$ is continuous, odd and strictly increasing.

Proof. We may suppose that c has unit speed and that $S = \Gamma\backslash H$, where Γ is a uniform subgroup of G which acts freely and properly discontinuously on H. Suppose that t_0 is the period of c and that C is a lift of c to H. Then there exists $g \in \Gamma$ such that $g_* \dot{C}(0) = \dot{C}(t_0)$. Since G acts transitively on $T^1 H$, by conjugating Γ by an element of G if necessary, we may suppose without loss of generality, by Lemma 2 (c), that $C(t) = e^{t \sin\theta_0} e^{i\theta_0}$ with $\cos\theta_0 = \kappa_0$, $0 < \theta_0 < \pi$ and, additionally, that $g(z) = az$, where $a = e^{t_0 \sin\theta_0}$.
For $|\kappa| < 1$, let c_κ be the projection to S of the curve $C_\kappa(t) := e^{t \sin\theta} e^{i\theta}$, where $\cos\theta = \kappa$, $0 < \theta < \pi$. Clearly, c_κ satisfies the first two conditions. By Lemma 2 (c), c_κ has constant geodesic curvature κ. Since $g_* \dot{C}_\kappa(0) = \dot{C}_\kappa(t_0 \sin\theta_0 / \sin\theta)$ and C_κ has unit speed, then c_κ is periodic and

$$\operatorname{length}(c_\kappa) = t_0 \frac{\sin\theta_0}{\sin\theta} = t_0 \sqrt{\frac{1-\kappa_0^2}{1-\kappa^2}}$$

Thus, the function $\kappa \mapsto \kappa \operatorname{length}(c_\kappa)$ has the required properties. □

Comments on the remarks.

(a) follows from Proposition 1. Next we comment on (b). If S is flat, then S is covered by a flat torus (see [8]) whose unit tangent bundle is again a flat torus and clearly has the DPG property. On the other hand, if S has constant curvature 1, then S is covered by a sphere, whose unit tangent bundle is isometric to $SO(3)$ endowed with a bi-invariant metric, all of whose geodesics are periodic (see also [4]). (c) is a consequence of [5], since T^1H has conjugate points. This follows from the proof of Myers' Theorem (see [2]), since if γ is the geodesic in $G \approx T^1H$ with initial velocity Z, then $\mathrm{Ricci}(\dot\gamma)$ is constant and positive. Indeed, γ is the orbit of the one-parameter group $t \mapsto \exp(tZ)$ of isometries of G, and $\mathrm{Ricci}(Z) > 0$ by Theorem 4.3 of [6]. Finally, a counterexample for (d) can be found for example in [3].

Proof of the Theorem.

Let Γ be the fundamental group of S and suppose that $S = \Gamma\backslash H$. Let $P : TT^1H \to T^1H$ and $\pi : TH \to H$ be the canonical projections. By abuse of notation we call also π the restriction of the latter to T^1H. Let $T'T^1H = \{\xi \in TT^1H \mid \pi_*\xi \neq 0\}$ and let $T'T^1S = \Gamma\backslash T'T^1H$. These are open dense subsets of TT^1H and TT^1S, respectively. Let now

$$F : T'T^1H \to \{(v, Y, \kappa) \in T^1H \times TH \times \mathbf{R} \mid Y \neq 0 \text{ and } \pi(v) = \pi(Y)\}$$

be defined by $F(\xi) = (P\xi, \pi_*\xi, \kappa(\xi))$, where $\kappa(\xi)$ is the (constant by Proposition 1) geodesic curvature of πV, V being the unique geodesic in G with initial velocity ξ. F is a diffeomorphism since it is differentiable and so is the inverse $F^{-1}(v, Y, \kappa) = \dot V(0)$, where V is the unique geodesic in T^1H such that $V(0) = v$, and $C := \pi V$ has constant geodesic curvature κ and satisfies $\dot C(0) = Y$ (see Proposition 1).

Fix $v_0 \in T^1H$ and $\eta \in T'_{\Gamma v_0}T^1S$. Suppose that η lifts to $\xi \in T'_{v_0}T^1H$ and that $F(\xi) = (v_0, Y_0, \kappa_0)$. We have to show that given $\varepsilon > 0$ and open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of v_0 and Y_0, in T^1H and TH respectively, then there exist κ with $|\kappa - \kappa_0| < \varepsilon$, $v \in \mathcal{U}$ and $0 \neq Y \in \mathcal{V}$, with the same footpoint, such that the geodesic V in T^1H with initial velocity $F^{-1}(v, Y, \kappa)$ projects to a periodic geodesic in T^1S. By the expression (1), it suffices to show that $c := \Gamma\pi V$ is periodic and $2\lambda\kappa t_0 \in 2\pi\mathbf{Q}$ for some positive number t_0 such that $\dot c(0) = \dot c(t_0)$, where λ, κ are as in Proposition 1.

Suppose that $|\kappa_0| \geq 1$. In this case choose $v = v_0$, $Y = Y_0$ and κ such that $|\kappa - \kappa_0| < \varepsilon$, $|\kappa| > 1$ and $2\kappa/\sqrt{\kappa^2 - 1} \in \mathbf{Q}$ (such a κ exists since the function $\kappa \mapsto 2\kappa/\sqrt{\kappa^2 - 1}$ is odd and strictly monotonic for $\kappa > 1$). Indeed, by Lemma 2 (a), $\dot c(0) = \dot c(t_0)$ holds for $t_0 = 2\pi/\lambda\sqrt{\kappa^2 - 1}$, since c has constant speed λ. Hence, $2t_0\kappa\lambda \in 2\pi\mathbf{Q}$ by the choice of κ.

If $|\kappa_0| < 1$, then by Lemma 4 there exists $(v_1, Y_1) \in \mathcal{U} \times \mathcal{V} \subset T^1H \times TH$ close to (v_0, Y_0), with $\pi(v_1) = \pi(Y_1)$, such that Γc_1 is periodic, where c_1 is the constant speed curve in H of constant geodesic curvature κ_0 with $\dot{c}_1(0) = Y_1$. By Lemma 5, since $\mathcal{V}$ is open, there exist κ with $|\kappa| < 1$, $|\kappa - \kappa_0| < \varepsilon$, and $(v, Y) \in \mathcal{U} \times \mathcal{V}$ close to (v_1, Y_1), with $Y \neq 0$ and $\pi(v) = \pi(Y)$, such that C projects to a periodic curve c in $\Gamma \backslash H$ with length ℓ satisfying $2\kappa\ell \in 2\pi\mathbf{Q}$, where C is the constant speed curve in H with constant geodesic curvature κ and initial velocity Y. If $t_0 = \ell/\lambda$, then $\dot{c}(0) = \dot{c}(t_0)$ and $2\lambda\kappa t_0 = 2\kappa\ell \in 2\pi\mathbf{Q}$. Consequently, for v, Y and κ as above, the geodesic in G with initial velocity $F^{-1}(v, Y, \kappa)$ projects to a periodic geodesic in T^1S. This completes the proof of the theorem. □

References

[1] D. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Instit. Math. **90**, 1967.

[2] J. Cheeger & D. Ebin, *Comparison Theorems in Riemannian Geometry*, North-Holland, 1975.

[3] P. Eberlein, *Geometry of 2-step nilpotent groups with a left invariant metric*, Ann. Sc. de École Normal Sup. **27** No. 5 (1994), 611-660.

[4] H. Gluck, *Geodesics in the unit tangent bundle of a round sphere*, L'Enseignement Math. **34** (1988), 233-246.

[5] W. Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type*, Annals of Math. **99** (1974), 1-13.

[6] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), 293-329.

[7] M. Salvai, *Spectra of unit tangent bundles of hyperbolic Riemann surfaces*, Ann. Global Anal. Geom. **16** (1998), 357-370.

[8] J. Wolf, *Spaces of constant curvature*, Publish or Perish, 1977.

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On Stochastic Parallel Transport and Prolongation of Connections.

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Abstract

Let $P(M,G)$ be a principal fiber bundle, ∇ a G-invariant CDO of P and Ψ a CDO of M. We prove that ∇ is projetable with projection Ψ iff there is a unique prolongation $\mathbf{H} \to \mathbf{H}^{\mathbf{r}}$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. We given an explicit expression for $\mathbf{H}^{\mathbf{r}}$ in terms of $\mathbf{H}$ and ∇.

The stochastic parallel displacement of a tensor along a random curve was considered by K. Itô [6]. Its natural generalization, the stochastic horizontal lifting in principal fiber bundles were studied by I. Shigekawa and others ([1], [8], [11], [10]).

The motivation for the present investigation is the discovery of P. Meyer [9] of a correspondence between the stochastic extensions of the equation of parallel transport of vectors and certain extensions to the tangent bundle TM of the connection ∇ on M. The stochastic parallel transports studied by P. Meyer are induced by 2-connections [1] of BM (the fiber bundle of bases of M) that are prolongations of ∇. These prolongations are of 1-connections to 2-connections of BM, and are given by $Gl(n,\mathbb{R})$-invariant connections of BM with projection ∇.

In this work we study these prolongations of 1-connections to 2-connections in the context of principal fiber bundles.

This paper is organized as follows, in 1. we prepare some notions concerning Schwartz geometry, 2-connections and martingales. In 2. we prove the main result of this work. Let $P(M,G)$ be a principal fiber bundle, ∇

a G-invariant covariant derivative operator without torsion of P and Ψ a covariant derivative operator without torsion of M. Then ∇ is projetable with projection Ψ iff there is a unique prolongation of 1-connections into 2-connections [1] of $P(M,G)$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. We given an explicit expression for $\mathbf{H}^r$ in terms of $\mathbf{H}$ and ∇. Finally, in section 3 we apply this results to diffusions given by Stratonovich equations, and discussed the special case of the principal fiber bundle of bases of a differential manifold with the G-invariant connections ∇^C and ∇^H [3].

1 Schwartz Geometry, 2-Connections and ∇-Martingales.

Throughout this paper, manifolds, maps and functions will always be assumed to be smooth. As to manifolds and stochastic differential geometry, we shall freely concepts and notations of Kobayashi-Nomizu [7] and Emery [4].

Now, we recall some fundamental facts about Schwartz second order geometry ([8], [9], [4], [10]) and martingales.

If x is a point in a manifold M, the second order tangent space to M at x, denoted $\tau_x M$, is the vector space of all differential operators on M, at x, of order at most two, with no constant term. If $dim M = n$, $\tau_x M$ has $n + \frac{1}{2}n(n+1)$ dimensions; using a local coordinate system (U, x^i) around x, every $L \in \tau_x M$ can be written in a unique way as

$$L = a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + a^k \frac{\partial}{\partial x^k} \qquad \text{with } a^{ij} = a^{ji}$$

(we use here and in other expressions in coordinates the convention of summing over the repeated indices). The elements of $\tau_x M$ are called second-order tangent vectors at x.

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ is canonically endowed with a vector bundle structure over M, called the second order tangent fiber bundle of M. We denote by $\Gamma(\tau M)$ the space of second order operator on M, that is, the space of sections of τM.

If M and N are manifolds and $\varphi : M \to N$ is a smooth mapping, it is possible to push forward second order tangent vectors by φ, given $L \in \tau_x M$

its image under φ is $\varphi_*(x)L \in \tau_{\varphi(x)}N$ given by

$$\varphi_*(x)L(f) \quad = \quad L(f \circ \varphi)$$

with f an arbitrary smooth function. We says that $\phi : \tau_x M \longrightarrow \tau_y N$ is a Schwartz morphism if there exists a smooth mapping $\varphi : M \longrightarrow N$ with $\varphi(x) = y$ such that $\phi = \varphi_*(x)$.

We know [8] that, we can associate with each covariant derivative operator without torsion (in short, CDO) ∇ of M a morphism $\Phi_\nabla : \tau M \longrightarrow TM$ defined in a local chart (U, x^i) of M by

$$\Phi_\nabla(a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + a^k\frac{\partial}{\partial x^k}) = (a^{ij}\Gamma^k_{ij} + a^k)\frac{\partial}{\partial x^k}$$

where $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\frac{\partial}{\partial x^k}$. We observe that Φ_∇ satisfies $\Phi_\nabla \circ i = Id_{TM}$ where $i : TM \longrightarrow \tau M$ is the inclusion.

Conversely, if $\Phi : \tau M \longrightarrow TM$ is a morphism of vector bundles such that $\Phi \circ i = Id_{TM}$ then we have defined a covariant derivative operator without torsion ∇^Φ by $\nabla^\Phi_X Y = \Gamma(XY)$ for all $X, Y \in \Gamma(M)$. Obviously, $\Phi_{\nabla^\Phi} = \Phi$ and $\nabla^{\Phi_\nabla} = \nabla$.

We remember the following proposition ([4], [5]).

Proposition 1 *Let M and N manifolds be endowed with CDO ∇ and Ψ respectively and $\varphi : M \longrightarrow N$ a smooth mapping. The following statements are equivalent:*

i) For every $x \in M$, $\Phi_\Psi \circ \varphi_*(x) = \varphi_*(x) \circ \Phi_\nabla$

ii) For every geodesic $g : U \longrightarrow M$, $\varphi \circ g : U \longrightarrow M$ is a geodesic.

iii) φ is affine.

Let M be a manifold endowed with a CDO ∇ and $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ a filtered probability space satisfying the usual conditions [4]. A continuous semimartingale X in M, is a ∇-martingale ([8], [4]) if, for every $\theta \in \Gamma(T^*M)$ with compact support

$$\int_0^t \langle \theta, \Phi_\nabla d_2 X\rangle \textit{ is a local martingale.}$$

where $\int_0^t \langle \theta, \Phi_\nabla d_2 X \rangle$ is the Itô integral of θ along X. Martingales, too, can be characterized in local coordinates. In fact, let (U, x^i) be a local chart of M a semimartingale $X = (X^i)$ is a ∇-martingale iff for some real local martingales (N^i),

$$X_t^i - X_0^i = N^i - \frac{1}{2}\int_0^t \Gamma_{jk}^i(X_s) d[N^j, N^k]_s$$

Now, we remember the definition of 2-connection [1]

Definition 2 *Let $P(M, G)$ be a principal fiber bundle. A family of Schwartz morphism $\mathbf{H} = \{H_p : p \in P\}$ is called a 2-connection if*

1) $H_p : \tau_{\pi p} M \longrightarrow \tau_p P$.

2) $\pi_* \circ H_p = id_{\tau_{\pi p} M}$

3) $H_{pg} = R_{g*} H_p$ for all $p \in P$ and $g \in G$ where R_g stands for the right action of G in P.

4) The mapping $p \to H_p L$ belongs to $\Gamma(\tau P)$ if $L \in \Gamma(\tau M)$.

We observed that by changing in the above definition τ for T, we get the classical definition of connection in principal fiber bundles, that we call 1-connection in this work. Obviously, every 2-connection $\mathbf{H} = \{H_p : p \in P\}$ induces a unique 1-connection $\mathbf{H}_R = \{H_p \mid_{T_{\pi p} M} : p \in P\}$ by restriction to the tangent space.

Let $P(M, G)$ be a principal fiber bundle, $\mathbf{H} = \{H_p : p \in P\}$ a 2-connection, X an M-valued semimartingale and Z a P-valued $\mathcal{F}_0$-random variable such that $\pi \circ Z = X_0$. We know [1] that the stochastic horizontal lift (s.h.l) of X initialized in Z is a P-valued semimartingale Y such that satisfies the following stochastic differential equation

$$\begin{aligned} d_2 Y &= H_Y d_2 X \\ Y_0 &= Z \end{aligned}$$

2 Prolongation of Connections and Stochastic Horizontal Lifts

Let us first introduce some definitions

Definition 3 *Let $P(M,G)$ be a principal fiber bundle and ∇ a CDO of P. We says that ∇ is G-invariant if $\Phi_\nabla(pg)\circ R_{g*} = R_{g*}\circ\Phi_\nabla(p)$ for all $p\in P$ and $g\in G$.*

Definition 4 *Let $P(M,G)$ be a principal fiber bundle and ∇ a G-invariant CDO of P. We says that ∇ is projectable if*

$$\Phi_\nabla(p)(Ker(\pi_*(p))) \subset Ker(\pi_*(p))$$

Example 5 *Let BM be the principal fiber bundle of bases of M, Ψ a CDO of M and Ψ^C (Ψ^H) the complete (horizontal) lift of Ψ to BM [3]. We have that Ψ^C and Ψ^H are projectable.*

The "projection" of ∇ by π is described in the following proposition.

Proposition 6 *Let $P(M,G)$ be a principal fiber bundle and ∇ a G-invariant CDO of P. Then ∇ is projectable iff there is a unique CDO Ψ of M such that π is affine. We says that Ψ is the projection of ∇.*

Proof: Let $L\in\tau_xM$ and $p\in P$ such that $\pi(p)=x$. Then there is $T\in\tau_pP$ such that $\pi_*(p)(T)=L$, we define $\Phi_\Psi(x)(L)$ by $\pi_*(p)(\Phi_\nabla(p)(T))$. Now, we prove that $\Phi_\Psi(x)(L)$ is well defined. For this let $g\in G$ and $S\in\tau_{pg}P$ such that $\pi_*(pg)(S)=L$, we have that

$$\begin{aligned}\pi_*(pg)(\Phi_\nabla(pg)(S)) &= \pi_*(pg)(\Phi_\nabla(pg)\circ R_{g*}\circ R_{g^{-1}*}(S))\\ &= \pi_*(pg)(R_{g*}\circ\Phi_\nabla(p)\circ R_{g^{-1}*}(S))\\ &= \pi_*(p)(\Phi_\nabla(p)\circ R_{g^{-1}*}(S))\end{aligned}$$

On the other hand, $\pi_*(p)(R_{g^{-1}*}(S)) = \pi_*(pg)(S) = \pi_*(p)(T)$, this is $R_{g^{-1}*}(S)-T\in Ker(\pi_*(p))$ and by hypothesis $\Phi_\nabla(p)\circ R_{g^{-1}*}(S)-\Phi_\nabla(p)(T)\in Ker(\pi_*(p))$, thus

$$\pi_*(p)(\Phi_\nabla(p)\circ R_{g^{-1}*}(S)) = \pi_*(p)(\Phi_\nabla(p)(T)$$

We conclude that $\Phi_\Psi(x)(L)$ is well defined. Obviously, $\Phi_\Psi : \tau M \longrightarrow TM$ is a morphism of vector bundles such that $\Phi_\Psi \circ i = Id_{TM}$, and define a CDO Ψ and as

$$\Phi_\Psi(\pi(p)) \circ \pi_*(p) \quad = \quad \pi_*(p) \circ \Phi_\nabla(p)$$

for all $p \in P$, we have that π is affine and Ψ is unique.
Conversely, given $L \in Ker(\pi_*(\mathrm{p}))$ we have that:

$$\pi_*(p)(\Phi_\nabla(p)L) \quad = \quad \Phi_\Psi(\pi(p))(\pi_*(p)L) \quad = \quad 0$$

□

Now, we give the definition of prolongation.

Definition 7 *Let $P(M,G)$ be a principal fiber bundle. An application φ from* 1*-connections into* 2*-connections of $P(M,G)$ is called a prolongation if $\varphi(\mathbf{H})_R = \mathbf{H}$ for every* 1*-connection $\mathbf{H}$ of $P(M,G)$.*

In [8] P. Meyer states that there is a canonical prolongation $\mathbf{H} = \{H_p : p \in P\} \longrightarrow \mathbf{H}^S = \{H_p^S : p \in P\}$, this prolongation is called the Stratonovich prolongation and is characterized by

$$H_p^S\{X,Y\} = \{HX, HY\}_p$$

where X and Y are local vector fields of M.

Now we state our main result.

Theorem 8 *Let $P(M,G)$ be a principal fiber bundle, ∇ a G-invariant CDO of P and Ψ a CDO of M. Then ∇ is projetable with projection Ψ iff there is a unique prolongation $\mathbf{H} \longrightarrow \mathbf{H}^r$ such that satisfies: Every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale.*

Proof: Let $\mathbf{H} = \{H_p : T_{\pi p}M \longrightarrow T_pP\}$ be a 1-connection of $P(M,G)$. Then there is a unique 2-connection $\mathbf{H}^r$ ={$\mathbf{H}^r_p$:¿½pM→¿ pP} of $P(M,G)$ such that

$$H_p^\nabla : \tau_{\pi p}M \longrightarrow \tau_p P \ is\ affine$$

In fact, by [5, Lemma 11] we have that

$$H_p^\nabla = (\exp_p^\nabla \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p)$$

then the map $p \to H_p^\nabla$ is smooth,

$$\pi_*(p) \circ H_p^\nabla = Id_{T_{\pi p}M}$$

and

$$\begin{aligned} R_{g*} \circ H_p^\nabla &= (R_g \circ \exp_p^\nabla \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p) \\ &= (\exp_{pg}^\nabla \circ R_{g*} \circ H_p \circ (\exp_{\pi p}^\Psi)^{-1})_*(\pi p) \\ &= (\exp_p^\nabla \circ H_{pg} \circ (\exp_{\pi pg}^\Psi)^{-1})_*(\pi pg) \\ &= H_{pg}^\nabla \end{aligned}$$

Therefore, we conclude that $\mathbf{H}^r$ is a 2-connection of $P(M,G)$ and as $H_p^\nabla |_{T_{\pi p}M} = H_p$ for every $p \in P$, we have that $\mathbf{H}^r$ is a prolongation of $\mathbf{H}$. Now, let X be a Ψ-martingale and Y a stochastic horizontal lift of X, then Y is solution of $d_2Y = H_Y^\nabla d_2X$, and by the Itô transfer principle [5, Theorem 12] we have that Y is solution of $d^\nabla Y = H_Y d^\Psi X$ (where d^∇ and d^Ψ are the Itô differential in relation to ∇ and Ψ respectively) and as X is a Ψ-martingale, we obtain that Y is a ∇-martingale.

Let ρ be a prolongation such that every stochastic horizontal lift of a Ψ-martingale is a ∇-martingale. Let (x^λ) and (x^λ, y^i) be local charts of M and P respectively. In these local charts

$$\Phi_\Psi\left(\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) = \tilde{\Gamma}^\lambda_{\mu\nu}\frac{\partial}{\partial x^\lambda}$$

$$\begin{aligned} \Phi_\nabla\left(\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right) &= \Gamma^\lambda_{\mu\nu}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{\mu\nu}\frac{\partial}{\partial x^i} \\ \Phi_\nabla\left(\frac{\partial^2}{\partial x^\mu \partial x^j}\right) &= \Gamma^\lambda_{\mu j}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{\mu j}\frac{\partial}{\partial x^i} \\ \Phi_\nabla\left(\frac{\partial^2}{\partial x^j \partial x^k}\right) &= \Gamma^\lambda_{jk}\frac{\partial}{\partial x^\lambda} + \Gamma^i_{jk}\frac{\partial}{\partial x^i} \end{aligned}$$

and [1, page 6]

$$\begin{aligned} \rho(\mathbf{H})\left(\frac{\partial}{\partial x^\lambda}\right) &= \frac{\partial}{\partial x^\lambda} + a^i_\lambda\frac{\partial}{\partial x^i} \\ \rho(\mathbf{H})\left(\frac{\partial^2}{\partial x^\lambda \partial x^\mu}\right) &= \frac{\partial^2}{\partial x^\lambda \partial x^\mu} + a^i_{\lambda\mu}\frac{\partial}{\partial x^i} + a^{ij}_{\lambda\mu}\frac{\partial^2}{\partial x^i \partial x^j} + \\ &\quad 2a^{i\nu}_{\lambda\mu}\frac{\partial^2}{\partial x^i \partial x^\nu} \end{aligned}$$

where $\mathbf{H}$ is a 1-connection locally given by $H\left(\frac{\partial}{\partial x^\lambda}\right) = \frac{\partial}{\partial x^\lambda} + a^i_\lambda\frac{\partial}{\partial x^i}$ and

$$\begin{aligned} a^{ij}_{\lambda\mu} &= \tfrac{1}{2}(a^i_\lambda a^j_\mu + a^i_\mu a^j_\lambda) \\ a^{i\nu}_{\lambda\mu} &= \tfrac{1}{2}(a^i_\lambda \delta^\nu_\mu + a^i_\mu \delta^\nu_\lambda) \end{aligned}$$

Let X be a semimartingale in M and Z a stochastic horizontal lift of X. Locally Z is given as (X^λ, Y^i), where

$$dY^i_t = a^i_\lambda dX^\lambda_t + \tfrac{1}{2}a^i_{\mu\nu}d[X^\mu, X^\nu]_t \tag{1}$$

Now, let X be a Ψ-martingale. In local coordinates X is expressed by

$$dX_t^\lambda = dM_t^\lambda - \tfrac{1}{2}\widetilde{\Gamma}^\lambda_{\mu\nu}(X_t)d[M^\mu, M^\nu]_t \tag{2}$$

where M^λ are local martingales. We obtain from (1) and (2) that

$$dY_t^i = a^i_\lambda dM_t^\lambda + \tfrac{1}{2}(a^i_{\mu\nu} - a^i_\lambda\widetilde{\Gamma}^\lambda_{\mu\nu})d[M^\mu, M^\nu]_t$$

On the other hand, $Z = (X^\lambda, Y^i)$ is a ∇-martingale iff

$$dY_t^i + \frac{1}{2}\Gamma^i_{\mu\nu}d[X^\mu, X^\nu]_t + \Gamma^i_{j\mu}d[Y^j, X^\mu]_t + \frac{1}{2}\Gamma^i_{jk}d[Y^j, Y^k]_t$$

and

$$dX_t^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu}d[X^\mu, X^\nu]_t + \Gamma^\lambda_{j\mu}d[Y^j, X^\mu]_t + \frac{1}{2}\Gamma^\lambda_{jk}d[Y^j, Y^k]_t$$

are local martingale. A direct computation, using the identities previously obtained leads that

$$a^i_\lambda dM_t^\lambda + \frac{1}{2}\left((a^i_{\mu\nu} - a^i_\lambda\Gamma^\lambda_{\mu\nu}) + \Gamma^i_{\mu\nu} + 2\Gamma^i_{j\mu}a^j_\nu + \Gamma^i_{jk}a^j_\mu a^k_\nu\right) d[M^\mu, M^\nu]_t \tag{3}$$

and

$$dM_t^\lambda + \left(\frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \widetilde{\Gamma}^\lambda_{\mu\nu}) + \Gamma^\lambda_{j\mu}a^j_\nu + \Gamma^\lambda_{jk}a^j_\mu a^k_\nu\right) d[M^\mu, M^\nu]_t \tag{4}$$

are local martingale. We obtain from (3) that

$$a^i_{\mu\nu} = a^i_\lambda\Gamma^\lambda_{\mu\nu} - (\Gamma^i_{\mu\nu} + 2\Gamma^i_{j\mu}a^j_\nu + \Gamma^i_{jk}a^j_\mu a^k_\nu)$$

This is $\rho(\mathbf{H}) = \mathbf{H}^{\mathfrak{r}}$. Converselly, we have that

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \widetilde{\Gamma}^\lambda_{\mu\nu}\\ \Gamma^\lambda_{j\mu} &= 0\\ \Gamma^\lambda_{jk} &= 0\end{aligned}$$

since (4) is true for every 1-connection $\mathbf{H}$. Therefore, ∇ is projectable with projection Ψ. □

The next proposition give an explicit expression for $\mathbf{H}^{\mathfrak{r}}$ in terms of $\mathbf{H}$ and ∇.

Proposition 9 *Let X and Y be local vector fields of M. Then*

$$H^{\nabla}\{X,Y\} = \{HX, HY\} - \omega^{H}\left(\nabla_{HX}HY + \nabla_{HY}HX\right)^{*}$$

where ω^{H} is the form of connection associated with **H** *and $* : \mathcal{G} \longrightarrow \Gamma(TP)$ is the homomorphism defined by the right action of G on P.*

Proof: Let X and Y be local vector fields of M, and set $C(X,Y) = H^{\nabla}\{X,Y\} - \{HX, HY\}$. Then $C(X,Y)$ is a vertical local field. In fact, we have that $\pi_*(H^{\nabla}\{X,Y\}) = \pi_*(\{HX,HY\}) = \{X,Y\}$, hence $C(X,Y)$ is vertical. And as

$$\begin{aligned} QH^{\nabla}\{X,Y\} &= H_R^{\nabla}\otimes H_R^{\nabla}(Q\{X,Y\}) \\ &= H\otimes H(Q\{X,Y\}) \\ &= H_R^{S}\otimes H_R^{S}(Q\{X,Y\}) \\ &= QH^{S}\{X,Y\} \end{aligned}$$

where Q is the squared gradient operator (In local coordinates $Q(a^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + a^k\frac{\partial}{\partial x^k}) = a^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$), we have that $C(X,Y)$ is a local vector field. Now, since

$$\begin{aligned} H\left(\Psi_X Y + \Psi_Y X\right) &= H^{\nabla}\left(\Phi_{\Psi}\{X,Y\}\right) \\ &= \Phi^{\nabla}\left(H^{\nabla}\{X,Y\}\right) \\ &= \Phi^{\nabla}\left(\{HX,HY\}\right) + C(X,Y) \\ &= \left(\nabla_{HX}HY + \nabla_{HY}HX\right) + C(X,Y) \end{aligned}$$

we have that

$$\begin{aligned} C(X,Y) &= \omega^{H}(C(X,Y))^{*} \\ &= \omega^{H}\left(H(\Psi_X Y + \Psi_Y X) - \nabla_{HX}HY - \nabla_{HY}HX\right)^{*} \\ &= -\omega^{H}\left(\nabla_{HX}HY + \nabla_{HY}HX\right)^{*} \end{aligned}$$

This completes the proof. □

3 Applications

i) Let $P(M,G)$ be a principal fiber bundle, $\mathbf{H} = \{H_p : p \in P\}$ a 1-connection and ∇ a projectable CDO of P. Let $A_0, A_1, \ldots, A_n$ be C^∞ vector fields on M

and $B_t = (B_t^1, ..., B_t^n)$ a standard Brownian motion, and $X_t(x)$ the solution of the following Stratonovich differential equation

$$\begin{aligned} dX_t &= A_0(X_t)dt + \sum_{i=1}^{n} A_i(X_t) \circ dB_t^i \\ X_0 &= x \in M \end{aligned} \tag{5}$$

Then the stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ in relation to $\mathbf{H}^r$ is given by the solution of

$$\begin{aligned} dY_t &= \left(HA_0 - \frac{1}{2}\sum_{i=1}^{n} \omega^H(\nabla_{HA_i}HA_i)^*\right)(Y_t)dt \\ &\quad + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \in P \end{aligned} \tag{6}$$

In fact, let Z_t be a solution of (6). Since $\pi \circ Z_t = X_t$ and the infinitesimal generator of Z_t is $H^\nabla\left(A_0 + \frac{1}{2}\sum_{i=1}^{n} A_i^2\right)$, by [2, Lemma 2.1] we have that Z_t is the stochastic horizontal lift of X_t in relation to $\mathbf{H}^r$.

ii) Let $E = E(M, \rho, F)$ be a vector bundle associated to $P(M, G)$ with fibre F, $\mathbf{H} = \{H_p : p \in P\}$ a 1-connection of $P(M, G)$ and ∇^E the CDO of E induced by $\mathbf{H}$. Let $Y_t(p)$ be the stochastic horizontal lift of $X_t(\pi p)$ in relation to $\mathbf{H}^r$, and $\eta_t(\pi p) = Y_t(p) \circ p^{-1} : E_{\pi p} \longrightarrow E_{X_t(\pi p)}$, where p is regarded as linear mapping $p : F \longrightarrow E_{\pi p}$. We have the following Itô formula for cross sections of E,

$$\begin{aligned} \eta_t(x)^{-1}\sigma(X_t(x)) - \sigma(x) &= \sum_{i=1}^{n}\int_0^t \eta_s(x)^{-1}\nabla^E_{A_i}\sigma(X_s(x))dB_s^i + \int_0^t \eta_s(x)^{-1} \\ &\quad \left(\nabla^E_{A_0} + \sum_{i=1}^{n}\frac{1}{2}\left(\left(\nabla^E_{A_i}\right)^2 - \frac{1}{2}\overline{\omega^H(\nabla_{HA_i}HA_i)}\right)\right)\sigma(X_s(x))ds \end{aligned}$$

Where σ is a cross section of E and $\bar{\ } : \mathcal{G} \longrightarrow \Gamma(TE)$ is the vertical homomorphism defined by $\overline{A}_e = \frac{d}{dt}\mid_{t=0} p\exp tA \cdot p^{-1}(f)$.

iii) Let BM be the principal fiber bundle of bases of M, ∇ a CDO of M, $\mathbf{H} = \{H_p : p \in P\}$ the 1-connection of BM associated with ∇. We have that ∇^C and ∇^H are projectable with projection ∇. Since $\nabla^H_{HX}HY = H(\nabla_X Y) + \frac{1}{2}\widetilde{R(X,Y)}$, where $R(X,Y)$ is the tensor of type $(1,1)$ defined by $R(X,Y)(Z) = R(X,Y)Z$ (R is the curvature tensor associated with ∇) and $\widetilde{R(X,Y)}$ is the vertical right invariant vector field of BM defined by

$\widetilde{R(X,Y)}_p = (p^{-1}R(X,Y)p)^*_p$ we have that the stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ (solution of (5)) in relation to $\mathbf{H}^{\mathbf{r}\ \mathbf{H}}$ satisfies

$$\begin{aligned} dY_t &= HA_0(Y_t)dt + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \end{aligned}$$

In the case of ∇^C we have that $\nabla^C_{HX} HY = H(\nabla_X Y) + \widetilde{R(-,X)Y}$, where $R(-,X)Y$ is the tensor of type $(1,1)$ defined by $R(-,X)Y(Z) = R(Z,X)Y$ and $\widetilde{R(-,X)Y}$ is the vertical right invariant vector field of BM defined by $\widetilde{R(-,X)Y}_p = (p^{-1}R(-,X)Yp)^*_p$([3, page 94]).

The stochastic horizontal lift $Y_t(p)$ of $X_t(x)$ (solution of (5)) in relation to $\mathbf{H}^{\mathbf{r}\ \mathbf{C}}$ satisfies

$$\begin{aligned} dY_t &= \left(HA_0 - \tfrac{1}{2}\sum_{i=1}^{n} \widetilde{R(-,A_i)A_i}\right)(Y_t)dt + \sum_{i=1}^{n} HA_i(Y_t) \circ dB_t^i \\ Y_0 &= p \end{aligned}$$

References

[1] P. Catuogno: *Second Order Connections and Stochastic Calculus.* Relatorio de pesquisa 31/95. IMECC. UNICAMP, 1995.

[2] P. Catuogno: *Composition and Factorization of Diffusions on Principal Fiber Bundles.* 4to Congreso Antonio Monteiro. UNS. Bahia Blanca. Argentina, 1997.

[3] L.A. Cordero, C. T. J. Dobson, M. de León: Differential Geometry of Frame Bundles. Kluwer Academic Publishers, 1989.

[4] M. Emery: Stochastic Calculus in Manifolds. Springer-Verlag, 1989.

[5] M. Emery: *On Two Transfer Principles in Stochastic Differential Geometry.* Sèminaire de Probabilitès XXIV. Lecture Notes in Mathematics **1426**, Springer 1990.

[6] K. Itô: *The Brownian Motion and Tensor Fields on Riemannian Manifolds.* Proc. Inter. Congr. Math. (Stockholm), 536-539, 1962.

[7] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry. Interscience. vol.**1** ,1963.vol. **2**, 1968.

[8] P.A. Meyer: *Géométrie Différentielle Stochastique.* Séminaire de Probabilités XV. Lecture Notes in Mathematics **851**, Springer 1981.

[9] P.A. Meyer: *Géométrie Différentielle Stochastique (bis).* Séminaire de Probabilités XVI. Lecture Notes in Mathematics **921**, Springer 1982.

[10] L. Schwartz: *Géométrie Différentielle du 2e ordre, Semimartingales et Équations Différentielle Stochastiques sur une Variété Différentielle.* Séminaire de Probabilités XVI. Lecture Notes in Mathematics **921**, Springer 1982.

[11] I. Shigekawa: *On Stochastic Horizontal Lifts.* Z. Wahrscheinlichkeitsheorie verw. Geviete **59**, 211-221, 1982.

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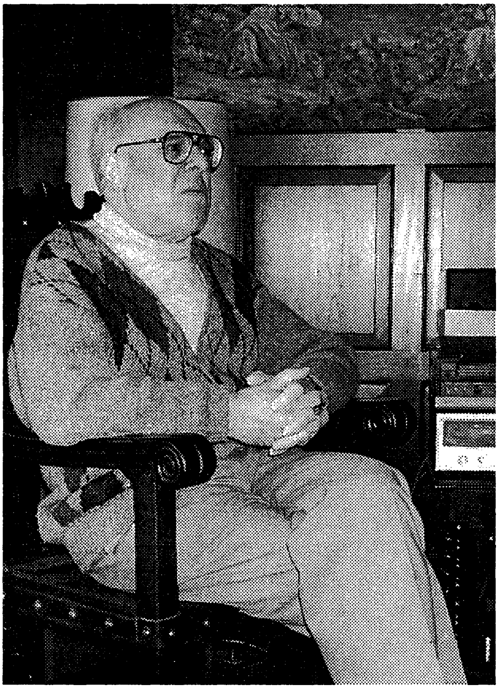

ORLANDO EUGENIO VILLAMAYOR
(1923-1998)

Yo lo sabía enfermo desde hacía muchos años, pero jamás hubiera pensado en una partida tan rápida. Villa era mi amigo y creo que yo le devolvía la amistad que él manifestaba. Un día de febrero me llaman por teléfono: Villa estaba internado en el Hospital *Foch de Suresnes*, cerca de París. Con mi familia lo fuimos a ver el domingo 15 de febrero por la tarde. Verlo fue para mí una inmensa alegría mezclada con una profunda tristeza. No era aceptable que me sentara a su lado sin hablar porque a él le costaba hacerlo. ¿Qué mejor que contarle un chiste de los que le gustaban? Se rió mucho y esto fue para mí una victoria. Durante la semana del 15 al 22, Odila (mi esposa) me decía casi todos los días: llamá para ver cómo está Villa. Llamé el lunes 23 por la mañana y ahí supe que él nos había dejado el domingo 22. En tales momentos la profunda tristeza que nos invade debe ser reemplazada por el recuerdo de los exultantes momentos pasados juntos. El miércoles 25 por la mañana, su esposa María rodeada de la familia y de tres de sus amigos (Andrea Solotar, Max Karoubi y yo), lo llevamos a su última morada en el Cementerio de *Neuilly* (cerca de París), al pie del Arco de la Defensa (*l'Arche de la Défense*.)

El hombre y el matemático. Dos matemáticos tuvieron una importancia fundamental en mi formación científica: Pierre Samuel y Orlando Eugenio Villamayor. Con Samuel he llevado a cabo mi tesis doctoral y con Villamayor la inmensa (para mí) aventura pos-doctoral. Nos conocimos en Montevideo en la Conferencia de Algebra de septiembre de 1965 organizada por Maurice Auslander (que yo había conocido en París y que pasaba un período en Uruguay) y Rafael Laguardia. En el primer día de la conferencia me preguntó: *¿Que hacés?* Le dije que trabajaba en álgebras de Clifford. Me contestó: *¡qué bueno! Yo también estoy en eso.* En realidad yo apenas conocía la definición de álgebra de Clifford pero la dinámica que Villa le imprimía a toda actividad de investigación hizo que en pocos años (1965-1971) publicáramos tres artículos de alguna importancia en este tema ([14], [20] y [23]). En una descripción de sus actividades científicas en el período 1959-1966, Villa habla de su grupo de investigación y dice: *"El grupo trabaja en forma cooperativa".* Era así como a él le gustaba trabajar. Así nació uno de los primeros artículos de la historia de la K-teoría que publicó con A. Nobile ([13]). La primera vez que fui a Buenos Aires (julio de 1968) por cuatro meses, su primer acto fue llevarme a Nuñez para presentarme a los colegas. Yo, como brasileño, hablaba *portuñol* (y lo hablo aún hoy ...) pero creía hablar español. Esto motivó que Gentile dijera: *"Che Villa, qué bien se le entiende el francés a Micali".* Buenos tiempos que nos traen el amargo sabor de la tristeza solamente con recordarlos.
En julio de 1973, Villa invitó a Armand Borel para dar un curso de un mes en la Universidad de Buenos Aires sobre *Grupos algebraicos.* Como siempre, Villa invitaba a los mejores especialistas en cada tema matemático que a él le parecía importante desarrollar en la Argentina. Así era el matemático Villamayor. Desgraciadamente, cuando Borel llega a Buenos Aires el país está en plena crisis política y la Universidad prácticamente cerrada. ¿ Qué hacer con Borel? No sería pensable enviarlo de vuelta y por ello nos pidió a Enzo Gentile y a mí que asistiéramos al curso y lo redactáramos. Así lo hicimos y en abril de 1974 el curso de Borel estaba redactado (en francés). Pero el texto nunca fue publicado.
Creo que la última vez que pasamos algunos buenos momentos juntos fue en el Coloquio de Mendoza (X Coloquio de Algebra) de 1992. Recuerdo que durante el mismo me encontré una tarde con él y María en la calle (el venía de un momento de cura en el hospital). Me agarró por el brazo y me dijo *"vamos a tomar un trago"* Así era el hombre Villamayor: lleno de atenciones para con sus amigos.
Veamos más en detalle los distintos temas sobre los cuáles trabajó Villamayor.

Teoría de Anillos. Este fue, quizás, el primer tema en el cual Villamayor trabajó. Carl Faith dice en uno de sus trabajos (ver Lecture Notes in Mathematics, Springer Verlag, Nº 49, pág. 130), citando a F. W. Anderson, que el siguiente resultado se lo debemos a él. *Para un anillo A, las siguientes condiciones son equivalentes: (i) todo A-módulo a derecha simple es inyectivo; (ii) todo ideal a derecha de A es intersección de ideales a derecha maximales.* La estructura de tales anillos, anillos de Villamayor, fue largamente estudiada en colaboración con G. O. Michler [27]. Por un resultado de I. Kaplansky, todo anillo conmutativo regular es un anillo de Villamayor. Recientemente; Carl Faith y Pere Menal (matemático catalán trágicamente desaparecido hace algunos años) caracterizaron los anillos de Villamayor por la condición del duplo anulador (ver Proc. A.M.S., vol. 123, Number 6, June 1995, pág. 1635-1637). Claro está que se podría decir mucho más pero sólo menciono este aspecto que me parece el más interesante.

Grupo de Brauer y Teoría de Galois. Estos fueron temas fuertes de sus primeros trabajos, inicialmente solo y posteriormente, después de sus primeros viajes a E.E.U.U., en colaboración con matemáticos de renombre como D. Zelinsky. Su sucesión larga de cohomología [38] en la cual aparece el grupo de Brauer (como quinto término) es una extensión importante de los trabajos de Chase - Harrison - Rosenberg. Solo ([4], [12]) o en colaboración con D. Zelinsky ([11], [17]), Villamayor ha hecho importantes contribuciones a la teoría de Galois para anillos.

Algebras de Clifford. Nuestros primeros pasos en la teoría de álgebras de Clifford se dieron en el momento cuando C. T. C. Wall publicaba su texto sobre el grupo de Brauer graduado (1964) y H. Bass su curso en el Tata (1967). Nosotros conocimos el texto de H. Bass cuando nuestro primer artículo [14] ya había sido publicado, pero esencialmente trabajamos en esa misma dirección, es decir, se trataba de determinar la estructura del funtor de Clifford. Nuestra literatura de base era el libro de N. Bourbaki (Formes sesquilinéaires et formes quadratiques) pero creo que fuimos más lejos estableciendo, por ejemplo, teoremas de periodicidad del tipo de Bott. Y éste fue, para Villamayor, otro enfoque que lo acercó a la K-teoría. Todo matemático que trabaja en teoría de formas cuadráticas y álgebras de Clifford es inmediatamente contaminado por una *"enfermedad"* que se llama *"la característica 2"*. Caben dos posibilidades: se supone que la característica del cuerpo es distinta de 2 (o que 2 es inversible si se trata de un anillo) y se va adelante, o se supone directamente que la característica es 2. Para los que no creen en este último caso, puedo testimoniar que aún allí se pueden decir cosas interesantes. Por ejemplo, si K es un cuerpo de característica 2, (V, f) un espacio cuadrático sobre K donde V es un K-espacio vectorial de dimensión finita y $f : V \rightarrow K$ una forma cuadrática, entonces (V, f) se descompone bajo la forma $(V, f) = (V_1, f_1) \perp (V_2, f_2) \perp (V_3, f_3)$ donde $f_1 : V_1 \rightarrow K$ es no degenerada, y en particular V_1 es de dimensión par, $f_2 : V_2 \rightarrow K$ es una forma aditiva y anisotrópica y $f_3 : V_3 \rightarrow K$ la forma nula. El álgebra de Clifford de (V, f) es el producto tensorial graduado de las álgebras de Clifford de las tres componentes y las álgebras de Clifford de la primera y tercera componentes son conocidas (una es un álgebra simple en el sentido graduado y la otra es un álgebra exterior). Queda pues por estudiar el álgebra de Clifford de la segunda componente y para ella tenemos el siguiente resultado ([46], [48]): *Sean K un cuerpo conmutativo de característica 2, (V, f) un K-espacio cuadrático donde V es un K-espacio vectorial de dimensión finita n y $f : V \rightarrow K$ una forma cuadrática aditiva y anisotrópica. Existe entonces una extensión puramente inseparable L de K de dimensión 2^s, $1 \leq s \leq n$, tal que el álgebra de Clifford $C_K(V, f)$ es K-isomorfa a la L-álgebra exterior $\Lambda_L(L^{n-s})$ (isomorfismo graduado). En particular el álgebra de Clifford $C_K(V, f)$ es conmutativa y tiene solamente a 0 y 1 como idempotentes.*

K-**teoría.** Como decía antes, uno de los primeros artículos de la historia de la K-teoría fue el que Villamayor publicó con A. Nobile [13]. En su primer viaje a Montpellier en febrero-marzo de 1968, organizamos (en marzo de 1968) una reunión de una semana sobre K-teoría y álgebras de Clifford. No recuerdo exactamente quiénes fueron los participantes de ese encuentro pero en las notas publicadas en ese entonces figuran los nombres de H. Bass, L. Gruson, M. Karoubi, D. Lehmann y Shih Wei-Shu además de Villa y yo. Ese encuentro se conoció como el K-Coloquio o más bien, como el Primer K-Coloquio pues hubo un Segundo K-Coloquio del 23 al 26 de febrero de 1970 que coincidió con el segundo viaje de Villamayor a Montpellier como profesor de intercambio en enero-febrero-marzo de 1970. Entre los participantes de ese segundo encuentro están A. Fröhlich, C. Contou-Carrère, M. Karoubi, M. Knus, A. Larotonda, J. Larotonda, D. Lehman, M. I. Platzeck, Ph. Revoy, N. Roby, C. A. Ruiz, C. B. Thomas, Shih Wei-Shu, H. O. Singh Varma, J. R. Strooker, J. P. Olivier además de Villa y yo. La idea era realizar un encuentro bi-anual para exponer los progresos realizados. Pero como todo, la teoría creció muy rápidamente y continuó, en Montpellier, con dos reuniones más sobre formas cuadráticas y álgebras de Clifford en 1975 y 1977. En los tres primeros meses de 1970 Villamayor dictó un curso de K-teoría en la Universidad de Montpellier, el cual tenía por finalidad explicar el desarrollo de la teoría hasta ese momento. Las notas manuscritas del curso todavía existen en Montpellier [60]. En ese período, como profesor de intercambio, Villamayor pidió licencia a la Universidad de Montpellier para viajar a Strasbourg (del 6 al 9 de febrero de 1970) donde una larga cooperación con M. Karoubi ya había empezado y que lo llevó al año siguiente (enero-febrero-marzo de 1971), como profesor de intercambio a la Universidad de Strasbourg. Su colaboración con M. Karoubi dio origen a una serie de importantes trabajos en K-teoría ([18], [21], [22], [24], [26]) y más tarde en homología cíclica [49]. También merecen ser señalados los trabajos que Villamayor realizó con J. R. Strooker ([34], [35]). Una primera versión de [35], bajo forma de pre-publicación, tenía por título *"Yet another K-theory ?"*.

Singularidades. La contribución de Villamayor en esta dirección fue muy importante. Con él colaboraron K. Mount (referencias [19], [29], [30], [31], [37], [42], [43]), A. Evyatar ([32]), O. E. Villamayor (h) ([40]) y N. Hipps ([37], [42]). La construcción algebraica de singularidades genéricas de Thom - Boardman es uno de sus importantes aportes. La aplicación de la K-teoría a la teoría de curvas algebraicas soluciona un problema parcialmente resuelto por H. Bass en el caso de curvas afines no singulares [25].

BACH - Buenos Aires Cyclic Homology Group. Esta fue una de las grandes aventuras de la homología cíclica. En 1986 Villamayor estuvo una semana en la Universidade de São Paulo (yo estaba aprovechando ahí el año sabático que me había concedido la Universidad de Montpellier) y en ese entonces hicimos un articulito en el cual calculábamos la homología de Hochschild de algunas álgebras de grupos [45]. Este fue, en cierta forma, el preludio de su subsecuente trabajo en homología cíclica ([44], [49], [50] y [52] al [56]). En el seno del grupo BACH, que él inspiró y al cual dio su colaboración hasta último momento, Villamayor reunió un grupo de brillantes y jóvenes matemáticos argentinos. Quizás sería aquí el momento de recordar un resultado de álgebra homológica obtenido en colaboración con M. L. Bruschi [10] y publicado en una revista de poca difusión, la Revista de la Facultad de Ciencias Físico-Matemáticas de La Plata en 1962.

El trabajo docente en la Universidad de Montpellier. La tercera visita de Villamayor a Montpellier fue como profesor asociado de abril a junio de 1973. Su permanencia en la Universidad de Montpellier continuó de octubre de 1973 a septiembre de 1974. En el año escolar 1973/1974, él reemplazó a una colega francesa, la matemática Monique Hakim, que se iba de Montpellier. Durante ese año escolar, Villamayor dictó dos cursos importantes. Uno sobre *"Geometría Afín"* [59] destinado a los estudiantes de licenciatura (son 266 páginas de un texto impreso por la Universidad) y un curso sobre *"Curvas Algebraicas"* (también para la licenciatura) en el cual seguía el conocido libro de R. Walker sazonado con un lenguaje moderno. Villamayor dejó en Montpellier el recuerdo de un matemático de primera línea, de un docente atento para con los estudiantes y de un hombre afable. Esto no quiere decir que su relación con sus colegas fuera siempre cordial. Sobre uno de ellos (que llamaremos acá X), no muy escrupuloso y que para publicar mucho achicaba sus textos, me dijo cierto día: *"X es como las moscas, muchas ... pero chiquititas."* Villamayor apreciaba trabajos completos, aunque estuvieran contenidos en pocas páginas.

Es todavía muy temprano para evaluar la influencia de Orlando Eugenio Villamayor en la matemática y, en particular, en la matemática latino-americana y argentina. Villamayor ha formado generaciones de buenos matemáticos en su peregrinación comenzada como profesor en la Universidad de Cuyo (1949/1952 y 1954/1956) y continuada por Córdoba (1953), La Plata (1956 y 1959/1960), Bahía Blanca (1961) y Buenos Aires (a partir de 1964). Su brillante carrera internacional dispensa toda forma de panegírico. Los matemáticos formados por él sembrarán a su vez las buenas semillas. Su recuerdo quedará vivo entre los que lo conocieron no solamente por sus excepcionales cualidades de matemático sino también por sus cualidades humanas y su manera muy especial de conservar sus amistades.

Artibano Micali

Publicaciones

[1] *Sur les équations et les systèmes linéaires dans les anneaux associatifs I,* C. R. Ac. Sc. Paris 240 (1955), 1681-1683.

[2] *Sur les équations et les systèmes linéaires dans les anneaux associatifs II,* C. R. Ac. Sc. Paris 240 (1955), 1750-1751.

[3] *On the theory of unilateral equations in associative rings,* Revista Matemática Cuyana 1 (1955), 1-40.

[4] *La théorie de Galois pour les anneaux associatifs,* Revista de la Facultad de Ciencias Físico-Matemáticas, La Plata 5 (1956), 173-184.

[5] *Sur une représentation matricielle de l'anneau d'endomorphismes d'un module quelconque,* Revista de la Facultad de Ciencias Físico-Matemáticas, La Plata 5 (1957), 185-190.

[6] *On the semisimplicity of group algebras I,* Proc. A.M.S. 9 (1958), 621-627.

[7] *On the semisimplicity of group algebras II,* Proc. A.M.S. 10 (1959), 27-31.

[8] *On weak dimension of algebras,* Pac. Journal of Math. 9 (1959), 941-951.

[9] *Fibrados vectoriales algebraicos,* Notas de Matemáticas 3, Universidad Nacional de la Plata, Departamento de Matemática, La Plata 1961.

[10] *Una formulación del teorema de los modelos acíclicos,* con M. L. Bruschi, Revista de la Facultad de Ciencias Físico-Matemáticas, La Plata 8 (1962), 35-38.

[11] *Galois theory for commutative rings with a finite number of idempotents,* con D. Zelinsky, Nagoya Math. Journal 27 (1966), 721-731.

[12] *Separable algebras and Galois extensions,* Osaka Journal of Math. 4 (1967), 161-171.

[13] *Sur la K-théorie algébrique,* con A. Nobile, Ann. École Normale Sup. 1, (1968), 581-616.

[14] *Sur les algèbres de Clifford,* con A. Micali, Ann. École Normale Sup. 1, (1968), 271-304.

[15] *Cubic functors and related K-theories,* Serie Impresiones previas N° 4, Departamento de Matemática de la Universidad de Buenos Aires, 1968.

[16] *Algebra lineal,* Monografías de Matemática de la Organización de los Estados Americanos, Washington, N° 5 (1969).

[17] *Galois theory with infinitely many idempotents,* con D. Zelinsky, Nagoya Math. Journal 35 (1969), 88-98.

[18] *Foncteurs K^n en Algèbre et en Topologie,* con M. Karoubi, C. R. Acad. Sc. Paris 269 (1969), 416-419.

[19] *Taylor series and higher derivations,* con K. Mount, Serie Impresiones previas N° 18, Departamento de Matemática de la Universidad de Buenos Aires, (1969).

[20] *Sur les algèbres de Clifford II,* con A. Micali, Journal für Reine und Ang. Math. 242 (1970), 61-90.

[21] *Sur la suite exacte d'une localisation en K-théorie,* con M. Karoubi, C. R. Acad. Sci. Paris Sér. A-B 271 (1970), 1049-1051.

[22] *K-théorie algébrique et K-théorie topologique I,* con M. Karoubi, Math. Scand. 28 (1971), 265-307.

[23] *Algébres de Clifford et groupe de Brauer,* con A. Micali, Ann. École Normale Sup. 4 (1971), 285-310.

[24] *K-théorie hermitienne,* con M. Karoubi, C. R. Ac. Sc. Paris 272 (1971), 1237-1240.

[25] *The functors K^n for the ring of a curve,* con M. I. Platzeck, Revista de la Unión Matemática Argentina 25 (1971), 389-394.

[26] *K-théorie algébrique et K-théorie topologique II,* con M. Karoubi, Math. Scand. 32 (1973), 57-86.

[27] *On rings whose simple modules are injective,* con G. Michler, Journal of Algebra 25 (1973), 185-201.

[28] *Sur le groupe de Witt,* con A. Larotonda y A. Micali, Symposia Mathematica XI (1973), 211-219.

[29] *On a conjecture of Y. Nakai,* con K. Mount, Osaka J. Math. 10 (1973), 325-327.

[30] *An algebraic construction of the generic singularities of Boardman-Thom,* con K. Mount, Pub. I.H.E.S. 43 (1974), 205-244.

[31] *Weierstrass points as singularities of maps in arbritrary characteristic,* con K. Mount, Journal of Algebra 31 (1974), 343-353.

[32] *Taylor series and divisibility,* con A. Evyatar, Israel Jr. of Math. 18 (1974), 309-319.

[33] *An algebraic construction of the generic singularities of Boardman-Thom,* with K. R. Mount, Inst. Hautes Études Sci. Pub. Math. N° 43 (1974), 205-244.

[34] *A spectral sequence for double complexes of groups,* con J. Strooker, Advances in Math. 15 (1975), 216-231.

[35] *Building K-theories,* con J. Strooker, Advances in Math. 15 (1975), 232-268.

[36] *Algebra multilineal,* con A. Micali, Monografías de Matemática de la Organización de los Estados Americanos N° 16 (1976).

[37] *Sur la polarisation des polynômes homogènes en caractéristique $p \geq 0$,* con K. R. Mount y N. Hipps, C. R. Ac. Sc. Paris 284 (1977), N° 22, 1433-1434.

[38] *Brauer groups and Amitsur cohomology for general commutative ring extensions,* con D. Zelinsky, Journal of Pure and App. Algebra 10 (1977/78), 19-55.

[39] *An extension to fields of positive characteristic of Mather's construction of the Thom-Boardman sequence*, Ann. Sci. École Norm. Sup. (4) 11 (1978),N° 1, 1-28.

[40] *A remark on terrible points,* con Orlando E. Villamayor (h), Revista de la Unión Matemática Argentina 29, N^{os} 1-2 (1979), 77-84.

[41] *On Jacobian extensions of ideals,* Bol. Soc. Brasil. Mat. 10 (1979), N° 1, 87-95.

[42] *Vertices and polarization for homogeneous polynomials,* con N. Hipps y K. R. Mount, Advances in Math. 37 (1980), 105-120.

[43] *Thom-Boardman singularities and monoidal transforms,* con K. R. Mount, Portugaliae Mathematica 40, Fasc. 2 (1981), 137-220.

[44] *Cyclic homology of* $K[\mathbb{Z}/2\mathbb{Z}]$, con G. Cortiñas, Revista de la Unión Matemática Argentina 33, Nros. 1-2 (1987), 55-61.

[45] *Homologie de Hochschild de certaines algèbres de groupes,* con A. Micali, Trabalhos do Departamento de Matemática N° 4, Universidade de São Paulo, São Paulo 1986.

[46] *Formes quadratiques sur un corps de caractéristique 2,* con A. Micali, Communications in Algebra 17 (1985), 299-313.

[47] *On ordinary and symbolic powers of prime ideals,* Journal of Algebra 103 (1986), N° 1, 256-266.

[48] *Algèbres de Clifford sur un corps de caractéristique 2,* con A. Micali, Fundamental Theories of Physics 47 (1992), 65-68, Kluwer Academic Publishers.

[49] *Homologie cyclique d'algèbres de groupes,* con M. Karoubi, C. R. Ac. Sc. Paris Série I Math. 311 (1990), N° 1, 1-3.

[50] *Cyclic homology of* $K[\mathbb{Z}/p\mathbb{Z}]$, con G. Cortiñas y J. Guccione, K-theory 2 (1989), 603-616.

[51] *Noncommutative Taylor series,* con O. Lezama, VIth. National Mathematics Conference (Bucaramanga 1994), Rev. Integr. Temas Mat. 12 (1994), N° 2, 107-121.

BACH - Buenos Aires Cyclic Homology Group

[52] *Cyclic homology of algebras with one generator,* K-theory 5 (1991), N^o 1, 51-69. J. A. Guccione, J. J. Guccione, M. J. Redondo, A. Solotar, and O. E. Villamayor.

[53] *Cyclic homology of hypersurfaces,* Journal of Pure Appl. Algebra 83 (1992), N^o 3, 205-218. J. A. Guccione, J. J. Guccione, M. J. Redondo and O. E. Villamayor.

[54] *Hochschild and cyclic homology of hypersurfaces,* Adv. Math. 95 (1992), N^o 1, 18-60. J. A. Guccione, J. J. Guccione, M. J. Redondo and O. E. Villamayor.

[55] *Cyclic homology of monogenic algebras,* Communications in Algebra 22 (1994), N^o 12, 4899-4904. J. A. Guccione, J. J. Guccione, M. J. Redondo, A. Solotar and O. E. Villamayor.

[56] *A Hochschild homology criterium for the smoothness of an algebra,* Comm. Math. Helv. 69 (1994), N^o 2, 163-168. A. Campillo, J. A. Guccione, J. J. Guccione, M. J. Redondo, A. Solotar and O. E. Villamayor.

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[57] *Taylor series and Wronskian,* Conferencia en el "Mathematiches Forschungsintitut Oberwolfach" en el congreso sobre "Ringe, Moduln und homologische Methoden" del 6 al 12 de mayo de 1973 (resumen en el fascículo del Instituto Oberwolfach.)

[58] *Notas de Geometría I,* Universidad de Nacional de Buenos Aires, Facultad de Ciencias Exactas y Naturales, sin fecha.

[59] *Géométrie Affine,* Université de Montpellier, 1973/1974.

[60] *K-théorie algébrique et K-théorie topologique,* curso dictado en el Departamento de Matemáticas de la Universidad de Montpellier en enero, febrero, marzo de 1970 (notas manuscritas por Philippe Revoy).

Revista de la
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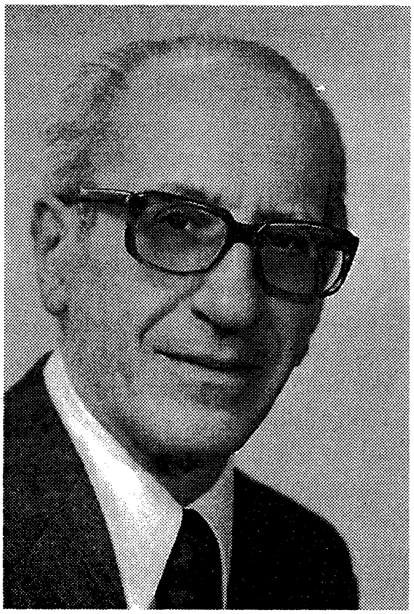

ALBERTO PEDRO CALDERÓN
Matemático

El 16 de Abril de 1998 falleció en Chicago el Profesor Alberto Pedro Calderón tras una corta enfermedad. Había nacido en Mendoza el 14 de Septiembre de 1920 en el seno de una tradicional familia cuyos orígenes en nuestro país se remontan a los primeros años de la colonización española. Cursó estudios secundarios en Suiza donde ya y por inspiración de uno de sus profesores desarrolló su vocación por la matemática. A su regreso y por sabio consejo de su padre ingresó en la Facultad de Ingeniería de la Universidad de Buenos Aires egresando en 1947 con el título de Ingeniero Civil. Su gusto por la matemática lo acercó a D. Julio Rey Pastor y más aún a Alberto González Domínguez que con fino instinto reconoció el talento y originalidad que había en Calderón. Después de una breve pero fructífera experiencia como ingeniero en la empresa Yacimientos Petrolíferos Fiscales fue designado ayudante de González Domínguez en la entonces Facultad de Ciencias Exactas, Físicas y Naturales. En 1948 asistió al seminario que dictó el insigne maestro Antoni Zygmund que se hallaba visitando la Facultad de Ciencias. Su sorprendente actuación en el seminario, dio una nueva y mucho más simple demostración del famoso teorema de Marcel Riesz sobre la función conjugada, hizo que Zygmund le propusiera ir a la Universidad de Chicago para trabajar bajo su dirección, donde recibió su doctorado en 1950.

En 1951, en colaboración con A. Zygmund, publicó en el Acta Mathematica el fundamental trabajo *"On the existence of singular integrals"* donde aplican sus resultados para extender un teorema de Kellogg sobre el potencial Newtoniano que a decir de Calderón contenía el germen de la aplicación de las integrales singulares a las ecuaciones diferenciales parciales. Estando en el Instituto Tecnológico de Massachusetts le encargaron que dictase un curso sobre ecuaciones en derivadas parciales, tema que no era de su especialidad. Fue así que entró en contacto con las ecuaciones diferenciales en derivadas parciales y en 1958 publica el trabajo *"Uniqueness in the Cauchy problem for partial differential equations"* que le dio fama y reconocimiento universal. El esperaba que fuese el método de las integrales singulares y no tanto el resultado lo que llamase la atención . No fue así por mucho tiempo, hasta que la evidencia de posteriores aplicaciones hizo de su método tema de estudio obligado en el área de la ecuaciones diferenciales. En 1959 vuelve a la Universidad de Chicago con el cargo de Professor of Mathematics, continuando su nunca interrumpida colaboración con Zygmund. En 1965 obtiene un resultado sobre conmutadores de integrales singulares que permite eliminar condiciones de Lipschitz sobre los coeficientes de las ecuaciones diferenciales, dando un notable grado de generalidad a resultados propios y ajenos conocidos. El método usado parecía de imposible aplicación para resolver el problema de los conmutadores de orden superior. Sin embargo, con el mismo orden de ideas publica en 1977 el trabajo *"On the Cauchy integral on Lipschitz curves and related operators"* que contiene la solución al problema de los conmutadores de orden superior como caso particular y que abrió una nueva área del análisis.

Lo hasta aquí dicho destaca sólo algunos logros de Calderón que quien escribe esta nota considera hitos en su obra matemática. Ha contribuído decisivamente a la teoría de valores límites de funciones armónicas y analíticas, a la teoría de interpolación de operadores con su llamado Método Complejo, a la teoría ergódica, a las series de Fourier, a las álgebras de Banach, a la teoría de los operadores pseudo diferenciales, a la teoría de los espacios de Hardy, a los problemas de contorno de ecuaciones elípticas. Durante su vida publicó 86 trabajos de investigación, el primero en colaboración con González Domínguez y Zygmund apareció en la Revista de la Unión Matemática Argentina en 1949.

Sus trabajos han sido siempre aportes de gran originalidad habiendo abierto nuevas áreas de la matemática cuya investigación ha atraído a numerosísimos especialistas en el mundo entero. Es altamente significativo que las referencias previas de sus resultados podrían reducirse a resultados propios anteriores, lo que marca el grado de originalidad de los mismos.

Su generosidad en dedicar tiempo y compartir sus ideas con sus discípulos era enorme y más importante aún la amistad y estímulo que les brindaba. Tuvo 27 discípulos que completaron tesis doctorales bajo su dirección. De éstos, 13 fueron argentinos que estudiaron con él, sea en Chicago o en universidades argentinas y muchísimos jóvenes matemáticos se dedicaron al análisis armónico y a las ecuaciones diferenciales directa o indirectamente por su influencia.

Fue Profesor en las Universidades de Ohio State University, Massachusetts Institute of Technology, The University of Chicago y Universidad de Buenos Aires. Además fue Investigador Superior de la Carrera del Investigador Científico y Técnico del Consejo Nacional de Investigaciones Científicas y Técnicas de la Argentina.

Recibió importantísimos premios entre los que se destacan el Bôcher Memorial Prize de la American Mathematical Society en 1979, el Wolf Prize in Mathematics de Israel en 1989 y el Steele Prize (fundamental research work category) también de la American Mathematical Society en 1989, entre otros.

Era miembro de la American Academy of Arts and Sciences de los Estados Unidos de América (1957), Académico Honorario de la Academia Nacional de Ciencias Exactas, Físicas y Naturales de la Argentina (1959), de la National Academy of Sciences de los Estados Unidos de América (1968), miembro correspondiente de la Real Academia Española de Ciencias (1970), miembro de la Academia Latino Americana de Ciencias, miembro extranjero asociado del Instituto de Francia (1984) , y de The Third World Academy of Sciences de Trieste, Italia (1984).

Era Doctor Honoris Causa de las Universidades de Buenos Aires, Technion de Haifa, Ohio State University y de la Universidad Autónoma de Madrid.

El Doctor Alberto Pedro Calderón es sin duda alguna uno de los más importantes matemáticos de este siglo. Su obra matemática será recordada y citada por siempre, habiéndose ganado un lugar entre los grandes matemáticos de todos los tiempos.

Supo también ganarse la estima y el afecto de cuantos lo trataron. En su conversación era profundo y ameno. En sus opiniones equilibrado. Jamás hacía críticas negativas de nadie. Los que tuvieron el privilegio de conocerlo y gozar de su amistad lo tendrán siempre presente mientras vivan y hallaran consuelo por su ausencia en su recuerdo.

Carlos Segovia Fernández

Curriculum vitae del Profesor Alberto P. Calderón.
(de los archivos del Instituto Argentino de Matemática)

Born in: Mendoza, Argentina, September 14, 1920.
Civil Engineering Degree, Universidad de Buenos Aires, Argentina. 1947.
Doctor of Philosophy in Mathematics, University of Chicago, U.S.A., 1950.
I. On the Ergodic Theorems. II. On the Behavior of Harmonic Functions at the Boundary. III. On the Theorem of Marcinkiewicz and Zygmund.

- Honors academies:
  - ◇ Member of the American Academy of Arts and Sciences, Boston, Massachusetts, 1958.
  - ◇ Correspondent Member (1939), and Member (1984), of the National Academy of Exact, Physical and Natural Sciences, Buenos Aires, Argentina.
  - ◇ Member of the National Academy of Sciences of the United States, Washington D.C., 1965.
  - ◇ Correspondent Member of the Royal Academy of Sciences, Madrid, Spain, 1970.
  - ◇ Member of the Latin American Academy of Sciences, 1983.
  - ◇ Foreign Associate ot the Institut de France, Paris, France, 1984.
  - ◇ Member of the Third World Academy oF Sciences, Trieste, Italy , 1984.
- Prizes:
  - ◇ *Latin American Prize in Mathematics*, awarded by IPCLAR (Instituto para la Promoción de las Ciencias, Letras, Artes y Realizaciones), Santa Fe, Argentina, 1969.
  - ◇ *Bôcher Memorial Prize*, awarded by American Mathematical Society, 1979.
  - ◇ *Consagración Nacional Prize*, Argentina, 1989.
  - ◇ *Wolf Prize*, awarded by the Wolf Foundation, Jerusalem, Israel, 1989.
  - ◇ *Steele Prize*, American Mathematical Society, 1989.
  - ◇ *National Medal of Science*, United States of America, 1991.
- Degrees, positions, scholarships:
  - ◇ *Doctor Honoris Causa*, University of Buenos Aires, Argentina, 1969.
  - ◇ *Doctor of Science, Honoris Causa* Technion. Haifa, Israel, 1989.
  - ◇ *Doctor of Science, Honoris Causa* Ohio State University, 1995.
  - ◇ *Louis Block Profesor of Mathematics*, University of Chicago, 1968-1972.

- ◇ *University Professor of Mathematics*, University of Chicago,1975-1985.
- ◇ *Honorary Professor*, University of Buenos Aires, 1975.
- ◇ *Rockefeller Foundation Fellow*, University of Chicago, 1949-1950.

- Teaching, visiting and research positions:
  - ◇ Visiting Associate Professor, Ohio State University, Columbus, 1950-1953.
  - ◇ Temporary Member, Institute for Advanced Study, Princeton, New Jersey, 1953-1955.
  - ◇ Associate Professor, Massachusetts Institute of Technology, 1955-1959.
  - ◇ Professor, University of Chicago, 1959-1968.
  - ◇ For other teaching positions see *Honors*.
  - ◇ Visiting Professor at various times at the following universities: University of Buenos Aires, Cornell University, Stanford University, National University of Bogotá, Colombia, Collège de France, Paris, University of Paris (Sorbonne), Autónoma and Complutense Universities, Madrid, University of Rome, Göttingen University.

- Other professional activities:
  - ◇ Former Associate Editor of the following journals: Transactions of the American Mathematical Society, Illinois Journal of Mathematics, Journal of Functional Analysis, Duke Mathematical Journal, Journal of Differential Equations, Advances in Mathematics.
  - ◇ Former member of the Council of the American Mathematical Society.
  - ◇ Consultant of the Organizing Committe of the International Congress of Mathematicians, Nice, France, 1970.
  - ◇ Former member of the Editorial Committee of the American Mathematical Society.
  - ◇ Member of the Research Career of the National Council of Scientific and Technical Research of Argentina.

- Publications

[1] Calderón, A. P., González Domínguez, A. and Zygmund, A. *Nota sobre los valores límites de funciones analíticas,* Revista de la Unión Matemática Argentina 14 (1949), 16-19.

[2] *On theorems of M. Riesz and A. Zygmund,* Proc. Am. Math. Soc. I (1930), 533-535.

[3] *On the behaviour of harmonic functions at the boundary,* Trans. Amer. Math. Soc. 68 (1950), 47-54.

[4] Calderón, A. P. and Zygmund, A. *On the theorem of Hausdorff-Young and its extensions, Contributions to Fourier Analysis,* Annals of Math. Studies 25. Princeton University Press, Princeton, N.J., (1950), 166-188.

[5] *On a theorem of Marcinkiewicz and Zygmund,* Trans. Amer. Math. Soc, 68 (1950), 55-61.

[6] Calderón, A. P. and Zygmund, A. *Note on the boundary values of functions of several complex variables. Contributions to Fourier Analysis,* Annals of Math. Studies, 25, Princeton University Press, Princeton. N. J., (1950), 145-165.

[7] Calderón, A. P. and Zygmund, A. *On singular integrals in the theory of the potential,* Proc. Int. Congress of Mathematicians, Vol. 1. (1950).

[8] *On the differentiability of absolutely continuous functions,* Revista Mat. Universitá di Parma 2 (1951),203-213.

[9] Calderón, A. P. and Zygmund, A. *On the interpolation of linear operations,* Studia Math. 12 (1951), 194-204.

[10] Calderón, A. P. and Zygmund, A. *On the existence of certain singular integrals,* Acta Math. 88 (1951), 85-139

[11] Calderón, A. P. and Klein, G. *On an extremum problem concerning trigonometrical polynomials,* Studia Math. 12 (1951), 166-169.

[12] Calderón, A. P. and Pepinsky, R. *On the phases of Fourier coefficients of positive real periodic functions, computing methods and the phase problem in X-Ray crystal analysis,* Dept. of Physics, Penn. State College, (1952), 339-349.

[13] Calderón, A. P. and Mann, H. B. *On the moments of stochastic integrals,* Sankhya 12 (1953), 347-350.

[14] *Solution of the scalar radiation problem for surfaces of revolution,* Ohio State Univ. Research Foundation, *Antenna Laboratory* A. F. 18 (600) 88, 478-17, (1953), 4 pp.

[15] *A general ergodic theorem,* Annals of Math. (2) 58 (1952), 183-191.

[16] Calderón, A. P. and Arens, R. *Analytic functions of Fourier transforms,* Segundo Symposium sobre algunos problemas matemáticos que se están estudiando en Latinoamérica, Centro de la UNESCO de Cooperación Científica para América Latina, Montevideo, Uruguay, (1954), 39-52.

[17] *Singular integrals,* Segundo Symposium sobre algunos problemas matemáticos que se están estudiando en Latinoamérica, Centro de la UNESCO de Cooperación Científica para América Latina, Montevideo, Uruguay, (1954), 319-328.

[18] *The multipole expansion of radiation fields,* Journal of Rational Mech. and Anal. 3 (1954), 523-537.

[19] Calderón, A. P. and Zygmund, A., *Singular integrals and periodic functions,* Studia Math. 14 (1954), 249-271.

[20] *On a problem of Mihlin,* Trans. Amer. Math. Soc. 78 (1955), 209-224.

[21] Calderón, A. P. and Arens, R., *Analytic functions of several Banach Algebra elements*, Annals of Math. 2 (1955), 204-216.

[22] *Sur les mesures invariantes,* C. R. Acad. Sci. Paris 240 (1955), 1960-1962.

[23] Calderón, A. P. and Devinatz, A. *On the Fourier-Stieltjes transforms,* Canadian J. Math. 7 (1955), 453-461.

[24] *Sur certaines courbes dans l'espace de Hilbert,* C. R. Acad. Sci. Paris 241 (1955), 539-541.

[25] *Sur certaines courbes à courbure constante dans l'espace de Hilbert*, C. R. Acad. Sci. Paris 241 (1955), 586-587.

[26] Calderón, A. P. and Zygmund, A. *A note on the interpolation of sublinear operations*, Amer. J. Math. 78 (1956), 282-288.

[27] *On singular integrals*, Amer. J. Math. 78 (1956), 289-309.

[28] *Ideals in Abelian group algebras*, Symposium on Harmonic Analysis and related integral transforms, Technical Report, Dept. of Math. Cornell Univ. Ithaca, N.Y. (1956), 12 pp.

[29] *Algebras of certain singular integral operators*, Amer. J. Math. 78 (1956), 310-320.

[30] *Singular integrals operators and differential equations*, Amer. J. Math. 79 (1956), 901-921.

[31] *On a problem of Mihlin*, Trans. Amer. Math. Soc. 84 (1957),559-560. Addenda to the paper.

[32] *Uniqueness in the Cauchy problem for partial differential equations,* Amer. J. Math. 80 (1958), 16-36.

[33] *Integrales singulares y sus aplicaciones a ecuaciones diferenciales hiperbólicas*, Cursos y Seminarios de Matemática, Fasc. 3, Univ. Buenos Aires (1960), 121 pp.

[34] Calderón, A. P., Spitzer, P. and Widom, H. *Inversion of Toeplitz matrices*, Illinois J. Math. 3 (1959), 490-498.

[35] Calderón, A. P. and Zygmund, A. *A note on local properties of solutions of elliptic differential equations,* Proc. Nat. Acad. Sci., U.S.A., 46 (1960), 1385-1389.

[36] *Lebesgue spaces of differentiable functions and distributions*, Proc. Symp. Pure Math. IV (1961), 39-49. Amer. Math. Soc., Providence Rhode Island.

[37] *Existence and uniqueness theorems for systems of partial differential equations*, Proc. Symp. Fluid Dynamics and Applied Math., University of Maryland (1961), 147-195, Gordon and Breach, New York.

[38] Calderón, A. P. and Zygmund, A. *Local properties of solutions of elliptic partial differential equations*, Studia Math. 20 (1961), 171-225.

[39] *On the differentiability of functions which are of bounded variation in "Tonelli's sense"*, Rev. Un. Mat. Arg. 20 (1962), 102-121.

[40] Calderón, A. P., Benedek, A, and Panzone, R. *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A., 48, 3, (1962), 356-365.

[41] *Intermediate spaces and interpolation,* Studia Math., Seria Specjalna, Z. I., (1963), 31-34.

[42] *Boundary value problems for elliptic equations*, Outlines of the Joint Soviet-American Symposium on Partial Differential Equations, (August 1963), 303-304.

[43] Calderón, A. P. and Zygmund, A. *Higher gradients of harmonic functions*, Studia Math. 25 (1964), 211-226.

[44] *Intermediate spaces and interpolation, the complex method*, Studia Math. 24 (1964), 113-190.

[45] *Commutators of singular integral operators*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1092-1099.

[46] *Spaces between L' and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), 273-299.

[47] *Singular integrals,* Bulletin of the A.M.S. 72 (1966), 427-466.

[48] *Estimates for integral operators,* A. F. Office of Scientific Research, 65-1866 (1966), 3-12.

[49] *Algebras of singular integral operators,* A. M. S. Proceedings of Symposia in Pure Math., 10 (1966), 18-55.

[50] Calderón, A. P., Weiss, M. and Zygmund, A. *On the existence of singular integrals,* A.M.S. Proc. of Symposia in Pure Math. 10 (1966), 56-73.

[51] *The analytic calculation of the index of elliptic equations,* Proc. Nat. Acad. Sci. U.S.A. 57 (1967), 1193-1194.

[52] *Ergodic theory and translation invariant operators,* Proc. Nat. Acad. Sci. U.S.A., 59 (1968), 349-353.

[53] *A priori estimate for singular integral operators*, Centro Internazionale Matematico Estivo.

[54] *Uniqueness of distributions*, Revista Unión .Mat. Arg. 25 (1970), 37-65.

[55] Calderón, A. P. and Vaillancourt, R. *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A. 69 (1972), 1185-1187.

[56] *On the boundedness of pseudo-differental operators*, J. Math. Soc. Japan 23 (1972), 274-278.

[57] *Estimates for singular integral operators in terms of maximal functions*, Studia Math. 44 (1972), 563-581.

[58] Calderón, A. P. and Zygmund, A. *On singular integrals*, Studia Math. 46 (1973), 297-299. Addendum to the paper.

[59] *A note on biquadratic forms*, Journal of Linear Algebra and Appl. 7 (1973), 175-177.

[60] Calderón, A. P. and Torchinsky, A. *Parabolic maximal functions associated with a distribution*, Advances in Math. 26 (1975), 1-63.

[61] *Lecture Notes on Pseudo-Differential Operators and Elliptic Boundary Value Problems I*, Cursos de Matemática, Instituto Argentino de Matemática, 83 pp.

[62] *An inequality for integrals*, Studia Math. 57 (1976), 275-277.

[63] *On an integral of Marcinkiewicz*, Studia Math. 57 (1976), 279-284.

[64] *Inequalities for the maximal function relative to a metric*, Studia Math. 57 (1976) 297-306.

[65] *On the Cauchy integral on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 1324-1327.

[66] Calderón, A. P. and Scott, R. *Sobolev inequalities for* $p > 0$, Studia Math. 62 (1978), 75-92.

[67] Calderón, A. P., Calderón, C. P., Fabes, E, Jodeit M., and Riviére, N. *Applications of the Cauchy integral on Lipschitz curves*, Bulletin Amer. Math. Soc. 84, (1978), 287-290.

[68] *Singular integrals with variable kernels*, Applicable Analysis 7 (1978), 221-238.

[69] Calderón, A. P. and Zygmund, A. *A note on singular integrals*, Studia Math. 65 (1979), 77-87.

[70] Calderón, A. P. and Torchinsky A., *Parabolic maximal functions associated with a distribution II*, Advances in Math. 25 (1977), 101-171.

[71] *An anatomic decomposition of distributions in parabolic* H^p *spaces*, Advances in Math. 25, (1977), 216-225.

[72] Calderón, A. P. and Álvarez Alonso, J. D. *Functional calculi for pseudo-differential operators, Fourier Analysis*, Proceedings of Seminar at El Escorial. Asociación Matemática Española, (1979), 3-61.

[73] *Commutator Singular Integrals on Lipschitz Curves and Applications*, Proc. Int. Congress of Mathematicians, Helsinki, (1980), 67-73.

[74] *On the Radon Transform and some of its generalizations*, Conference on Harmonic Analysis in honor of Antoni Zygmund II, (1981), 673-689. Wadsworth Mathematics Series.

[75] *On an inverse boundary value problem*, Sociedade Brasileira de Matemática, Atas 12 (1980), 67-73.

[76] Calderón, A. P. and Capri, O. N. *On the convergence in L' of singular integrals*, Studia Math. 78 (1984), 321-327.

[77] Calderón, A. P. and Álvarez Alonso, J. D. *Functional calculi for pseudodifferential operators*, II, Studies in Appl. Math., Advances in Math. suplementary studies 8, 27-72.

[78] *Boundary value problems for the Laplace equation in Lipschitzian domains, Recent Progress in Fourier Analysis*, North Holland Mathematics Studies III, Notas de Matemática 101, 38-48.

[79] *Integrales singulares y operadores pseudodiferenciales. Historia y perspectiva*, Anales Acad. Nac. Ciencias Exac. Fis. y Nat., Buenos Aires, 38, 33-45.

[80] *Reflexiones sobre el aprendizaje y enseñanza de la Matemática*, Revista de Educación Matemática, Unión Matemática Argentina 3, Nro. 1 (1987), 3-13.

[81] *Presentation of Dr Eduardo Zarantonello as correspondent member of the Academia Nacional de Ciencias Exactas, Físicas y Naturales*, Buenos Aires, Argentina. Annals of the Academia, 40 (1988).

[82] *Presentation of Dr. Misha Cotlar as correspondent member of the Academia Nacional de Ciencias Exactas, Físicas y Naturales*, Buenos Aires, Argentina. Annals of the Academia, 41 (1989).

[83] *Presentation of Dr. Carlos Segovia Fernández as correspondent member of the Academia Nacional de Ciencias Exactas, Físicas y Naturales*, Buenos Aires, Argentina. Annals of the Academia, 42 (1990).

[84] Calderón, A. P. and Sagher, Y. *The Hilbert Transform of the Gaussian*, Proceedings of the 1989 International Conference on a.e. Convergence in Probability and Ergodic Theory, Evanston, Illinois, Academic Press.

[85] Calderón, A. P.,Bellow A. and Krengel, U. *Hopf's ergodic theorem for particles with different velocities and the " strong sweeping out property"*, Canad. Math. Bull., 38 (1), (1995), 11-15.

[86] Calderón, A. P., and Bellow A. *A weak type inequality for convolution products*, to appear.

Tesis dirigidas por el Dr. Alberto Calderón

- Robert T. Seeley, University of Massachusetts, Boston, Ph.D. 1959, Massachusetts Institute ol Technology, *Singular Integrals on Compact Manifolds.*
- Irwin S. Bernstein, City College, CUNY, Ph.D. 1959, M.I.T., *On the Unique Continuation Problem of Elliptic Partial Differential Equations.*
- Israel Norman Katz, Washington University, Dept. of. Systems, Science and Math. St. Louis, Missouri, Ph.D. 1959, M.I.T. *On the Existence of Weak Solutions to Linear Partial Differential Equations.*
- Jerome H. Neuwirth, University of Connecticut, Storrs, CN, Ph.D. 1959, M.I.T., *Singular Integrals and the Totally Hiperbolic Equation.*
- Earl Berkson, University of Illinois, Urbana, IL, Ph.D. 1961, University of Chicago, *I. Generalized Diagonable Operators. II. Some Metrics on the Subspaces of a Banach Space.*
- Evelio Tomás Oklander, Deceased, Ph.D. 1964, University of Chicago, *On Interpolation of Banach Spaces.*
- Cora S. Sadosky, Howard University, Washington, D.C., Ph.D. 1965, University of Chicago, *On Class Preservation and Pointwise Convergence for Parabolic Singular Operators.*
- Stephen Vági, De Paul. University, Ph.D. 1965, University of Chicago, *On Multipliers and Singular Integrals in L^p Spaces of Vector Valued Functions.*
- Nestor Rivière, Deceased, Ph.D. 1966, University of Chicago, *Interpolation Theory in S-Banach Spaces.*
- John C. Polking, Rice University Ph.D. 1966, University of Chicago, *Boundary Value Problems for Parabolic Systems of Differential Equations.*
- Umberto Neri, University of Maryland, College Park, MD, Ph.D. 1966, University of Chicago, *Singular Integral Operators on Manifolds.*
- Miguel De Guzmán, Univ. Complutense de Madrid, Madrid (3), Spain, Ph.D. 1967, University of Chicago, *Singular Integral Operators with Generalized Homogeneity.*
- Carlos Segovia Fernández, Universidad de Buenos Aires, Ph.D. 1967, University of Chicago, *On the Area Function of Lusin.*
- Alberto Torchinsky, Indiana University, Bloomington, IN, Ph.D. 1972, University of Chicago, *Singular Integrals in Lipschitz Spaces of Functions and Distributions.*

- Keith William Powers, Ph.D. 1972, University of Chicago, *A Boundary Behavior Problem in Pseudo-differential Operators.*
- Robert R. Reitano, Senior Financial Officer for John Hancock, Ph.D. 1976, M.I.T., *Boundary Values and Restrictions of Generalized Functions with Applications.*
- Josefina Dolores Alvarez Alonso, Florida Atlantic University, Boca Raton, FL, Ph.D. 1976, Universidad de Buenos Aires, *Pseudo Differential Operators with Distribution Symbols.*
- Telma Caputti, University of Buenos Aires, Ph.D. 1978, University of Chicago, *Lipschitz Spaces.*
- Carlos Kenig, University of Chicago, Ph.D. 1978, University of Chicago, *H^p Spaces on Lipschitz Domains.*
- Angel Eduardo Gatto, De Paul University, Ph.D. 1979, Universidad de Buenos Aires, *An Atomic Decomposition of Distributions in Parabolic H^p Spaces.*
- Cristian E. Gutiérrez, Temple University, Ph.D. 1979, Universidad de Buenos Aires, *Continuity Properties of Singular Integral Operators.*
- Kent Merryfield, California State Univ., Long Beach, Ph.D. 1980, University of Chicago, *H^p Spaces in Poly-Half Spaces.*
- Michael F. Christ, UCLA, Ph.D. 1982, University of Chicago, *Restriction of the Fourier Transform to Submanifolds of Low Codimension.*
- Gerald Cohen, Ph.D. 1982, University of Chicago, *Hardy Spaces: Atomic Decomposition, Area Functions, and Some New Spaces of Distributions.*
- Maria Amelia Muschietti, Universidad Nacional de La Plata, Argentina, Ph.D. 1984, Universidad Nacional de La Plata, *On Complex Powers of Elliptic Operators.*
- Marta Urciuolo, Universidad Nacional de Córdoba, Argentina, Ph.D. 1985, Universidad de Buenos Aires, *Singular Integrals on Rectifiable Surfaces.*

XLVIII REUNION ANUAL DE COMUNICACIONES CIENTIFICAS DE LA UNION MATEMATICA ARGENTINA Y XXI REUNION DE EDUCACION MATEMATICA.

En el Centro Regional Universitario de la Universidad Nacional del Comahue, Bariloche, desde el lunes 21 de septiembre hasta el viernes 25 de septiembre de 1998, se realizaron la XLVIII Reunión Anual de Comunicaciones Científicas y la XXI Reunión de Educación Matemática y el X Encuentro de estudiantes de matemática.

Hubo en total 577 participantes. Se dictaron cursos de perfeccionamiento para docentes de nivel primario, secundario y universitario, y seis cursos para los alumnos de Licenciatura en Matemática.

Las actividades de la XXI Reunión de Educación Matemática comenzaron el lunes 21. Durante su transcurso se dictaron 12 cursillos sobre temas variados. Del 23 al 25 se exibieron paneles sobre la enseñanza de la matemática y hubo veintiseis comunicaciones.

La XLVIII reunión anual de comunicaciones científicas se inició el miércoles 23 de septiembre con la inscripción de los participantes efectuándose por la tarde el acto inaugural en el Hotel Panamericano. En esa oportunidad hicieron uso de la palabra el Dr. Jorge Solomín y la Lic. Raquel Santinelli por la Comisión Organizadora Local.

La figura del Ingeniero Orlando E. Villamayor, desaparecido este año, fue recordada en las palabras de la Dra. María Julia Redondo. De igual manera, el Dr. Carlos Segovia Fernández destacó la trayectoria del Dr. Alberto Calderón, fallecido también este año.

Se procedió a la entrega de los premios del concurso *Néstor Rivière*: el primer premio correspondió a los Sres. Damián Pinasco, Juan Pablo Pinasco y Román Sasyk alumnos de la U.B.A, y el segundo premio al Sr. Miguel Pauletti alumno de la U.N.L.

A continuación la Dra. Eleonor Harboure presentó una semblanza del Dr. Néstor Rivière. Después de un cuarto intermedio el Dr. Juan A. Tirao pronunció la conferencia *Dr. Julio Rey Pastor* sobre el tema *Teoría de representaciones y de Invariantes de Grupos de Lie reductivos.* La jornada culminó con un vino de honor ofrecido a los participantes en el Hotel Panamericano.

Los días 24 y 25 se expusieron las comunicaciones científicas (se presentaron 168) distribuídas en:

- Convexidad y Geometría Analítica.
- Ecuaciones en derivadas parciales.
- Teoría de aproximación.

- Fractales, Teoría de la medida. Estadística.
- Lógica y Geometría.
- Geometría Diferencial.
- Análisis Armónico.
- Ecuaciones Diferenciales Ordinarias, Aplicaciones de la Matemática.
- Análisis Numérico.
- Algebras Asociativas. Algebra Lineal.
- Análisis Funcional. Teoría de Operadores.
- Teoría de Lie. Teoría de Números.
- Física Matemática.
- Optimización. Teoría de Control. Teoría de Juegos.

Las conferencias ofrecidas durante la reunión fueron: *El Teorema de Tichonoff para formas débiles* a cargo del Dr. Xavier Caicedo (Colombia), *Parientes del Teorema de la Corona*, a cargo del Dr. Daniel Suárez (U.B.A.), *Los problemas geométricos como recurso didáctico* a cargo del Dr. Fausto Toranzos (U.B.A.), *El uso de la vieja mayéutica para rejuvenecer la enseñanza de la matemática* a cargo del Dr. Roberto A. Macías (U.N.L.).

El día 25 a las 17 horas tuvo lugar la Asamblea Anual de Socios de la U.M.A.

El congreso se clausuró el viernes 25 a las 20 horas con la conferencia *Dr. A. González Domínguez* a cargo del Dr. Jorge Solomín sobre el tema *Simetrías y Anomalías en Mecánica Clásica.* Finalmente el vice-presidente de la U.M.A. Dr. Jorge Solomín, agradeció a los presentes por su participación y a todos los que colaboraron para el desarrollo de la reunión.

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The title of the manuscript should be written in capital letters and 4 lines below the beginning of the writing space. The name of the author (s) should follow the title, then the institutional data and finally the Abstract in English which shall not exceed of 200 words. Keywords and/ or acknowledgments of financial supports should be written in a footnote on the first page.
At the end of the last page, at least 4 blank lines should be left in order to write the reception dates.
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INDICE

Volumen 41,
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